Proceedings of the Second International Conference on Mathematics Textbook Research and Development.

7–11 May 2017

Rio de Janeiro

EDITED BY GERT SCHUBRING, LIANGHUO FAN & VICTOR GERALDO

Rio de Janeiro: Instituto de Matemática, Universidade Federal do Rio de Janeiro, 2018

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Proceedings of the Second International Conference on Mathematics Textbook Research and Development (ICMT-2)

Editors
Gert Schubring, Lianghuo Fan & Victor Giraldo

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Alef 3, Wege zu Mathematik, ed. Heinrich Bauersfeld. Handbuch zum Lehrgang, Hannover: Schroedel, 1972;
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PREFACE

This volume documents that research on mathematics textbooks has successfully established as a proper research area within mathematics education. A first significant event in this area has been the in the ***International Conference on School Mathematics Textbooks (ICSMT)***, held in Shanghai in 2011. Of particular importance and international impact was the special issue published by *ZDM Mathematics Education* with the theme of "textbook research on mathematics education" (Fan, Jones et al. 2013). Thanks to the progress of research in this area, the first ***International Conference on Mathematics Textbook Research and Development*** (ICMT-2014) took place at the University of Southampton (UK), from 29 to 31 July 2014. About 180 participants, from 30 different countries, attended ICMT-2014 – now to be called ICMT-1, since it was decided there to organise a sequel conference, the ***II International Conference on Mathematics Textbook Research and Development*** / ***II Conferência Internacional em Pesquisa e Desenvolvimento de Livros Didáticos de Matemática*** (ICMT-2).
It was held from 7 to 11 May 2017, at the Federal University of Rio de Janeiro (UFRJ) and at the Federal University of the State of Rio de Janeiro (UNIRIO), Brazil.
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with the support of the Brazilian funding agencies CNPq and CAPES and of the Brazilian Mathematics Education Society (**SBEM**), the Brazilian Society of Mathematics (**SBM**), and the Brazilian Society of Applied and Computational Mathematics (**SBMAC**).

The Conference had been attended by more than 200 participants – from all the five continents. The ICMT-2 comprised 5 plenary lectures, organised three thematic symposia with 20 contributions and eight workshops, among them six especially for Brazilian teachers. The oral communications were organised in nine thematic sections, with together 66 communications, and a poster session with 17 contributions.
In the preparation of ICMT-2, special emphasis has been given to two dimensions for textbook analysis: it is linguistics, for analysing the text of textbooks, and in particular for the role of signs and, as the particularity of mathematical texts, for diagrams. Likewise, historicity is another peculiarity of mathematical texts and so there was emphasis on history of mathematics. Both dimensions had been represented by plenary talks and by thematic sections.

The contributions submitted after the Conference for the Proceedings have been peer-reviewed. Alas, we regret the delay in publishing the Proceedings, but one has to the unexpected burdens in editing imposed by the great number of submitters who did not care to apply the template.

At present, already the ***III International Conference on Mathematics Textbook Research and Development*** is being prepared, for September 2019 in Paderborn (Germany), revealing that the initiative of 2014 turned into a series of conferences showing the productive character of the research area.

in the November of 2018
Gert Schubring
Lianghuo Fan
Victor Giraldo

PLENARY TALKS

TEXTBOOKS FOR MILLIONS: THE BRAZILIAN MATHEMATICS TEXTBOOK ASSESSMENT PROGRAM

João Bosco Pitombeira de Carvalho

Abstract

We review, very briefly, the history of government textbook policies in Brazil as a background to our discussion of the Brazilian national mathematics textbook assessments for PNLD, Programa Nacional do Livro Didático (National textbook program). It started for the 1997 school year and evolved gradually to assess textbooks for public elementary, middle and high schools. We discuss the program´s main features, its positive aspects, the problems it faces, its relationship with publishers and indicate what remains to be done.

Introduction

Let´s start with some numbers. For the 2016 school year (February 2016-November 2016, the Ministry of Education´s *Programa Nacional do Livro Didático* (PNLD - National Textbook Program) bought and distributed textbooks for (almost) all public school students from Grade 1 (6 years old children) to Grade 5 (11 years old students). This cost roughly US$ 418,500,000.00 and approximately 128,600,000 textbooks were distributed. For the following school year, 2017, the Government bought textbooks for students from Grade 6 through Grade 9. In addition, for the 2018 school year it bought and distributed textbooks for high school students (1^{st} through 3^{rd} years). TABLE 1 shows the global numbers of acquired books, serviced schools, recipient students and the total costs from the 2014 through 2017 school years. .

Table 1

PNLD	Books	Schools	Students	Investments[1] (US$)
2014	157,134,808	121,279	39,403,259	406,000,000.00
2015	144,291,373	123,947	30,601,3441	454,200,000.00
2016	128,588,730	121,574	34,513,075	418,500,000.00
2017	152,351,763	117,053	29,416,511	370,260,220.00

FNDE- Estatísticas do PNLD (organized by the author)

If we now look only at mathematics textbooks, the number of books bought for the 2015 school year (PNLD-2015) was 7,555,759 distributed by six approved collections (which we call A, B, C, D, E and F), out of 15 submitted for assessment (Figure 1).

[1] Very approximate values.

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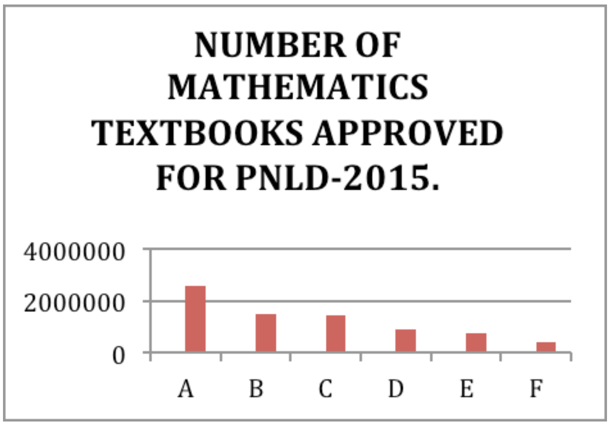

Figure 1 – Number of mathematics textbooks bought for 2015 (FNDE – Estatísticas do PNLD, organized by the author)

A very important part of PNLD is the mandatory assessment of all books presented by publishers for possible acquisition by the Government. Here, we will not discuss the logistic and administrative aspects of PNLD, only this assessment, which started with PNLD-1997. In addition, we will deal mainly with mathematics textbooks.

It might seem a dreary subject to study a textbook assessment program. Notwithstanding, we claim it is important, since this successful program has definitely influenced, for the better, the quality of Brazilian mathematics textbooks and made research on them a respectable academic pursuit. Also, because of the amounts of government acquisitions for PNLD as a whole, the relationship between publishers and Brazilian governments changed greatly since 1997.

Ensino Básico, the formal mandatory school education in Brazil for children and youngsters from six to seventeen years old, is presently divided into *Ensino Fundamental* that lasts nine years – 1st through 9th grade – and *Ensino Médio*, which lasts three years. *Ensino Fundamental* corresponds to the elementary (first five grades) – EF1 – and middle (last four grades) – EF2 – schools of many countries, and *Ensino Médio* corresponds to high school (secondary school). School years run from February through November, with a middle-year break in part of July.

Brazil is divided in 26 *Estados* (states) and a *Distrito Federal* (Federal District). Each state is divided in *municípios* (counties). Sates are responsible for *Ensino Médio*, and counties for *Ensino Fundamental*.

A word of explanation: the numbering of PNLD – for example PNLD 1997, PNLD, 2011, PNLD 2017 – refers always to the school year in which the books will be used. Of course, these books are assessed previously. This should always be kept in mind, to avoid confusion and misunderstandings.

The official site of FNDE – *Fundo Nacional de Desenvolvimento da Educação*, at www.fnde.gov.br, which makes available information about Brazil's textbook policies and programs is an essential information source for anyone interested in PNLD in the last twenty years, including its assessments. Another valuable information source is *Memorial do PNLD* (http:www.cchla.ufrn.br/pnld/) at the *Universidade Federal do Rio Grande do Norte*, which is collecting and preserving the assessments' memory.

The success of the mathematics assessments during the time span covered (PNLD 1997 through PNLD 2018) was in great part due, I believe, to a competent and dedicated group of persons from several institutions that have, in the last 20 years, continuously or not, given their time, knowledge and know-how to assure fair and reliable assessments of mathematics textbooks. Among them, I would like to mention Adriano Pedrosa de Almeida, Marilena Bittar, Bruno Alves Dassie, Iole Druck, Verônica Gitirana, Paulo Figueiredo Lima, Mônica Mandarino, Elisabeth Belfort Moren and Elvira Nadai. It has been a privilege to share, each year, many weeks of productive work not only with them but with more than 250 persons, from all over the country, that have taken part in the assessments.

This paper studies the mathematics textbooks assessments for PNLD 1997 through PNLD 2018. Starting with PNLD 2019, many important changes that worry educators have been instituted.

The author of this paper was actively involved with the mathematics textbooks assessments since 1993, when he coordinated a pilot assessment, which will be mentioned later, until 2017. During all these years, he kept extensive documentation that he has constantly used while writing this paper, which studies the assessments for PNLD from 1997 through PNLD 2018. Some of the statements made in this paper, like "in these two assessments, the improvement of textbook quality due to the assessment pressure is obvious" (p. 8), are based on his careful reading and comparison of the relevant assessment reports and checklists. They give valuable information on how authors and publishers react, along the years, to the assessments.

The author wishes to thank the editors and the referees, whose suggestions improved this paper.

A short history of textbook policies in Brazil

Before 1808, the Portuguese Crown forbade the printing of books in Brazil, and exerted strict censorship to avoid the introduction in Brazil of liberal, impious or subversive ideas (Soares 2013, p. 40). Starting in 1808, when the Portuguese Crown moved to Brazil, after Napoleon´s invasion of the Iberian Peninsula, we see mathematics books translated and printed by the *Impressão Régia*, the State official publishing house, for the professional courses just created in Brazil. We do not know much about the elementary mathematics textbooks used during the first half of the 19th century, partly because of the badly organized system of "aulas régias", created in the 18th century to take the place of Jesuit schools.

After independence from Portugal, in 1822, more precisely after 1830, Brazilians started writing elementary mathematics textbooks, instead of using translations, mainly from the French language. The first elementary mathematics textbook written by a Brazilian and published in Brazil was the *Compêndio de Aritmética*, by Cândido Baptista de Oliveira, in 1832. Of course, all books had to be approved by the ruling authorities before being printed and sold.

In 1837, we witness the establishment of Colégio Pedro II, in Rio de Janeiro, with the first regular and sequential secondary education courses in the country and that was instituted as the model establishment for high schools. If a provincial high school followed Pedro II´s curricula and adopted its textbooks, it was officially accredited and its students had, up to the 1870ties, the right to enter post secondary schools (law, medicine and engineering) without passing an entrance examination.

It seems that the first regulation dealing specifically with textbook censorship was instituted in 1849, in the City of Rio de Janeiro, the seat of the Crown, the *Município da Côrte* .[2] Among other things, it stipulated that textbooks used in schools had to be approved by the provincial governor (Soares 2013 p. 42). Other provinces followed suit. In 1854, an educational reform created, in Rio de Janeiro, the general inspectorship of primary and secondary education, which was in charge, among many other duties, to "review, correct, order to be corrected or substitute, when needed, the textbooks used in public schools" (Soares 2013, p. 43). The approved legislation also encouraged teachers and "learned persons" to write textbooks, with money rewards to the authors of selected works. The prize was awarded automatically if a textbook was adopted by *Colégio Pedro* II. In poorly paid occupations, this was indeed a powerful inducement for the production of mathematics textbooks not only by mathematics teachers but also by engineers and the military. In the *Município da Corte*, the government also took upon itself the task of distributing the approved textbooks, with no cost to the students. Information about this for the empire provinces is scant and unreliable.

We will not dwell upon changes in textbook policies for the remaining years of the empire and for the early republic. The general inspectorship remained responsible for supervising textbooks. An ever present unifying factor in textbook production were the curricula of *Colégio Pedro* II.

[2] Brazil, differently from all other South American countries, did not become a republic after its independence from Portugal in 1822: It became an empire which lasted till 1889, when the republic was instituted.

In 1930 a revolution brought Getúlio Vargas into power, who undertook a sweeping centralizing reform of Brazil. His first education minister, Francisco Campos, reorganized secondary and higher education in 1931. In 1937, a new minister of education, Gustavo Capanema, created the *Instituto Nacional do Livro* (INL, National Book Institute) with the task of publishing important books, a national encyclopaedia and a national dictionary. Besides, it should promote the establishment of a national network of public libraries. In the following year, 1938, the *Comissão Nacional do Livro Didático* (CNLD, National Textbook Committee) was instituted, responsible for all legislation concerning the production and circulation of textbooks (Decreto-Lei n° 1.006/38). CNLD was the first national committee to control all textbook production in Brazil (Filgueiras 2008).
The *Comissão* was initially composed by seven members, chosen by the President, and "renowned for their pedagogical expertise and unimpeachable moral principles" (Decreto-Lei n° 1.006/38, 9, §1). They could not have any commercial relationship with publishing houses and, at first, textbooks they wrote could not be presented for assessment. Teachers and school principals were free to choose any book approved by the committee. Rejected works could not be used in either private or public schools.
The law (Decreto-Lei n° 1.006/38, 20) listed 11 politic and ideological criteria for assessing textbooks and only five didactical and pedagogical ones (Oliveira, Guimarães, Bomény 1984, p.35). The last stated that textbooks could not present scientific or technical errors, violate fundamental pedagogical ideas and must follow the didactical principles officially adopted. All elementary school textbooks had to be written in Portuguese. [3]
Among the duties of the *Comissão* were (SOARES and ROCHA 2005, p 89):

a) Assess the books presented and approve or disapprove them.
b) Stimulate textbook production and provide advice about the importation and adaptation of foreign works.

In 1945, the government issued a law consolidating the several legal statutes about textbook assessment, production and use passed from 1938 on. Among other stipulations, the law stated unequivocally that only teachers or school principals could choose textbooks for school use. Meanwhile, the commission continued to struggle with the task of assessing an ever increasing amount of books (Dassie 2012, p. 96). An official list of approved books was issued only in 1947.
In 1966, during Brazil's military dictatorship, the Ministry of Education (MEC) and the United States Agency for International Development (USAID) signed an agreement that instituted the *Comissão do Livro Técnico e Livro Didático* (COLTED), to finance and organize the production, publication and distribution of textbooks. This provided MEC with funds to distribute 51 million books in a three year period. Besides, many books were translated or written and distributed by the program. This agreement was very important, because it marks the beginning of large scale free book distribution by the government.
Five years later, in 1971, MEC created a new program to substitute COLTED, *Programa do LivroDidático para o Ensino Fundamental* (PLIDEF – Elementary and middle school textbook program), run by *Instituto Nacional do Livro* (INL – National Book Institute).
From then on, the task of buying and distributing textbooks, and sometimes other school materials, was in charge of a succession of federal agencies, like *Fundação Nacional do Material Esco*lar (FENAME – National Foundation for School Materials) and *Fundação de Assistência ao Estudante* (FAE, Student Assistance Foundation). At last, in 1985, PNLD was instituted by law (Decreto Lei n°.91542), with several important characteristics, among which we stress the following: Textbooks will be chosen by teachers and the federal government assumes the cost of distributing books for all students in the first two grades of the public school system and community owned schools. We call

[3] This stipulation was aimed at the networks of ethnic schools in Brazil, mainly the German ones. They were forced to follow the official curricula issued by MEC, and to use only textbooks written in the Portuguese language (Ferreira 2008).

attention that there was no assessment of the books teachers could choose. Publishers presented lists of the books they were willing to sell and these lists were consolidated into a catalogue from which teachers had to select their textbooks. This was an excellent occasion to empty the publishing houses' storage rooms of unsold books.

The details of Brazil's textbook policies, programs and federal agencies from1938 through 1984 can be found in (Filgueiras 2011). In the period 1985-1997 there were no national assessments and we witness a clear loss of textbooks quality, as shown by the results of the national assessments that started with PNLD 1997.

Taking into account the many complaints of teachers and educators in general about the very poor quality of textbooks, MEC instituted, in 1993, a commission with two tasks: firstly, to establish criteria for the assessment of textbooks bought by PNLD.[4] Secondly, to assess the 10 most bought textbooks for each school year, from grade 1 through 4.

The report of this commission (MEC, FAE, PNLD 1994) was staggering. In mathematics for example, 54 books were examined, of which only seven (13%) passed the criteria established by the commission. In other areas, the situation was even worse. Summing up its findings, the mathematics group wrote (MEC, FAE, PNLD 1994, p. 61):

> "[T]he mathematics group was surprised by the poor quality of the texts, the repetition of the same mathematical errors in almost all books, the very poor illustrations, wrong language and disrespect of the child intelligence, due to ridiculous or senseless contextualization. (...) There is imprecise or obscure language, which makes its understanding by the student difficult (...)"

The media had a field day and quoted extensively from the report, which was supposed to be distributed to all schools. Publishers, through their professional associations, protested forcefully. A high MEC official said that the report was very pessimistic, like all academic studies, and it was better to have a bad book than no book at all. He did not stop at words and halted distribution of the report. It seemed the case was closed, that things would go on as always.

The History of the Assessment Program

In 1994, MEC decided to institute an assessment as a mandatory part of PNLD. Of course, there were several reasons for doing so, besides genuine concern about the quality of the textbooks used by millions of children at school. At the time, Brazil negotiated substantial loans for educational programs with international agencies, like the World Bank. This institution stressed the importance of good textbooks to compensate for poorly trained teachers, mentioning that in many countries these books impose *de facto* curricula and are very cheap (Torres 2000, p. 135). The bank also recommended that textbook production to be left to privately owned publishing houses and that the government should publish guides (catalogues with comments) to help teachers to select their textbooks. Besides, international organizations began to insist on accountability and program evaluation. We also mention that such an assessment would generate very positive media coverage for MEC,[5] because of widespread criticism of textbooks quality.

The legal basis for the assessments is provided by the Brazilian Constitution (1988) and by the National Education Act (*Lei de Diretrizes e Bases da Educação Nacional*), of 1996. The first declares that public education of good quality has to be provided by the State; the second affirms that it is the State duty to provide assistance to students, with didactic materials, transportation, food and health services. Besides, MEC is, by law, in charge of supervising the country´s educational system, including the watching over its quality. The government successfully used these legal facts

[4] The mathematics commission had five members, indicated by scientific and educational associations: Anna Franchi, Iara Augusta da Silva, João Bosco Pitombeira de Carvalho (coordinator), Martha Maria de Souza Dantas and Tânia Maria Mendonça Campos.

[5] During the first years of the assessment program, MEC "fed" the press with the worst and most colorful errors found in the assessed textbooks. These "pearls", as they were informally called, sometimes made first page headlines in some of the most widely read newspapers in Brazil.

when *Associação Brasileira de Editores de Livros Escolares* (ABRELIVROS, Brazilian association of textbook publishers), went to court in 1996 against the assessment, claiming that the government violated schoolteachers right to choose the textbooks they would be working with.
We describe and comment, next, the assessments carried out for PNLD 1997 through PNLD 2000, which we might consider the consolidation period of the assessment program. Of course, we cannot cover all details and present only the mainlines of the program evolution.

Consolidation of the program: the first assessments, PNLD 1997 – PNLD 2000

Planning for PNLD 1997 at MEC started in 1994. It was carried out in 1995 and 1996, and the books were in classrooms of the first four grades, all over the country, in February 1997. For the first time since 1985, publishers could not choose what they were willing to sell to the government. Now, they could sell only books successfully assessed by the MEC.
The assessment preparation involved, first, the choice, by MEC, of the coordinators for the assessment groups, one for each school discipline; many persons who had participated in the 1993 pilot assessment were chosen as coordinators for the different school subjects. After this, each area coordinator, jointly with MEC's officials directly responsible for the assessment, proposed a set of criteria, which were subsequently discussed with textbook authors and publishers, and researchers in the teaching of the different school subjects (MEC, FAE, CENPEC 1996, p. 165). This preparation also involved the planning of the whole assessment process, with the discussion and definition of the role of each participant. The result of all these preparations was consolidated in a public call for books, addressed to publishers. The assessment criteria for the mathematics assessment were based on the ones established in 1993, of course with improvements.
One of the major concerns during the first assessment preparation was the choice of assessors. In the case of mathematics, care was taken to select assessors with varied backgrounds, coming from different regions of the country. The assessors had to agree to a confidentiality clause until the assessment results were published and to state they had had no dealing with textbook publishers or authors of books presented for assessment during the preceding three years. Obviously, they could not be authors of books being assessed.
During the preparation for PNLD 1997 it was decided which documents the assessors had to write and how they would proceed to do so. These arrangements have been followed, with variations, until PNLD 2018: Each book is assessed by a team of two experts that, independently, fill out a detailed individual checklist which covers all items the assessors are supposed to consider. Next, in a meeting, each team works together and consolidates their individually filled checklists into a joint checklist and propose that the book be accepted or excluded. For all books, the pair of its assessors write a detailed report, called *parecer*, listing the reasons for approval or exclusion. In the case of accepted books, they also write a short report, called a *resenha* (summary) to be included in the catalogue called *Guia do Livro Didático* (textbook guide) sent to all schools, and from which teachers choose their textbooks.
The difficulties of writing a *resenha* were discussed at length. How to make it an effective help to teachers? Is it a condensed form of the textbook report or an independent text? What to select in the report to be included in the *resenha*? How to transform the technical and academic language of a report into a text understood by and helpful to school teachers?
In PNLD 1997, books could be excluded (EXC), not recommended (NR), recommended with restrictions (RR) or recommended (REC).Teachers could choose books from the three last categories. The compromise to allow teachers to choose not recommended books was made to give teachers time to adapt themselves to the new standards of quality, because the1993 pilot assessment had shown that most of the textbooks in use at that time would be excluded in any forthcoming assessment. Indeed, in PNLD 1997, the excluded and not recommended books represented a little more than 50% of the total! Also, only 27,5 % of the books were approved without any restriction, as shown by Table 2.

Table 2		
Results of the Mathematics Assessment for PNLD 1997		
	Number of books	(%)
Recommended (Rec)	25	27.5
Recommended with Restrictions (RR)	16	17.6
Not Recommended (NR)	37	40.7
Excluded (Exc)	13	14.3
Totals	91	100

(MEC/FNDE 2000, results organized by the author)

Publishers exacted from MEC the promise that the list of the excluded books would not be published, and that a new assessment for the first four grades would happen for the following school year, 1998.

In the assessment for PNLD 1998, the categories used were strongly recommended (RD) recommended, (REC), recommended with restrictions (RR), not recommended (NR) and excluded (EXC). The strongly recommended category (RD, *recomendado com distinção*) was created to induce teachers to choose exceptionally good books. This happened, but most teachers went back, in the following years, to more conventional and down to earth books. The results were the following.

Table 3		
Results of the Mathematics Assessment for PNLD 1998		
	Number of books	(%)
Strongly Recommended (RD)	6	6.7
Recommended (REC)	26	28.9
Recommended With Restrictions (RR)	25	27.8
Not Recommended (NR)	28	31.1
Excluded (EXC)	5	5.6
Totals	90	100

MEC/FNDE 2000, results organized by the author

To compare the results of this assessment with those of PNLD 1997, we group together the strongly recommended books and recommended books, as shown in FIGURE 2.

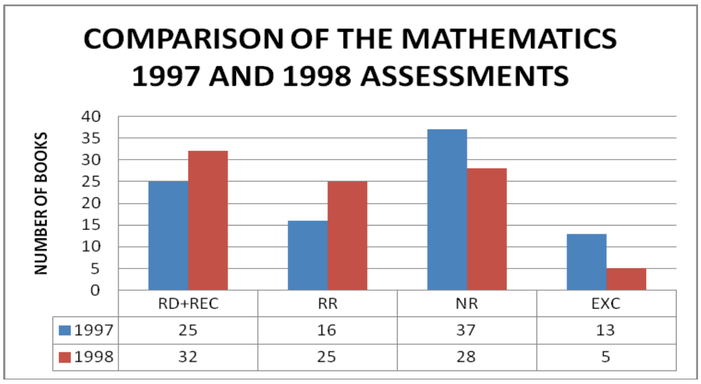

Figure 2 (MEC/FNDE 2000, results organized by the author)

First, we notice a strong decrease of excluded books, which is understandable, since many excluded books in PNLD 1997 were not re-submitted in PNLD 1998. The same is true for the not recommended books. In addition, some books were revised or updated, with a corresponding upgrade in their classification. In these two assessments, the improvement of textbook quality due to the assessment pressure is obvious. [6]

If we compare the choices made by teachers of all school subjects for PNLD 1997 and PNLD 1998, we have the percent results shown in TABLE 4.

Table 4		
Teachers´ Choices In PNLD 1997 and PNLD 1998, for all School Subjects, in Percentual Values		
	1997	1998
RD	---	21.88%
REC	19.64%	14.64%
RR	8.46%	22.15%
NR	71.90%	41.33%

Guia de Livros Didáticos, PNLD 2000.

We see that, when we consider all school subjects, the “shift” towards the lower part of the categories’ spectrum is even stronger than in mathematics. Unfortunately, so far, this characteristic of the mathematics area has not been sufficiently studied to allow explanation. Carvalho and Lima (unpublished) suggested this was due to a more consensual view of school mathematics in Brazil, nurtured by the many congresses and meetings of mathematics education. The divergent points of view of the “new math” movement were long gone and there have not been “maths wars” recently in Brazil.

Starting with PNLD 1998, the three categories, RD, REC, RR were identified by stars in the *Guia de Livros Didáticos*, with respectively, three, two and one star. Publishers objected vehemently to this and a protracted battle on the subject was eventually won by them; since PNLD 2005, there have been only two categories, approved or excluded.

Then, we have the first assessment of books for grades 5 through 8, in PNLD 1999, with the results shown in TABLE 5.

We see, once again, that the excluded books make up almost half of the total of the submissions. A noteworthy evolution of the assessment was the extinction of the category of not recommended books, which was really a contradiction in terms.

[6] These conclusions are based on the personal files of the author, who coordinated the 1997, 1998, 1999 and 2000 assessments.

Table 5 PNLD 1999 – Assessment of Mathematics Textbooks for Grades 5 through 8		
	Books	%
RD	4	5,6
REC	16	22,2
RR	18	25,0
EXC	34	47,2
Totals	72	100

(MEC/FNDE 2000, results organized by the author)

Next, we have PNLD 2000, again an assessment of books for the first four grades. Its main distinguishing feature was that, from that year on, publishers could present only complete collections for assessment, even though the exclusion of one of their volumes did not disqualify the whole collection. Besides, with the growing demands on the area coordinators, adjunct coordinators were added. Furthermore, the assessment criteria were refined: collections could now be excluded because of bad or inappropriate didactical methodologies. The assessment results are shown in TABLE 6.

Table 6 PNLD 2000 – Assessment of Mathematics Textbooks for Grades 1 through 4.		
	Number of books	%
RD	16	13.1
REC	24	19.7
RR	38	31.1
EXC	44	36.1
Totals	122	100

(Personal files of the author)

In the same year, 2000, the MEC established the *Comissão Técnica do Livro Didático* (Textbook Technical Committee), on the one hand to advise MEC on textbook policy, on the other hand to supervise, for MEC, the carrying out of the assessments. The following year, 2001, this committee issued a very important document, *Recomendações para uma política pública de livros didáticos* (Batista 2001) (Recommendations for textbook public policies). The report proposed a comprehensive approach to a policy of didactic materials, not only textbooks: dictionaries, setting up of school libraries, supplementary reading material, maps, literary works adapted for use with children and youngsters, use of several media in the teaching and learning process, etc. The *Comissão Técnica* considered all these as a galaxy with the textbook at its centre. These proposals slowly implemented. In addition, in 2001, the *Comissão Técnica* organized a pilot research on the choice and use of textbooks used in public schools. This was supposed to be followed by a comprehensive research, which so far has not been carried out.

The assessments for PNLD 1997 through PNLD 2000 were carried out directly by MEC. In 2001, it decided to delegate the assessment to a Brazilian private organization with expertise in the assessment and evaluation of educational programs. The *Comissão Técnica do Livro Didático* strongly opposed this. Instead, the *Comissão Técnica* defended forcefully that public universities should carry out the assessments, supervised by the *Comissão Técnica* and MEC. This idea prevailed and, starting with PNLD 2002 until PNLD 2018, the assessments were carried out by public universities; in the case of mathematics this has been done, until PNLD 2018, by Universidade Federal de Pernambuco (UFPe).

The reason for this recommendation was threefold: Firstly, by their very nature, university departments, institutes or schools would not hide their assessing methodologies. Secondly, this would foster research on textbooks in the universities. In addition, thirdly, if it were required that the selected universities choose members for their assessment teams not only among their personnel, but from all over the country, this would create, in Brazil as a whole, an assessment community in universities, educational research institutions, laboratory schools and in state and county education boards.

It was very fortunate, particularly for mathematics, that MEC followed *Comissão Técnica*'s advice and delegated the assessment execution to public universities. While in other areas, research on the respective textbooks was already active in Brazil, in mathematics not much had been done before PNLD (UNICAMP 1989). The assessments fostered the development, in the country as a whole, of a mathematics textbook assessment culture and made research on mathematics textbooks a serious subject in several mathematics education departments, schools of education and mathematics departments. Many doctoral and MSc dissertations have been written on this subject from 2000 on, after PNLD had grown roots.[7]

During this consolidation period, from PNLD 1997 through PNLD 2000, the assessing criteria were discussed with the assessors convened each year and progressively refined. Many lasting features were established: how many persons assess each textbook; the reports the assessors have to write, the structure of the catalogue that is sent to all schools. At first, the mathematics assessment had only a coordinator; starting with PNLD 2000, we have a coordination group with a variable number of advisors. In mathematics, since these first assessments, the final reports the experts write are reviewed again and again by all the coordinating group and the mathematics representative in the *Comissão Técnica*. One person assumes the role of devil's advocate, vigorously defending the excluded collections and attacking the approved ones. This has guaranteed a very high quality; so far, no attempts by authors and publishers to challenge the mathematics assessments have succeeded.

Expansion and flowering: PNLD 2002 – PNLD 2013

PNLD 2002, for grades 1 through 4, was the first assessment performed by UFPe. Publishers

Table 7
PNLD 2002
Assessment of Mathematics Textbooks for Grades 1 through 4.

	Number of books	%
RD	12	17,65%
REC	20	29,41%
RR	20	29,41%
EXC	16	23,53%
Totals	68	100%

(Personal files of the author)

[7] An ongoing project undertaken by this author and other persons involved with the mathematics assessments has already listed almost 130 MSc or PhD dissertations dealing with mathematics textbooks or PNLD policies, and there are more to locate.

presented 17 collections, each one with four books. For the first time, each collection was assessed as a whole: if one of its books was disqualified, the whole collection was excluded. Of the 68 books presented, 34 had never been assessed before. The results are given in Table 7.
It is interesting to compare the results of the two first assessments of books for grades 1 though 4, respectively for 1999 and 2002 (Figure 3).

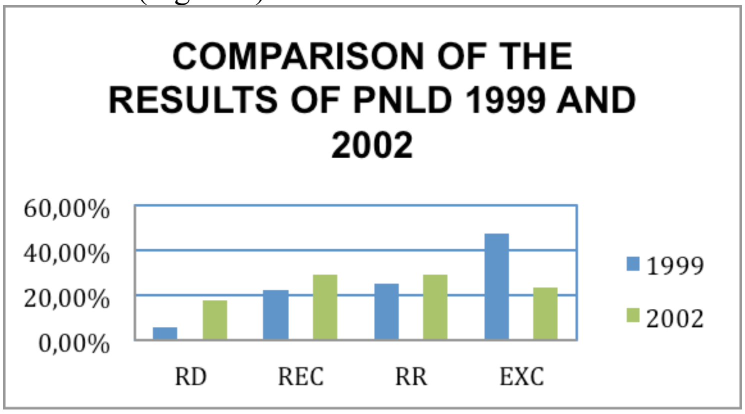

Figure 3 (Personal files of the author)

We notice a considerable percent increase in the best category (RD) from 5.6% to 17.7%. There were modest percent increases in the REC and RR categories (respectively from 22.2% to 29.4% and from 25% to 29.4%), and a very good percent reduction in the number of excluded books, from 47.2% to 23,6%.
From that year on, the textbook policies proposed in the report written by the *ComissãoTécnica* in 2001, (Batista 2001) were gradually implemented. FNDE started the distribution not only of textbooks for elementary, middle and high schools, but also of textbooks for adult education classes and for rural schools, dictionaries, geographic atlas, supplementary reading material for students, books for school libraries, all of them assessed.[8]
Starting with PNLD 2005, MEC finally bowed to publishers, and stopped ranking approved textbooks: From that year on, there were only two categories, approved and excluded books. Assessment of high school books started in 2004, as a pilot project, not part of PNLD. The following year, this program became nationwide. In 2009, high school mathematics textbooks were assessed again, this time by *Sociedade Brasileira de Matemática* (Brazilian Mathematical Society). Three years later, all textbooks assessments were incorporated into PNLD, and UFPe carried out PNLD 2012.
PNLD 2010, for elementary school (EF1), had a new structure. In that year, this school level was increased from four to five years. The first two years of this period had classes only in Portuguese language and *alfabetização matemática*, that is, mathematical literacy, basic mathematics for six and seven years old children. In 2013, this was increased for 3 years. From PNLD 2010 on, all assessments of textbooks for elementary school (EF1) have been divided in those two parts: collections for these initial years (first, 2 years; later on3 years), and collections for the final years (first, 3 years; later on 2 years).
A very important fact was a law (Law 7084, of January 27, 2010), passed in 2010, which made the pedagogical assessment of books for PNLD mandatory. It also created the category of conditionally approved collections, that is, collections that can be approved if authors and publishers correct minor mistakes found during the assessment process. Besides, publishers could now appeal against the exclusion of a collection. We mention again that in mathematics, so far, no appeals have been successful.

[8] This expansion was reversed, from 2014 on: now, FNDE buys just textbooks.

Changes and curtailments: PNLD 2014 – PNLD 2018

In an attempt to "modernize" the teaching and learning processes, MEC decided to distribute digital textbooks, starting with PNLD 2014. Publishers could submit traditional collections or collections that, besides the printed books, had pdf versions with links to multi-media material, called "digital objects". In mathematics, the quality of these objects was very bad, and just a few digital collections were approved. This experience continued until PNLD 2017, with some variations. Because of lack of funds, the digital collections approved for PNLD 2016 and PNLD 2017 were not bought and for PNLD 2018 only paper collections could be submitted. It is a pity that MEC stopped buying digital materials: in the same way that paper textbooks improved slowly since the beginning of the assessments, we hoped to see improvement in the new digital collections. In this period, the mathematics assessment coordination developed, with considerable success, an innovative methodology for the assessment of digital collections.

Starting with PNLD 2016, institutions had to compete to carry out the assessments. In mathematics, there was very little competition. In the other areas, competition slowly diminished. At first, institutions had little idea of the program complexities and of the requirements they had to satisfy and were disqualified and did not renew their attempts. Also, the government was taking some very disquieting decisions concerning PNLD, and there were rumours that much more was to come. Thus, for PNLD 2018, in some areas, it was so difficult to find institutions willing to compete that the deadline for applications had do be extended and members of the *Comissão Técnica* had to convince groups to compete.

Political developments in Brazil culminated in a change of government in 2016. The pressures on the assessments became stronger with the new government and PNLD as a whole came under heavy attack. It was considered too expensive; it would be better to allocate the money to the several state boards of education (Secretarias Estaduais de Educação) and let them choose and buy books as they saw fit. Also, pressure from conservative groups increased constantly. They had strident complaints against the approved textbooks for science and social studies. Some of these groups were against the presentation, in science books, of evolution as a scientific theory; it should at most be mentioned as a hypothesis, they said. In social studies, a more controversial area, some groups were against presenting, for example, Mao Zedong's accomplishments in uniting china. Others were squarely against discussing gender issues, as done in some textbooks. All these groups had access to the media and influence with congressional representatives, who often complained of distortions in the assessments, claiming it was in the hands of irresponsible university professors who had no direct experience with children and youngsters' education. It was said that the assessments carried out by the universities had ideological bias and the Federal Government had no control of its results. In addition, because of the severe Federal Government financial crisis the assessment budget was drastically cut. The *Comissão Técnica* that had been an advisory board on textbook policies became just a *pro-forma* ratifying body of policies formulated without its participation. As a result, in 2017, the government passed Law 9099. As stressed by government officials, control of the assessment was thereby taken away from the universities and concentrated in MEC, starting with PNLD 2019. A disquieting fact was the inclusion of confessional schools among the ones serviced by PNLD.

For PNLD 2018, MEC set up a data bank from which half of the assessors had to be chosen by lot (until then, assessing experts were chosen by the universities that carried out the assessments). In mathematics, since PNLD 2000 a patient process of weeding out resulted in a very good group of assessors, into which, each year, new members were added, while others were discarded. As of PNLD 2019 all the assessors must be chosen by lot from the data bank.

Besides, the *Comissão técnica*, whose members are all chosen by the Minister of Education, was very enlarged, and will include from PNLD 2019 on, members from several scientific, educational and professional organizations, some of them with specific agendas to fight for. Until now, books had to last three years, excepting the ones for first grade. As a bow to publishers, from PNLD 2019 on books shall be disposable, that is, students may write answers on them, and thus they cannot be handed over to next year's students.

This history of twenty years of the mathematics textbook assessment program does not, as all histories, dwell upon all details. We now describe some particular points of the program, sometimes commenting their evolution.

Some specific points of the assessments

The assessments are part of *Programa Nacional do Livro Didático* (PNLD – National Textbook Program), a long chain of steps designed to deliver textbooks to (almost) all public schools in the country before the first school day of each year (First workday in February). The assessment is funded by *Fundo Nacional de Desenvolvimento da Educação* (FNDE – National Education Development Fund), created in 1968, that finances many MEC, among them the distribution of books and other school materials.

Since PNLD 1997, this chain of steps begins when FNDE issues a call for publishing houses interested in selling textbooks. This is a legally binding very long document, which, among many things, deals with the technical characteristics the books must have, like size, paperweight, maximum number of pages, among others; it also stipulates the legal framework, like copyright rights, the deadline for books submission, and so on. And it includes the assessment criteria, both general, for all school disciplines, and the specific ones, for each area.

Assessment criteria

The two groups of criteria have varied slightly along the years, and we summarize them as follows.

General criteria that the collections must conform to (freely translated):

a) Comply with the laws and other legal documents related to Brazilian education.
b) Obey the ethical and democratic principles that underlie a republican and just society.
c) The collection methodology should be in accordance with the methodological principles propounded in the Teacher´s Manual.
d) Present correctly concepts, information and procedures.
e) Promote interdisciplinarity.
f) The editorial project must be in line with the collection´s didactical, scientific and pedagogical goals.

The laws and legal documents mentioned in a) define the goals of elementary, middle and high school. Besides, a) and b), jointly, forbid any kind of discrimination whatsoever. In addition, the collections must also show the variety and the richness of the several cultures that make up Brazilian society, in particular the contribution of African Brazilians. Besides, we have the children and youngsters statute (*Estatuto da criança e do adolescente,* Law 8069 of July 13th 1990) that specifies their rights and need of protection and Senior citizens act (*Estatuto do Idoso*, Law 10741 of October 1st 2003).

The specific criteria for mathematics are (freely translated):

- The collection must present all fields of school mathematics, adapted to the student's cognitive abilities, (for elementary and middle schools (EF1 And EF2); and numbers and operations, geometry, measurements, algebra data analysis and probability, for High school – Ensino Médio).
- The collection must stimulate the development of basic cognitive abilities by the student, such as observation, comprehension, argumentation, analysis, synthesis, communication of mathematical ideas, memorization.
- The collection must stress concept development and the power of mathematics for solving problems.

- The teacher´s manual must show the didactical choices available to the teacher and how to select, if he wants to, the topics that will be presented and in which order this might be done.
- The teacher´s manual must present detailed answers to all problems and exercises and guide the teacher on how to make the best use of them.
- The collection cannot advertise goods, commercial services or brands of any kind.

The bolts and nuts of the assessment

From PNLD 1997 through PNLD 2018, the starting point of each mathematics assessment was a meeting of all persons involved in the project. The group studied the assessment criteria, learned how to fill out the checklists and how to write the required reports. After this, each one received the collections she or he was going to assess. Usually, they had one month to work with each collection.

Each expert filled out a long checklist, which covered the contents and methodology of both the student book and the teacher´s manual, and proposed that the collection be accepted, excluded or should be corrected before being accepted. After this, the two experts met and filled out a detailed joint checklist and wrote a joint report stating clearly the reasons for their decision. If the collection was approved or should be corrected, the team wrote, jointly, a short summary, called *resenha* of their report to be included in a catalogue sent to all schools in the country. These *resenhas* had a common structure, and were divided in sections that dealt with specific features of the book so that the teacher could have an idea of how the author deals with the mathematics content and what are its methodological characteristics. There was also a discussion of the teacher´s manual and suggestions on how to use the book in the classroom. If the team in charge of a collection did not agree on a decision concerning its status, the coordination asked someone else to examine the collection.

After the assessors turned in their reports, the coordinating group started to organize all this material. Three difficulties were always present:

a) What is the boundary between a serious error which will automatically exclude the whole collection and a minor mistake that can be corrected by authors and publishers?
b) How can one be sure that problems which excluded a collection are not present in approved ones?
c) For approved collections, does the *resenha* match the detailed report? Is there any disagreement between them? Does it faithfully present the collection to teachers looking for the textbook they will use? Is it really helpful for the teacher?

These difficulties were tackled by the coordinating group with intensive work during several months, reading over and over all checklists, reports, *resenhas*, cross-checking them with the actual textbooks.

The catalogue of approved textbooks

The *Guia de Livros Didáticos*,[9] has changed considerably during these twenty years of the assessments, from PNLD 1997 through PNLD 2018. At first, it contained just an introduction describing its purpose and a commented list of the approved collections.[10] Now, it consists of an introductory message to teachers and texts that: stress the importance of choosing a good textbook; discuss the role of mathematics in school and society; present the assessment criteria; describe the *resenhas'* structure and how to use them efficiently to help to select a textbook; discuss the characteristic of the approved collections. In addition, of course, the *resenhas* for all the approved collections.

[9] All the catalogues can be downloaded from www.fnde.gov.br/programas/livro-didatico/guias-do-pnld.

[10] We remind that from PNLD 2000 on, only complete collections could be presented by publishers and that from PNLD 2002 on, if a book in a collection was excluded, the whole collection was *ipso facto* excluded.

A *resenha*, in PNLD 1997 a simple text that described the textbook and pointed out its strong and weak points, evolved along the years until it reached its very structured form in PNLD 2018: first, comes a small presentation of the collection, stressing its good points and its eventual weaknesses, followed by a list of the collection's contents, year by year. After this, we find several sections, as follows:

1 – *Content organization*, which discusses the distribution, for each year, of the big areas of school mathematics (numbers, geometry, measurements and data handling, for elementary and middle school; numbers, algebra, geometry, statistics and probability, for high school).

2 – *Content presentation*, which discusses how the collection presents each big are of school mathematics.

3 – *Didactical methodology*, which discusses the didactical choices made by the collection's authors.

4 – *Contextualization and interdisciplinarity*, which discusses if they are genuine or artificial, just a pretext to satisfy the requirements set by the *Edital*.

5 – *Contributions to the student citizenship*, which discusses whether the collection contributes to a critical consciousness of society's problems and promotes the principles of a just, democratic society, without prejudices of any kind.

7 – *Editorial project and language use and correction*, which discusses whether the editorial project promotes readability, the contents are easily identified, the illustrations are useful or just ornaments, graphs and maps are accurate and follow the prescribed norms for their presentation, and whether the language is correct and there is a variety of textual types.

8 – *Teacher's Manual*, which discusses whether it presents the authors' conception of school mathematics and its role in society, the pedagogical and didactical choices made, whether the *Manual* is really helpful to the teacher, both in content and for planning his course, and whether it has the answers to all proposed exercises, has extra projects and activities and bibliographical references for further study by the teacher.

9 – *In the classroom*, which discusses the care the teacher must have to make good use of the book.

What have been the effects of the mathematics assessment program?

First and paramount, the improvement in the quality of the textbooks used throughout the country by millions of children and youngsters attending the public school system. In the early days of the assessment program, there were books which stated, for example, that a quadrilateral is a figure formed by four angles or that presented situations in which the total of percent shares exceeded 100%. These extreme cases do not occur anymore. Of course, there are no perfect books, even the ones we ourselves write.

Nationwide textbooks assessments in Brazil stopped in 1985 (Filgueiras 2011). From that year until 1997, there were no assessments at all. The quality of mathematics textbooks became very low as shown by the pilot assessment of 1993 and by PNLD 1997, and steadily improved thereafter, as shown by the examination of PNLDs assessments reports.[11] This is very strong indirect evidence of the effects of PNLD's assessment on the quality of mathematics textbooks. Zúñiga (2007), investigated how authors changed and adapted their books to meet the stronger and stronger requirements of PNLD.

A second effect of the assessment program is the growing importance given to textbook studies in education and mathematics education research groups all over Brazil, as we have already said.

Thirdly, as also already told, the establishment of a growing community of teachers and researchers that have been members of the assessment. Textbook assessment has become part of many courses for prospective mathematics teachers.

Has the assessment program guaranteed that schools receive "perfect books"? No. The process is regulated by a legally binding document (*Edital*) which has explicit criteria for approval. If a book

[11] Personal files of the author.

does not clearly violate one or more of these criteria, it cannot be excluded, even if the assessment group thinks it should not be used in schools. We are dealing here with a situation completely different from the analysis of one or a few textbooks for a Master's or PhD degree. Assessment is a technical, difficult activity that requires trained personnel. Many mathematicians or mathematics teachers do not understand this, and claim that the assessment program approves books with errors, and that it is inconceivable that MEC put in the hand of children or teachers books that have such and such mistakes. Nevertheless, let us remember that mathematical rigor is contextualized: it depends on to whom and when you are speaking or writing.

PNLD and Publishers

Textbook publishing is big business in Brazil, due to the great number of books the government buys each year. It represents, year after year, more than 50% of the total amount of books published and sold in Brazil (Cassiano 2005, 2007). The money involved has attracted, in the last decade, international publishers (Cassiano 2007). Also, it is a market very dependent on government acquisitions.

Brazil is among the top 10 countries in book sales (Cassiano 2007, p. 96), and textbooks represent around 50% of books sold. For the years 2015 and 2016, we have the results shown in Table 8:

Table 8

Books Printed in Brazil

	2015	2016
Textbooks	221,214,936	220,458,397
Total	446,848,572	427,188,993

FIPE, 2016. Table organized by the author

For the total of textbooks sold in the same years, we have the results shown in Table 9, which shows that the textbook industry is greatly dependent on government acquisitions:

Table 9

Textbooks Sold in Brazil

	2015	2016
to the Government	128,622,634	147,631,141
To Others	50,772,492	47,962,585

FIPE, 2016. Table organized by the author

Prior to 1997, publishers had a perfect situation: an assured market, which grew steadily, because of the educational policies to promote inclusion, and no assessment of what they sold. Therefore, it was to be expected that they would react strongly against the first PNLD book assessment (Munakata 1997). In mathematics, there were no direct attacks to the assessment quality, but to the fact that the names of renowned authors were dragged in the mud and that some of them were rashly punished because of childish mistakes that had no influence on the student learning process, etc.

In Brazil, there have been few studies of the publishing of mathematics textbooks. We mention, Zúñiga's doctoral dissertation (Zúñiga 2007), that shows how authors and publishers changed their collections to fit PNLD's assessment requirements. Cassiano's dissertation (Cassiano 2007), does not specifically study mathematics textbooks, but how textbook publishing changed in Brazil in the last 20 years, particularly during 21st century, with the growing concentration of publishers and the presence of big international firms. Meksenas ([1992) and Munakata (1997) have studied the publishing of textbooks in general.

If we compare the first elementary school books assessment, for PNLD 1997 with the assessment of the same school grades for PNLD 2016, we see a great decrease in the number of publishers that submitted books, from 35 to 11! Even if we take into account the fact that some small publishers stopped presenting books, due to lack of success, there is still decrease. If we compare the numbers of publishers that presented books for the first three assessments with the publishers for 2016, we see a reduction from 16 to 11. The same number of publishers (11) presented books for PNLD 2018. Of the 15 collections assessed for PNLD 2018 (high school) only three were new collections. The situation for the other school grades is similar (personal files of the author). This shows the existence of a general pattern: publishers have some collections approved in past assessments that are presented repeatedly, and, each time, a few new ones that in most cases are not successful.[12] There are several reports made by persons hired by publishing houses to evaluate manuscripts for production that their recommendations are not taken into account. Since there are no studies, at least in Brazil, of how publishing houses choose the manuscripts they publish, one cannot speculate on this fact. According Venezky (1992) the same is true for other countries.

How do authors and publishers react to the errors or small mistakes found in the collections and that can be corrected? With a few exceptions, they do not really try to improve their works. The most common reaction is to delete errors or paste in a local correction. Informally, we might say that publishers have "tamed" the mathematics assessment program. By this, we mean that, due to the stability and reliability of the mathematics assessments, they know that collections approved in the past will not be disqualified, so they do not see the need to improve them, and send each time, a few new collections, in the hope – usually wrong – that they might be approved. So, there is very little hope for renovation.

One new feature is the growing presence of books authored by publishing houses. This practice was introduced recently, in the 21st century, by foreign publishers and is increasing. There have been, also, adaptations of foreign textbooks to Brazil, with very bad results.

Some big education firms have also started to sell "educational packages" to states and counties, with textbooks, tests and exams, teacher training courses, and so on. Since states counties are not forced to accept the books freely distributed by MEC,[13] more and more are buying these packages, in the hope that they will solve their school problems, particularly now, when there are state or national tests to assess school quality. These packages do not pass any assessment, and sometimes the quality of their didactic material is very, very low. So far, this new development has not been objectively studied, there has been only anguished bemoaning by educators.[14]

Final Remarks

We have listed above some accomplishments and characteristics of the mathematics assessments for PNLD, which definitely improved the quality of mathematics textbooks. Let us mention some things that remain to be done.

First of all, there is no research on the way books are chosen and used by teachers. The *Comissão Técnica* has repeatedly asked that this be done, with little avail. The assessment process has been continuously refined, but what are the points that make a teacher choose a book? And how do teachers use their chosen textbooks? In 2001, there was a pilot research on these two issues, but a really comprehensive study has never been undertaken by MEC. Also, the assessments have never been evaluated! As a matter of fact, the whole PNLD, a very expensive and complex program has never been evaluated!

There are many critics of PNLD and of its assessments. Discarding proposals of persons who felt they were unjustly harmed by the assessments (Sampaio 2010, 2012) there are those that have raised important questions about PNLD as a whole and about its assessments for example **(**Britto 2011**)**. A recurrent issue is the centralization-decentralization dispute. Why not let states and

[12] Personal files of the author, covering all the assessments, from PNLD 1997 through PNLD 2018.

[13] If a county or state does not want to receive PNLD's textbooks they must notify MEC. They cannot receive these books and, at the same time, pay for some of these packages. Some counties have been sued by the Government because of this practice.

[14] For an exception, see (Britto 2011).

counties select and buy, with funds provided by the central government, the textbooks they will use? (Britto 2011). In the first years of the assessment, two states decided not to accept books bought by PNLD. One, for political reasons; the other, because it had its own didactic materials, including textbooks, that were tailored to the state's educational and pedagogical planning. The state that had opted, for political reasons, not to receive books evaluated by MEC, when faced with the complexity of the task of assessing books, negotiating their acquisition and distributing them to all schools, gave up when the political complications were past.[15] The other state has had a complex story in its relationship with PNLD (Romanini 2013).

What changes should be done in the assessment program? Little thought has been given to this issue. One recurring proposal is that MEC institute a certifying procedure; publishers would present their collections, any time, that would be assessed and approved, or not, for a certain period. This would work more or less like the procedures of the Federal or State agencies that support research. So far, there have not been made comparative, objective studies on this point. The present model has undergone successive improvements, until it reached, in the case of mathematics, a very good "technical level", as of PNLD 2018. This does not mean that there could not exist other, possibly better ways, to assess mathematics textbooks.

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A MULTIMODAL APPROACH FOR THEORISING AND ANALYSING MATHEMATICS TEXTBOOKS

KAY O'HALLORAN

Introduction

Mathematics textbooks play a critical role for the construction of mathematical knowledge through the ordering, presentation and explanation of mathematical concepts and problems and by providing solutions to those problems. As Usiskin (2013) claims, textbooks are "a vehicle for learning mathematics", where "the only other vehicle of comparable importance is the teacher" (p. 716). The centrality of mathematics textbooks for teaching and learning mathematics remains unchanged in the digital age. For example, O'Halloran, Beezer, and Farmer (2018) show that students work through online mathematics textbooks according to the coverage of the mathematical content in class, with greater use of the textbook prior to tests and exams. In other words, mathematics textbooks (hardcopy, pdf and online) are still "the centerpiece of a course" (Usiskin 2013, p. 715).

It is not surprising that mathematics textbooks play a key role in teaching and learning mathematics. Modern mathematics evolved as a written discourse and thus mathematics knowledge is constructed in this format, albeit often today in digital form. In addition, mathematics has a systematically organised hierarchical knowledge structure within different areas and topics (see Bernstein 1999, 2000). Mathematics textbooks play a critical role in organising and linking this content. This is evident in Figure 1(a), where the student is instructed: "You may recall studying quadratic equations in Intermediate Algebra. In this section, we review those equations in the context of our next family of functions: the quadratic functions" (Stitz & Zeager 2013, p. 188). Similarly, in "Worked Example 6 " in Figure 1(b) the reader is instructed to "recall" and pay "attention" to specific content in the instructions highlighted by the red banner (see right hand side of the page).

It is also immediately evident from Figures 1(a) and 1(b) that mathematical knowledge is constructed using language, images and symbolism. The aim of this paper is to investigate how a multimodal approach to mathematics textbooks, where the functions of language, images and symbolism are taken into account in the construction of mathematical knowledge, sheds further light on the nature of mathematical reality and the problems associated with teaching and learning mathematics. As noted by Albert Einstein, mathematical symbolism and images are key to deriving mathematical ideas:

> “The words of language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The physical entities which seem to serve as elements in thought are certain signs and more or less clear images (...). Conventional words or other signs have to be sought for laboriously only in a second stage, where the associative play already referred to is sufficiently established and can be reproduced at will” Albert Einstein, extracted from a letter to mathematician Jacques Hadamard, cited in Cairo (2013, p. 141)

In what follows, different multimodal approaches are briefly reviewed in order to situate the approach adopted here, which is based on Halliday's systemic functional theory (e.g. Halliday, 2008, 2009a, 2009b). The approach involves the two concepts of 'system' – the architecture and underlying organisation of language, image and symbolism which enables these three resources to fulfil various functions in mathematics – and 'text' – the actual choices from these systems which are found in mathematics textbooks. From this perspective, the symbolism and images are theorised

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as systems which have evolved to fulfil to certain functions in mathematics through their underlying organization and strong semantic links with each other.

2.3 QUADRATIC FUNCTIONS

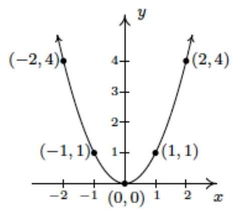
You may recall studying quadratic equations in Intermediate Algebra. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

Definition 2.5. A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

The most basic quadratic function is $f(x) = x^2$, whose graph appears below. Its shape should look familiar from Intermediate Algebra – it is called a **parabola**. The point $(0, 0)$ is called the **vertex** of the parabola. In this case, the vertex is a relative minimum and is also the where the absolute minimum value of f can be found.



$f(x) = x^2$

Much like many of the absolute value functions in Section 2.2, knowing the graph of $f(x) = x^2$ enables us to graph an entire family of quadratic functions using transformations.

Example 2.3.1. Graph the following functions starting with the graph of $f(x) = x^2$ and using transformations. Find the vertex, state the range and find the x- and y-intercepts, if any exist.

1. $g(x) = (x + 2)^2 - 3$
2. $h(x) = -2(x - 3)^2 + 1$

Solution.

1. Since $g(x) = (x + 2)^2 - 3 = f(x + 2) - 3$, Theorem 1.7 instructs us to first *subtract* 2 from each of the x-values of the points on $y = f(x)$. This shifts the graph of $y = f(x)$ to the *left* 2 units and moves $(-2, 4)$ to $(-4, 4)$, $(-1, 1)$ to $(-3, 1)$, $(0, 0)$ to $(-2, 0)$, $(1, 1)$ to $(-1, 1)$ and $(2, 4)$ to $(0, 4)$. Next, we *subtract* 3 from each of the y-values of these new points. This moves the graph *down* 3 units and moves $(-4, 4)$ to $(-4, 1)$, $(-3, 1)$ to $(-3, -2)$, $(-2, 0)$ to $(-2, 3)$, $(-1, 1)$ to $(-1, -2)$ and $(0, 4)$ to $(0, 1)$. We connect the dots in parabolic fashion to get

Figure 1(a) College Algebra (Stitz & Zeager, 2013, p. 188)

Worked Example 6

(Solving a Quadratic Equation by Graphical Method)

The variables x and y are connected by the equation $y = 2x^2 - 5x - 6$.

(i) Complete the table for $y = 2x^2 - 5x - 6$.

x	-2	-1	0	1	2	3	4
y							

(ii) Draw the graph of $y = 2x^2 - 5x - 6$ for $-2 \leqslant x \leqslant 4$.

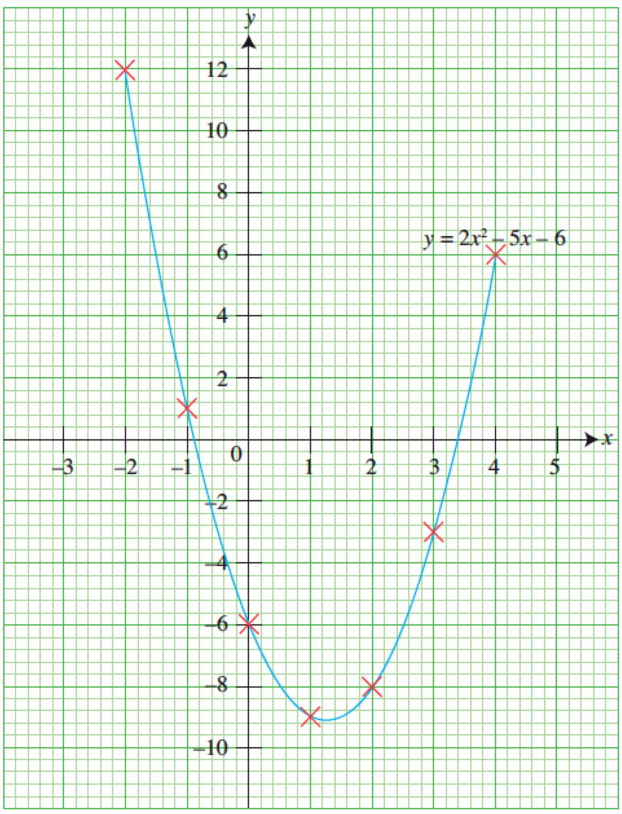
(iii) Hence, solve the equation $2x^2 - 5x - 6 = 0$.

Solution:

(i)

x	-2	-1	0	1	2	3	4
y	12	1	-6	-9	-8	-3	6

(ii)



(iii) From the graph, the x-coordinates of the points of intersection of $y = 2x^2 - 5x - 6$ and the x-axis (i.e. $y = 0$) are $x = -0.9$ and $x = 3.4$.

$\therefore$ The solutions of the equation $2x^2 - 5x - 6 = 0$ are $x = -0.9$ and $x = 3.4$.

RECALL

The solution of a pair of simultaneous linear equations is given by the coordinates of the *point of intersection* of the graphs of the two equations.

ATTENTION

For $2x^2 - 5x - 6 = 0$, the value of $b^2 - 4ac$ is $73 > 0$. Hence, there are two real solutions.

ATTENTION

The answers obtained by the graphical method can only be accurate up to half of a small square grid. In Worked Example 6, the solutions are accurate to the nearest 0.1.

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Figure 1(b) New Syllabus Mathematics 3 (Yeo et al., 2016, p. 13)

That is, the two resources work together with language to construct mathematical knowledge, with each resource playing a specific role. After considering mathematics in this manner, mathematics textbooks are explored in order to show how language, symbolism and images work together to create mathematical knowledge. The nature of linguistic constructions in mathematics textbooks is an integral part of this discussion. Lastly, the subsequent problems which arise in teaching and learning mathematics are investigated.

Multimodal Approaches

Multimodal analysis is a rapidly developing field that has largely arisen in linguistics and language-related fields in order to study how language combines with other resources (e.g. images, gestures, body proxemics and movements, sound and so forth) to create meaning (Bateman, 2014; Jewitt, 2014; Jewitt, Bezemer, & O'Halloran, 2016). This research agenda is particularly critical in today's digital world where language is typically accompanied by images and other resources such as those found in videos and other dynamic media. Jewitt et al. (2016) explain that there are three main approaches to multimodality; namely, the systemic functional approach, social semiotics, and conversation analysis. The first two approaches are based on Halliday's (1978) social semiotic theory of language, which was first extended to images and displayed art by Kress and van Leeuwen (2006) and O'Toole (2011) respectively. The systemic functional approach incorporates different levels of analysis (i.e. the materiality of the text, the 'grammatical' and discourse systems, the genre, and the contexts of the situation and culture). The social semiotic approach also involves analysis of the text, which is viewed in terms of social practices. The systemic functional approach and the social semiotic approach use the same fundamental principles established by Michael Halliday and thus are very similar in terms of theory and analytical approach. The conversational analysis approach to multimodality (e.g. Goffman, 1967) is concerned with 'mutually elaborating semiotic resources' and the sequential organisation and coordination of semiotic activity in social interactions (see Jewitt et al., 2016, p. 91). There are a variety of other multimodal approaches (e.g. geo-semiotics, multimodal (inter)action analysis, multimodal ethnography, multimodal corpus analysis, multimodal reception analysis (see Chapter 6 in Jewitt et al., 2016)), in addition to cognitive approaches to non-verbal and multimodal metaphor (Forceville & Urios-Aparisi, 2009). However, the various approaches are not discrete: for example, systemic functional approaches to multimodal corpus analysis (i.e. big data) are being developed (O'Halloran et al., 2016).

In what follows, a systemic functional multimodal approach (O'Halloran & Lim, 2014; O'Halloran, Tan, & Wignell, 2018 in press) to mathematics textbooks is adopted to investigate the functions and underlying organisation of language, images and symbolism and the contributions of each resource for the construction of mathematical knowledge. This work has been underway since the 1990s (e.g. Chapman, 1992; Morgan, 1996, 2006). For example, Morgan (2006, p. 237) claims that social semiotics and systemic functional linguistics "provide tools that allow a principled description of the language of the texts being studied but also structure interpretation of the functioning of the texts within their contexts of production and consumption". The areas which have been investigated include the multisemiotic nature of mathematics texts (e.g. Lemke, 2003; O'Halloran, 1999, 2005, 2008, 2015), mathematics classroom discourse (O'Halloran, 2000), the historical evolution of mathematics discourse (O'Halloran, 2014), and mathematics, grammar and literacy (O'Halloran, 2007a, 2007b). Some of these findings, particularly those relating to mathematical symbolism, are reviewed in the ensuing discussion. Today, multimodal approaches are a major trend in mathematics education research as researchers increasingly investigate the functions of language, symbolism and images (e.g. Moschkovich, 2010). For example, Schleppegrell (2007) advocates that the multimodal (or multisemiotic) nature of mathematical constructions must be taken into account to understand the linguistic difficulties of learning mathematics. In light of these findings, mathematical symbolism and images are investigated as resources which have been specifically developed in collaboration with language to construct mathematical knowledge.

Language, images and Symbolism

Natural language has a range of functions which have evolved over time (e.g. Halliday, 2003, 2004). That is, language is used to construct our experience of the world in terms of happenings and events (*experiential meaning*). Language is used to logically connect those happenings, for example in terms of additive relations, time, causality and consequence (*logical meaning*). In addition, language is used to enact, maintain and challenge social relations through the expression of affect, social distance and status (*interpersonal meaning*). Lastly, language has resources for organising these meanings (*textual meaning*). The four 'metafunctions' of language (i.e. experiential, logical, interpersonal and textual) are realised through a range of grammatical systems which Halliday has documented in detail (e.g. Halliday & Matthiessen, 2014). Martin and colleagues have extended these descriptions to discourse systems which operate at the rank of paragraph and text (e.g. Martin & Rose, 2007).

Mathematical symbolism evolved from language, but it developed a different form of underlying organisation because it was designed to fulfill different functions. That is, the symbolism was designed to describe patterns and relations (experiential meaning), and thus encoded mathematical entities, operations and relations in ways which made it possible to derive results (logical meaning). As such, the ways in which mathematical symbolism constructs mathematical reality differ from the ways which natural language constructs everyday reality. It is possible to track these differences by tracing the historical evolution of mathematics discourse. For example, as described in history of mathematics (see Nesselmann 1842), algebra initially developed as a form of rhetoric using statements and words. The language-based form of algebraic reasoning was dominant until the 16th century, when some elements were symbolised. Following this, syncopated algebra that consisted of linguistic statements with symbolic elements was developed. In the 16th century, François Viète (1540–1603) developed a symbolic algebra where all elements (numbers, variables, operations and relations) were symbolised and organised according to specific rules. However, René Descartes (1596–1650) created a major breakthrough in the 17th century with the development of Cartesian geometry where geometrical problems were solved algebraically. In this process, language, symbolism and images became fully integrated, providing access to the semiotic potential of each resource and the semantic expansions that took place with movements between them. From here, the symbolism was fully developed as a semiotic resource (i.e. a system of signs), providing the basis for modern mathematics.

This raises several questions: i.e. how does the symbolism differ in relation to the functions and architecture of natural language? And how do these differences in the symbolism relate to mathematical images? These questions are explored by considering the metafunctions of language which by default formed the basis for the development of the symbolism.

1. *Experiential meaning* (content): Natural language is used to construct human experience and thus has underlying systems for structuring thought and reality across the various dimensions of human life. On the other hand, mathematics is used to construct certain areas of human experience only; the description of patterns and relations (i.e. number, quantity and space). However, knowledge in these areas expanded through various innovations. That is, the symbolism was designed to encode complex configurations of mathematical entities, operations and relations in a simple and unambiguous fashion, making it possible to use and manipulate the symbolic descriptions to derive results. A variety of techniques were developed to achieve this goal. For example, general symbols were used to represent mathematical participants and operations, and certain mathematical operations were ellipsed (e.g. multiplication). Brackets and rules of order were developed so that the mathematical operations unfolded in a particular order which was not necessarily in the sequential order in which the operations were written. This is a marked departure from language where thought and reality are structured around linear sequences of happenings, where each happening consists of a limited number of entities and a single process. Furthermore, it was possible to represent the symbolic descriptions (in most cases) in graphs, diagrams and other visual forms. This was a significant advance because the human visual system functions so that the mathematical representation is viewed a whole together with its constituent parts, following Gestalt theory (Koffka, 1935). This opens up a new vista for understanding and deriving mathematical

results. Namely, symbolic reality can be visualised, resulting in new representations of mathematical ideas and concepts that are otherwise unattainable: i.e. "the whole is *something else* than the sum of its parts" (Koffka, 1935, p. 176). In summary, mathematical symbolism provides the means for exactly describing the relations and interactions of multiple entities in what is in essence a new version of reality. Importantly, this new reality can be visualised, thus making it possible to move beyond the realm of its constituent parts.

2. *Logical meaning* (logical relations): The symbolism was designed to be unambiguous and easily manipulated in order to reorganise mathematical operations and relations, increasing the power of the resource as a tool for logical thinking. Significantly, these descriptions combine with visual representations to provide insights beyond those possible with the symbolism and language alone. The symbolism is the resource through which results are derived and problems are solved, but mathematical images play an indispensable role in this process, given that they contribute new representations of mathematical reality. That is, both the symbolism and the images are powerful tools for thinking, as Einstein points out (cited in Cairo, 2013, p. 141).

3. *Interpersonal meaning* (enactment of social relations): Language is used to negotiate social relations, hence there are a variety of linguistic systems for this purpose (e.g. speech functions to give and request information and goods and services; positive and negative polarity; modality systems for expressions of probability, usuality, willingness and obligation; and expressions of attitude, judgement and affect). However, mathematical symbolism was not designed to deal with the complex interplay of human relations, hence the systems for interpersonal meaning found in language were either constrained or removed altogether. For example, mathematics largely consists of statements (with information) and commands (to undertake actions). Expressions of probability, usuality, willingness and obligation (e.g. might, could, would, should) are replaced with probability statements and statistical formulations. As a result, mathematical statements and commands are absolute with positive and negative polarity (i.e. "is" and "is not"). Mathematical images are similar, given the connections to the symbolism and the nature of the relations which are portrayed. Together, the interpersonal stance of mathematics is one of certainty, unlike everyday language with its various expressions of certainty, doubt and obliqueness.

4. *Textual meaning* (organisation of the message): Mathematical symbolism is a written resource so space can be utilised to make meaning (e.g. division, exponents, subscripts and superscripts). The use of space as a system for making meaning, together with the spatial layout of mathematical content, assists with the economy of the expression in mathematics. Furthermore, the advantages of symbolism (economy of expression, simplicity and clearly defined meanings) were combined with the advantages of the visual image (representation of the complete mathematical relations and the constituent parts), resulting in mathematical formulations which are specifically organised to provide immediate and unequivocal access to experiential and logical meanings of the mathematical content.

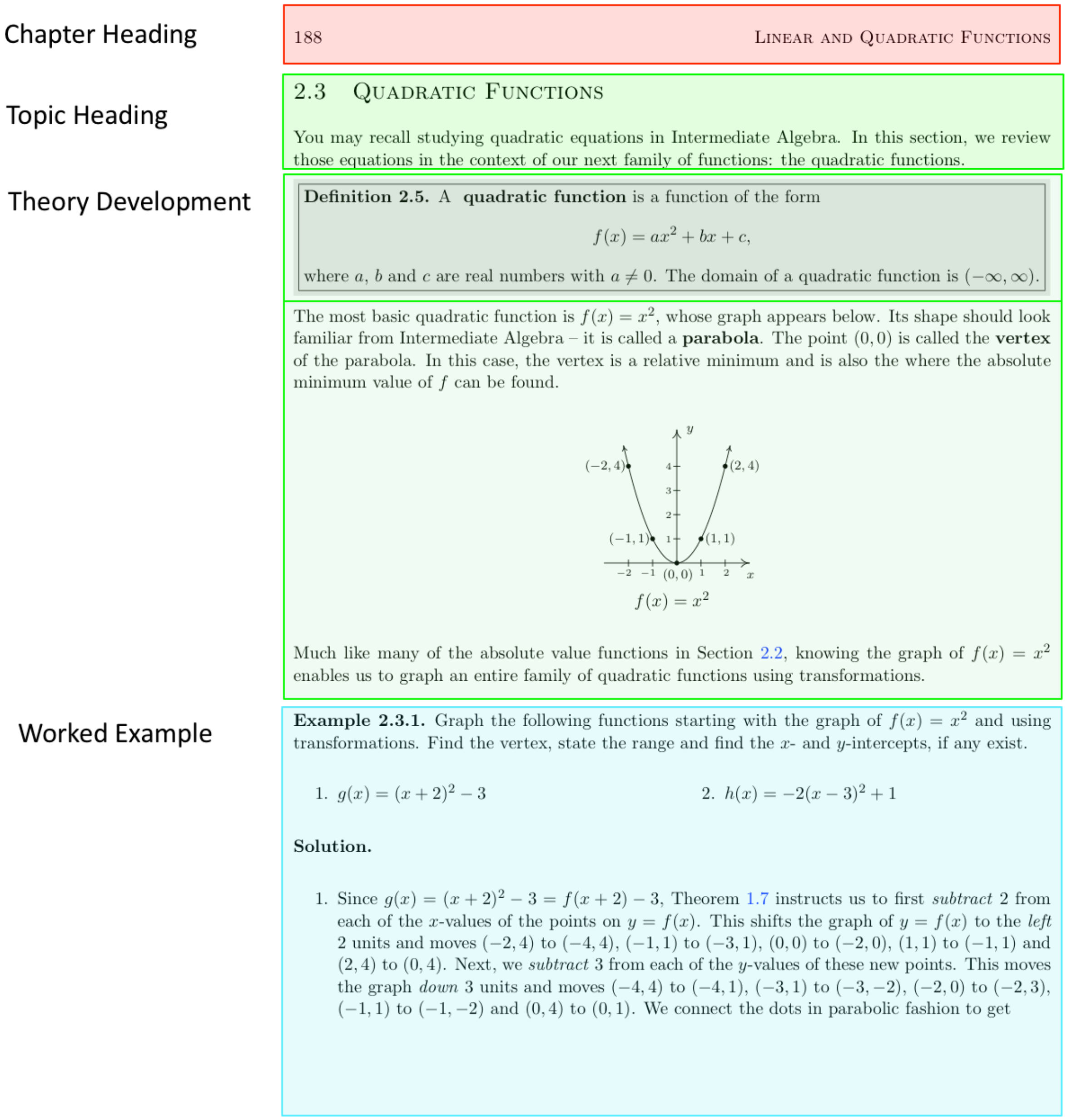
Together, language, image and the symbolism are formidable tools for constructing mathematical reality which focuses on experiential meaning and logical relations with little variation in interpersonal meaning. The link between the symbolism and images is critical because the mathematical relations can be seen as a whole, providing vital insights when results are derived symbolically. That is, the patterns are seen visually, but they are manipulated symbolically. Language is used to contextualise the mathematical knowledge which is derived symbolically and visually.

Mathematical knowledge is constructed by accessing the potential of language, images and symbolism, together with the ability to move from one semiotic resource to another. The significance of the moves between the three resources is well recognised: "In *any* problem solving situation, you are actually translating information from one form to another", with alternative paths for getting from words to symbols (Collingwood, Prince, & Conroy, 2017, pp. ix-x). However, the semantic expansions which occur in these translations are less understood. For example, in Figure

1(a), *linguistically* "a function" is a noun (an entity). However, *symbolically* a function is a complex configuration of mathematical entities and operations: e.g. $f(x) = ax^2 + bx + c$. *Visually*, a function is a curve consisting of a series of points which correspond to the various values of x and y. The three semiotic versions – the noun (linguistic), the complex of mathematical operations and relations (symbolic) and the curve (visual) – are choices from the three different semiotic resources. It is possible to give exact translations of each choice, but each choice is not semantically equivalent: i.e. a noun ↔ a configuration of mathematical operations and relations ↔ a visual entity with parts. These representations of the mathematical content are different and they are used for different purposes. In order to investigate this further, extracts from mathematics textbooks are considered below, where mathematics textbooks are considered as a genre consisting of sections and subsections with clusters of linguistic, symbolic and visual elements. After exploring mathematical textbooks and the movements between the three resources in more detail, the implications of the multimodal approach for teaching and learning mathematics are considered.

Mathematical Textbooks

The systematic organisation of mathematical knowledge is evident in the structure of mathematical textbooks. That is, mathematics consists of different areas and topics, each of which are documented in various mathematics textbooks. The textbooks consist of chapters with sections and subsections. Each chapter has a section heading with sub-sections: e.g. topic headings, theory development (with definitions, laws, theorems and other results), theory application (with applications of the mathematical content), worked examples, exercises, and solutions (in most cases). Examples of subsections from university and school mathematics textbooks are displayed in Figures 2(a) and 2(b) respectively. Other elements of mathematics textbooks include historical anecdotes, photographs of real-life applications of mathematics (e.g. bridges and buildings) and other elements designed to attract interest. In this way, mathematics textbook writers attempt to provide variations of interpersonal meaning in order to engage readers. Regardless, the structure and content of mathematics textbooks (with linguistic, symbolic and visual elements) are well-defined and immediately recognisable to anyone who has studied mathematics. Einstein explains that language only comes into play in mathematical thinking once the content is "sufficiently established [symbolically and visually] and can be reproduced at will" (cited in Cairo, 2013, p. 141). The mathematical content in mathematics textbooks is reproduced knowledge, thus the role of language in this context is first investigated below.



188 LINEAR AND QUADRATIC FUNCTIONS

2.3 QUADRATIC FUNCTIONS

You may recall studying quadratic equations in Intermediate Algebra. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

Definition 2.5. A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

The most basic quadratic function is $f(x) = x^2$, whose graph appears below. Its shape should look familiar from Intermediate Algebra – it is called a **parabola**. The point $(0, 0)$ is called the **vertex** of the parabola. In this case, the vertex is a relative minimum and is also the where the absolute minimum value of f can be found.

$f(x) = x^2$

Much like many of the absolute value functions in Section 2.2, knowing the graph of $f(x) = x^2$ enables us to graph an entire family of quadratic functions using transformations.

Example 2.3.1. Graph the following functions starting with the graph of $f(x) = x^2$ and using transformations. Find the vertex, state the range and find the x- and y-intercepts, if any exist.

1. $g(x) = (x + 2)^2 - 3$
2. $h(x) = -2(x - 3)^2 + 1$

Solution.

1. Since $g(x) = (x + 2)^2 - 3 = f(x + 2) - 3$, Theorem 1.7 instructs us to first *subtract* 2 from each of the x-values of the points on $y = f(x)$. This shifts the graph of $y = f(x)$ to the *left* 2 units and moves $(-2, 4)$ to $(-4, 4)$, $(-1, 1)$ to $(-3, 1)$, $(0, 0)$ to $(-2, 0)$, $(1, 1)$ to $(-1, 1)$ and $(2, 4)$ to $(0, 4)$. Next, we *subtract* 3 from each of the y-values of these new points. This moves the graph *down* 3 units and moves $(-4, 4)$ to $(-4, 1)$, $(-3, 1)$ to $(-3, -2)$, $(-2, 0)$ to $(-2, 3)$, $(-1, 1)$ to $(-1, -2)$ and $(0, 4)$ to $(0, 1)$. We connect the dots in parabolic fashion to get

Figure 2(a) Structure of Mathematics Textbooks (Stitz & Zeager, 2013, p. 188)

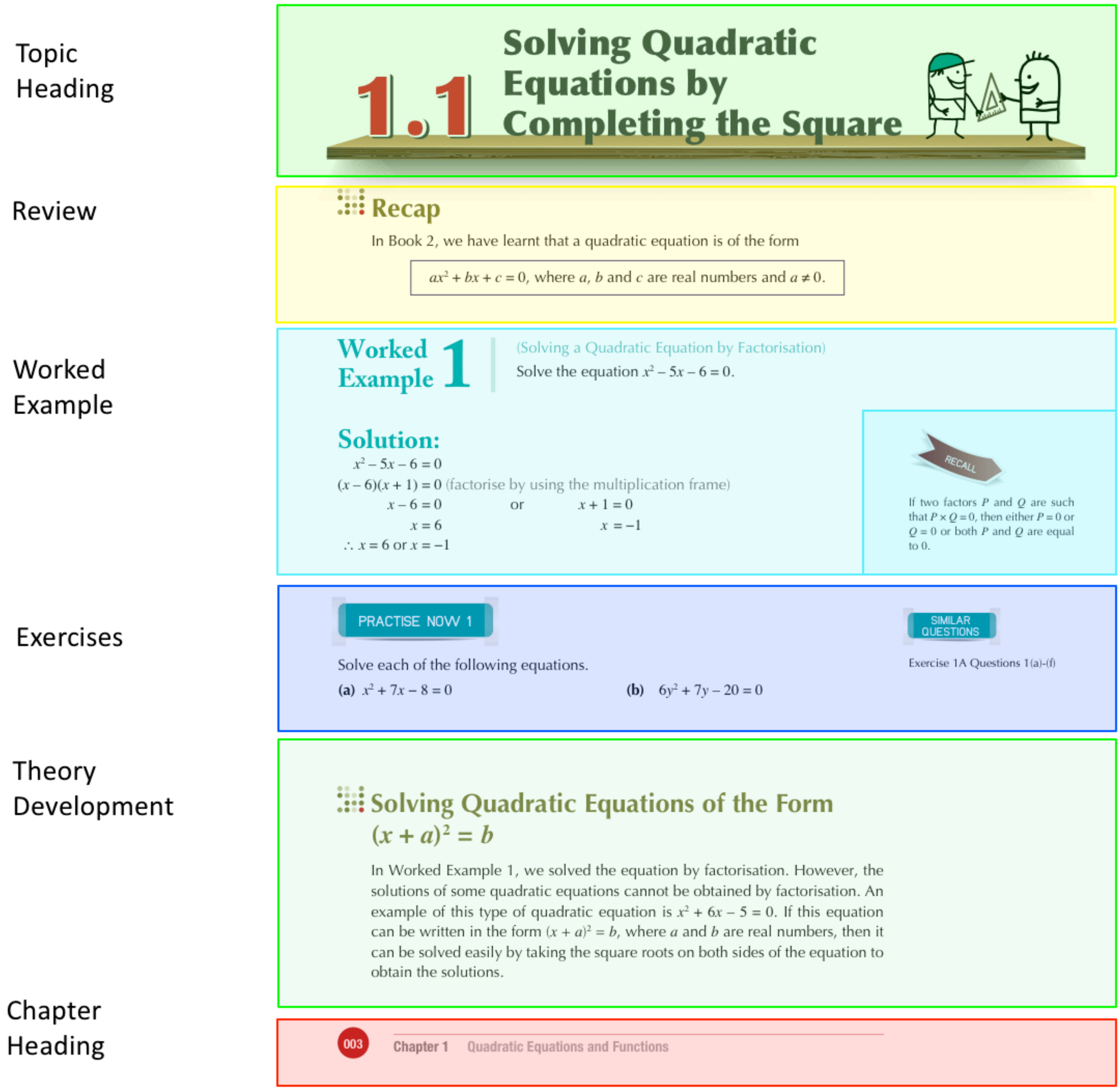


Figure 2(b) Structure of Mathematics Textbooks(Yeo et al., 2016, p. 13)

The various subsections of a mathematics textbook have distinct linguistic patterns in terms of experiential meaning. For example, topic headings where new content is first introduced are likely to contain mental processes (e.g. thinking, recalling) and material processes (actions). Similarly, worked examples are likely to contain material processes (e.g. actions). However, when mathematical theory is being developed and drawn upon, relational processes (which relate one entity to another) are likely to dominate. For example, the various process types from Stitz and Zeager (2013, p. 188) are displayed in Figure 3(a), with mental processes (green), material processes (red), and relational processes (yellow). These process types are displayed in relation to the subsections in Figure 3(b), showing the clusters of process types according to the subsection type. Importantly, clusters of relational processes are found when the mathematical content is developed (e.g. theory development subsection). Similar patterns of experiential meaning are displayed in Figures 3(c) and 3(d) which is a high school mathematics textbook (Yeo et al., 2016, p. 13). The various clusters of experiential meaning in the subsections of mathematics textbooks reveal how mathematical content is introduced, reviewed and developed. That is, students are instructed to carry out actions in certain sections (topic headings introducing new content, the instructions in worked examples) but those actions involving working with mathematical content which is largely relational in nature, as seen in the theory development subsection (Figure 3(b)) and the solutions to problems (Figure 3(d)). Moreover, the symbolism consists of mathematical operations and relational processes (O'Halloran, 2005, 2015). In what follows, the relational processes (coloured yellow in Figures 3(a) to (d)) are investigated further because these linguistic constructions are key to understanding mathematical content.

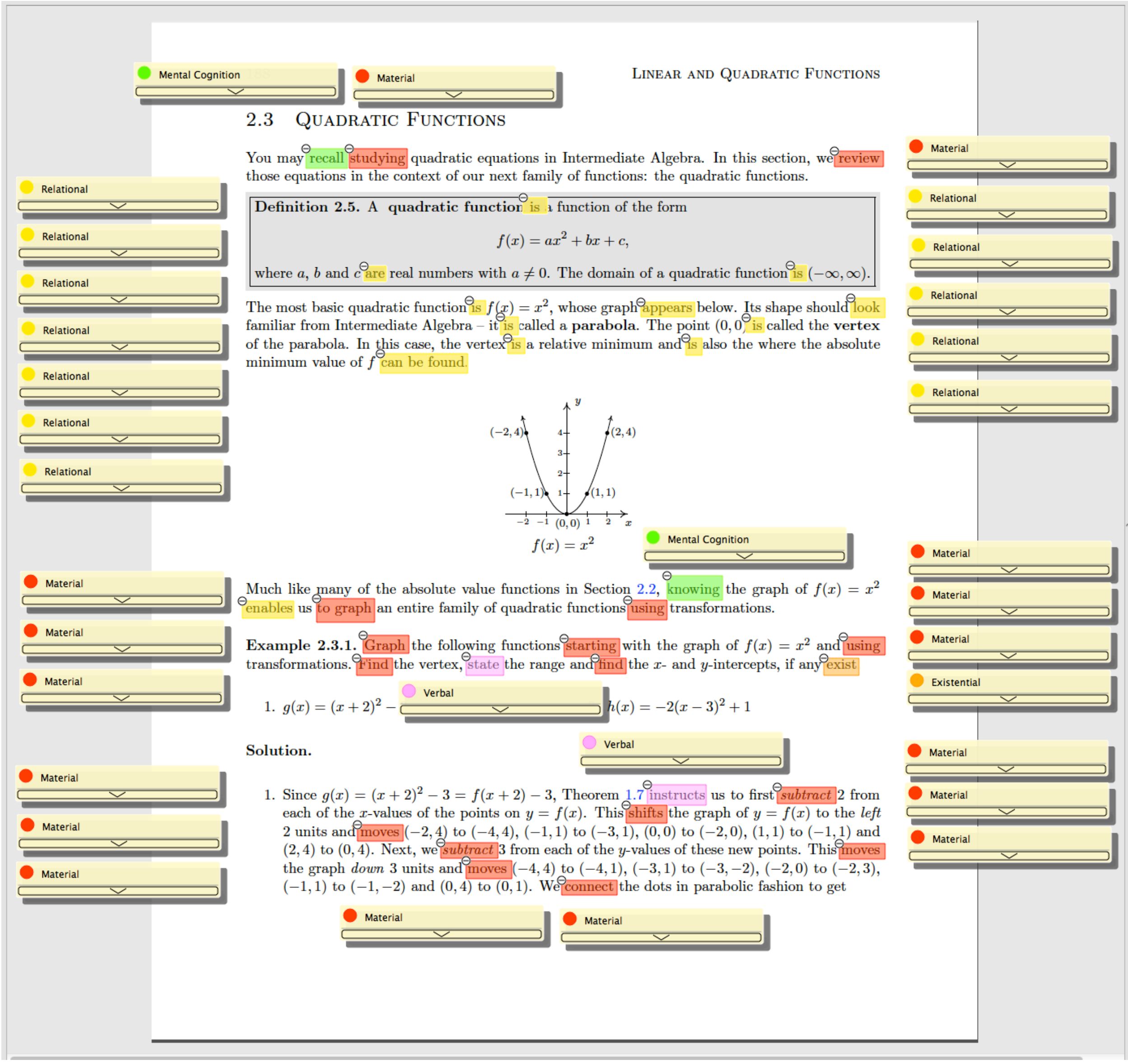


Figure 3(a) Linguistic Patterns (Stitz & Zeager, 2013, p. 188)

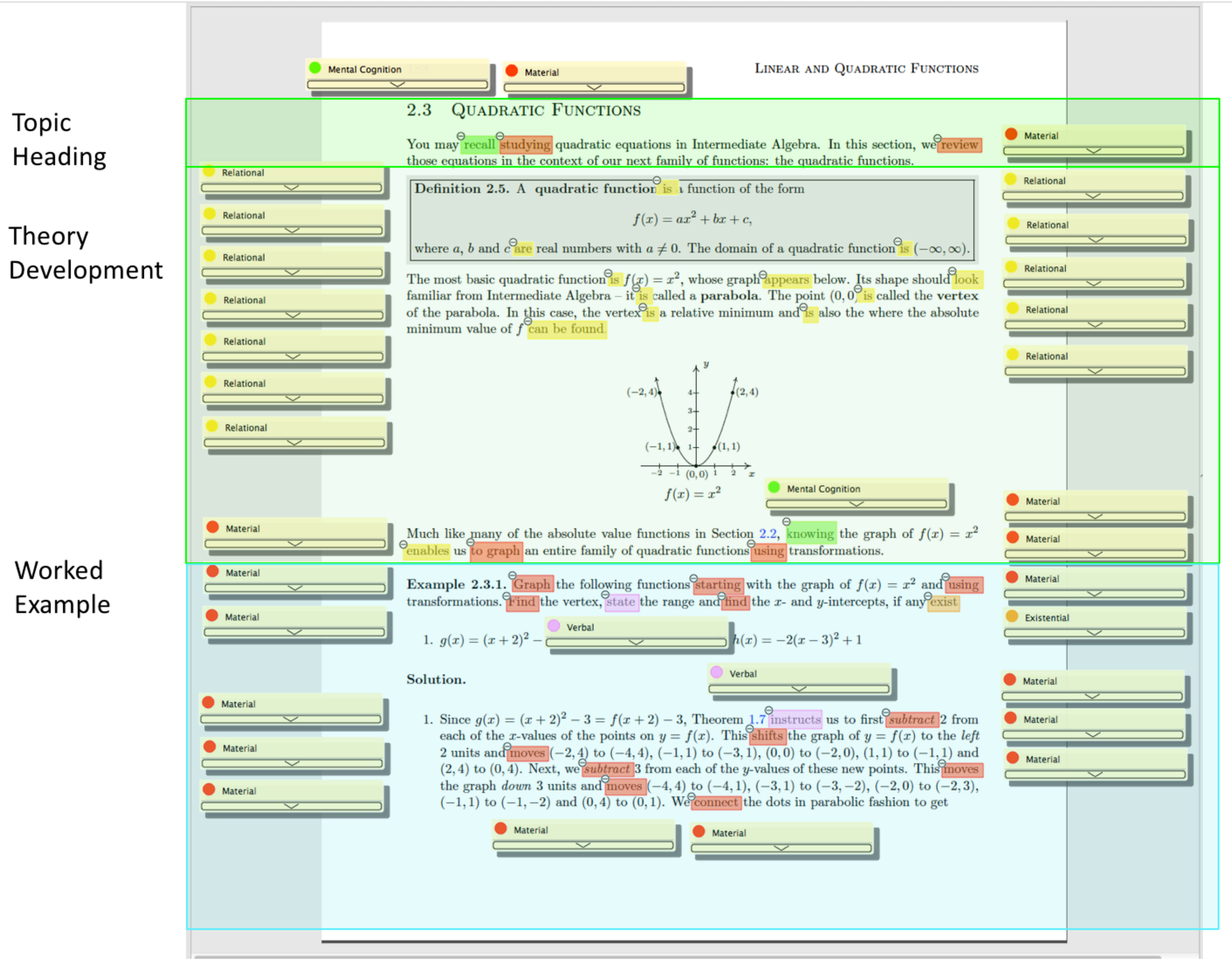


Figure 3(b) Linguistic Patterns in Sections (Stitz & Zeager, 2013, p. 188)

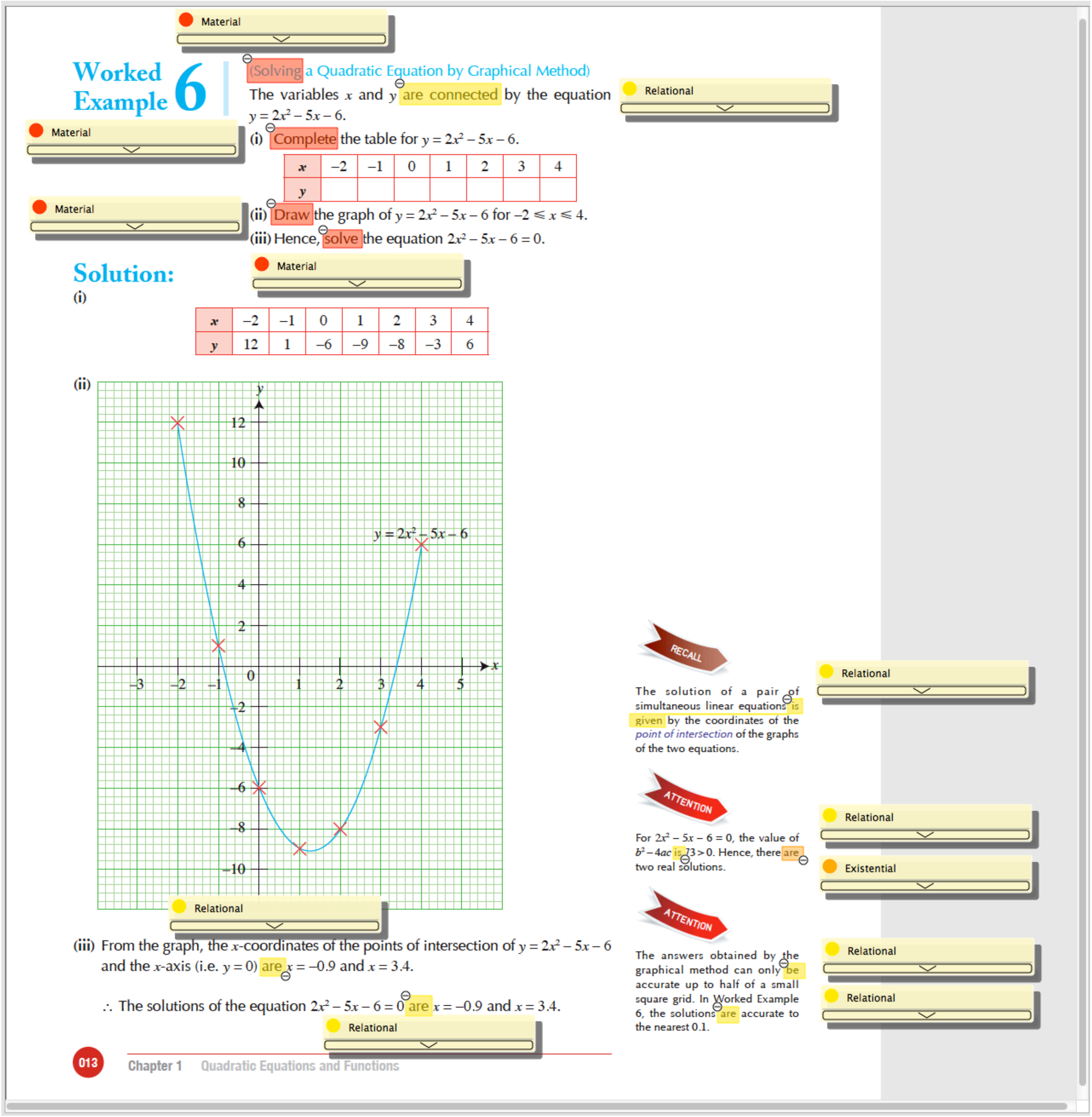


Figure 3(c) Linguistic Patterns (Yeo et al., 2016, p. 13)

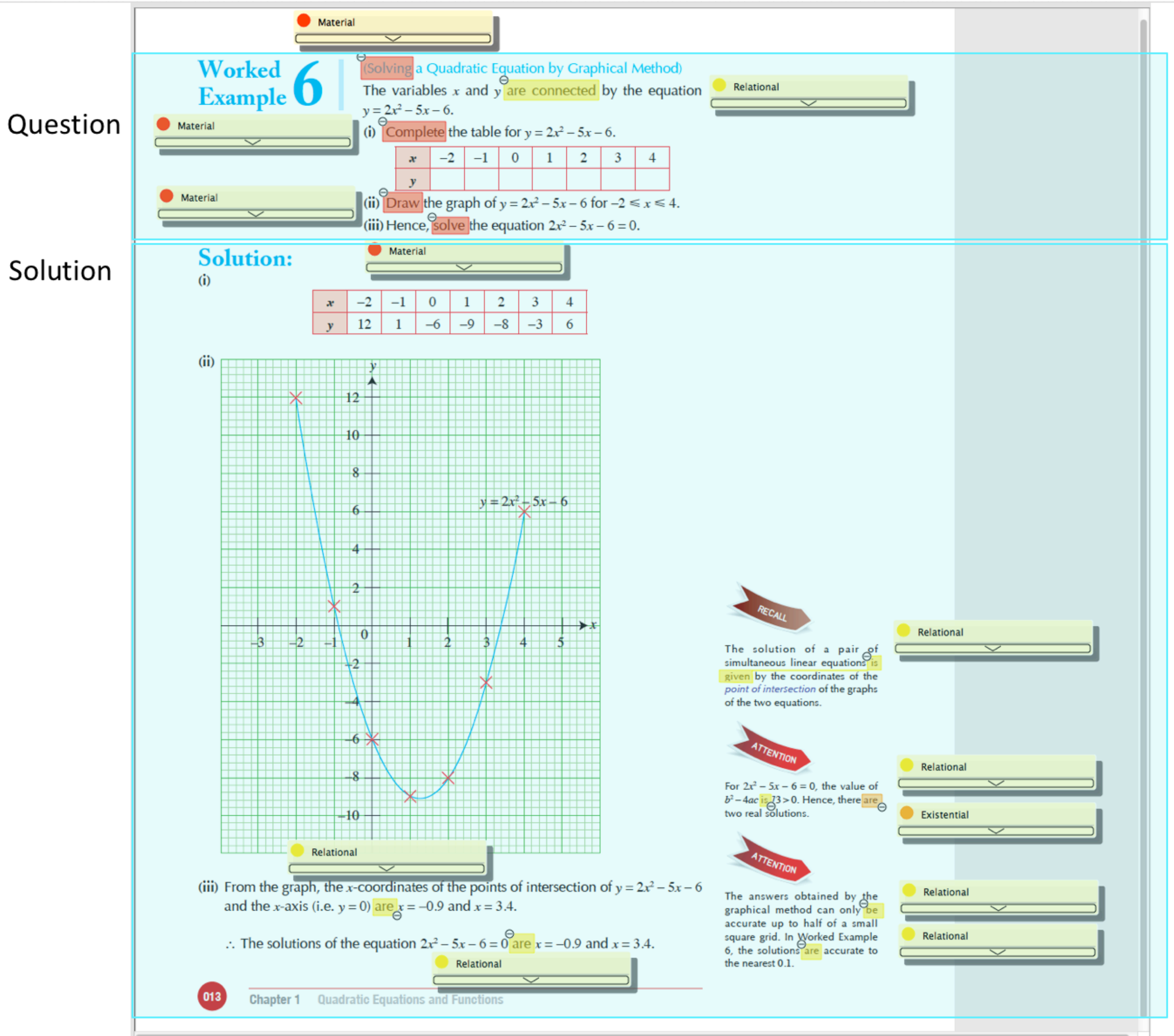


Figure 3(d) Linguistic Patterns in Question and Solution (Yeo et al., 2016, p. 13)

The linguistic construction "The solution of a pair of simultaneous linear equations is given by the coordinates of the point of intersection of the graphs of the two equations" (Yeo et al., 2016, p. 13) (see Figure 3(c)) contains three elements, as displayed in Table 1(a):

- The relational process: "is given" (in the passive form "is given by").
- Two entities: "The solution of a pair of simultaneous linear equations" and "the coordinates of the point of intersection of the graphs of the two equations"

Table 1(a): Encoding experiential meaning in language

The solution of a pair of simultaneous linear equations	is given (by)	the coordinates of the point of intersection of the graphs of the two equations.
Entity (noun)	Relational Process (verb)	Entity (noun)

The experiential content is encoded linguistically within the two entities (i.e. the two nouns) which are equated with each other through the process "is given by". As can be seen in Table 1(b), the first noun "the solution of a pair of simultaneous linear equations" involves two levels of postmodification of the head noun "the solution". A similar pattern is seen in the second noun, "the coordinates of the point of intersection of the graphs of the two equations" where there are four levels of postmodification. Moreover, some elements of these two nouns (e.g. "the solution", "equations" and "intersection") are derived from processes (i.e. "solve", "equate" and "intersect").

Table 1(b): Encoding experiential meaning in a noun

The solution	of a pair	of simultaneous linear equations
Head noun	Postmodifier	
		Post-Postmodifier

the coordinates	of the point	of intersection	of the graphs	of the two equations
Head noun	Postmodifier			
		Post-Postmodifier		
			Post-Post-Postmodifier	
				Post-Post-Post-Postmodifier

The example illustrates how experiential meaning is encoded in nouns in mathematical writing. This involves a shift from processes (such as solve, equate and intersect) to nouns (solution, equation and intersection) with various postmodification (and potentially premodification) elements. Halliday (1998, 2006, 2006 [1993]) refers to this semantic shift from processes (or happenings) to entities as "grammatical metaphor", where meaning is compressed into nouns. This has several important implications. First, mathematical writing is very dense and thus difficult to read, write and understand. Second, this grammatical strategy in scientific writing is fundamentally different to that found in mathematical symbolism where experiential meaning is encoded as embedded groupings of mathematical processes and entities. This is illustrated in the example: "The variables x and y are connected by the equation $y = 2x^2 – 5x – 6$", displayed in Table 2.

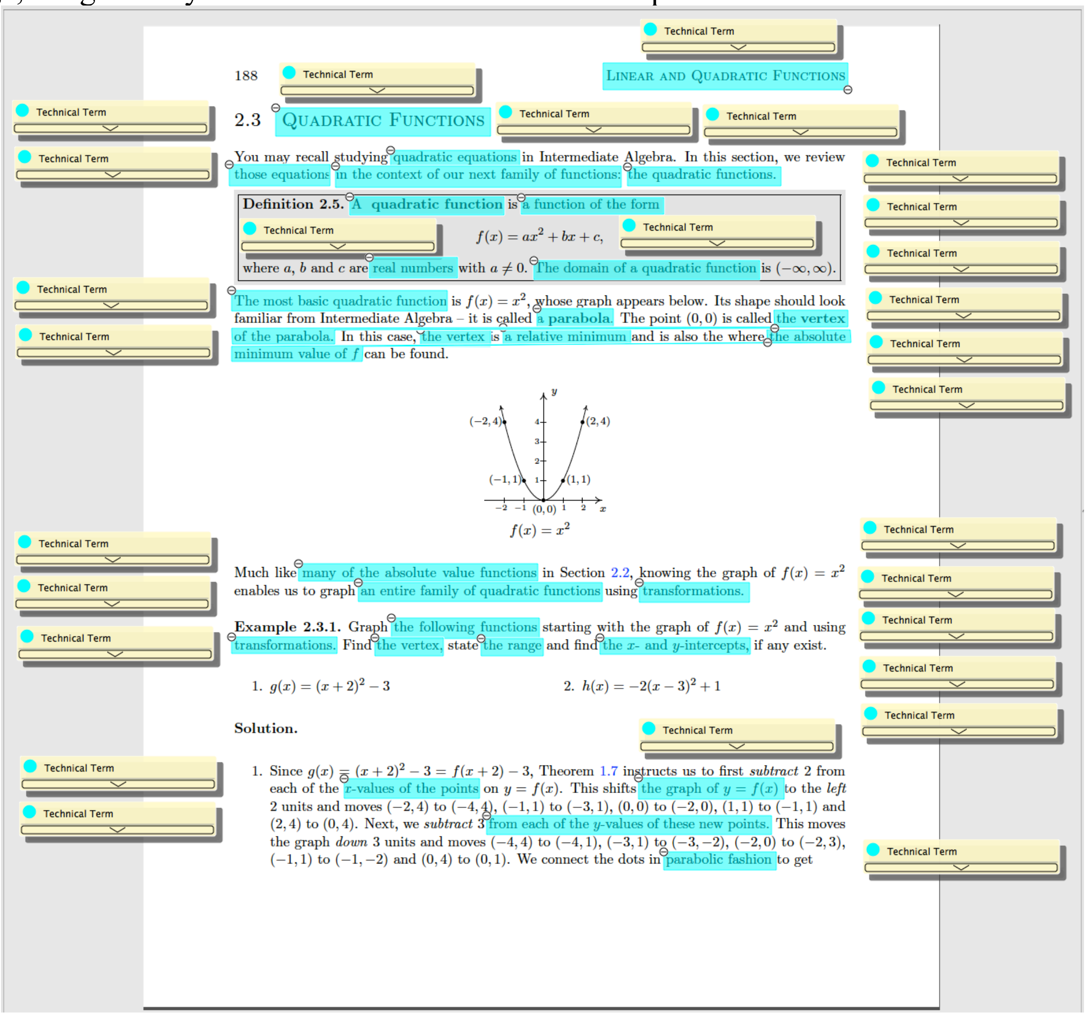
Table 2: Encoding experiential meaning in language

The variables x and y	are connected (by)	the equation $y = 2x^2 – 5x – 6$
Entity (noun)	Relational Process (verb)	Entity (noun)

In Table 2, the two entities are related to each other, and each has a symbolic expression that defines the entity (i.e. "the variables ***x* and *y***" and "the equation $\boldsymbol{y = 2x^2 – 5x – 6}$"). In the entity "the equation $y = 2x^2 – 5x – 6$", the linguistic element "the equation" is a noun, derived from the process "equate". In contrast, the symbolic element "$y = 2x^2 – 5x – 6$" is a complex of mathematical entities and processes. This is illustrated by the square brackets ([[....]]) which group together mathematical symbolic entities with the associated mathematical operations:

y = [[[[2x[[*x*x*x*]]]]– [[5x*x*]] – 6]]

The level of embedding of configurations of mathematical entities and processes in "$y = 2x^2 – 5x – 6$" is quite extensive, as seen by the double brackets. This grammatical strategy of embedding in the symbolism can be contrasted with the strategy of encoding meaning in nouns found in language. The implications of the differing methods for encoding experiential meaning in natural language and the symbolism are discussed in relation to teaching and learning mathematics. Before doing so, it is worth noting that mathematics has many technical terms in the form of nouns. These technical terms are organised into taxonomies: that is, (a) this is a part of this (e.g. a line segment is part of a triangle) and (b) this is a type of this, (e.g. a quadratic function is a type of function). For example, the technical terms in Stitz and Zeager (2013, p. 188) and Yeo et al. (2016, p. 13) are coloured blue in Figures 4(a) and 4(b). As these examples illustrate, mathematical writing has many technical terms, and each term has a specific meaning in relation to others in accordance with the accompanying taxonomy. This adds to the complexity of reading and understanding mathematical content. Before turning the implications of the nature of mathematical writing, the shifts between language, image and symbolism are considered in more depth.


188 LINEAR AND QUADRATIC FUNCTIONS

2.3 QUADRATIC FUNCTIONS

You may recall studying quadratic equations in Intermediate Algebra. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

Definition 2.5. A quadratic function is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

The most basic quadratic function is $f(x) = x^2$, whose graph appears below. Its shape should look familiar from Intermediate Algebra – it is called **a parabola**. The point $(0, 0)$ is called **the vertex of the parabola**. In this case, the vertex is a relative minimum and is also the where the absolute minimum value of f can be found.

Much like many of the absolute value functions in Section 2.2, knowing the graph of $f(x) = x^2$ enables us to graph an entire family of quadratic functions using transformations.

Example 2.3.1. Graph the following functions starting with the graph of $f(x) = x^2$ and using transformations. Find the vertex, state the range and find the x- and y-intercepts, if any exist.

1. $g(x) = (x + 2)^2 - 3$
2. $h(x) = -2(x - 3)^2 + 1$

Solution.

1. Since $g(x) = (x + 2)^2 - 3 = f(x + 2) - 3$, Theorem 1.7 instructs us to first *subtract* 2 from each of the x-values of the points on $y = f(x)$. This shifts the graph of $y = f(x)$ to the *left* 2 units and moves $(-2, 4)$ to $(-4, 4)$, $(-1, 1)$ to $(-3, 1)$, $(0, 0)$ to $(-2, 0)$, $(1, 1)$ to $(-1, 1)$ and $(2, 4)$ to $(0, 4)$. Next, we *subtract* 3 from each of the y-values of these new points. This moves the graph *down* 3 units and moves $(-4, 4)$ to $(-4, 1)$, $(-3, 1)$ to $(-3, -2)$, $(-2, 0)$ to $(-2, 3)$, $(-1, 1)$ to $(-1, -2)$ and $(0, 4)$ to $(0, 1)$. We connect the dots in parabolic fashion to get

Figure 4(a) Technical Terms (Stitz & Zeager, 2013, p. 188)

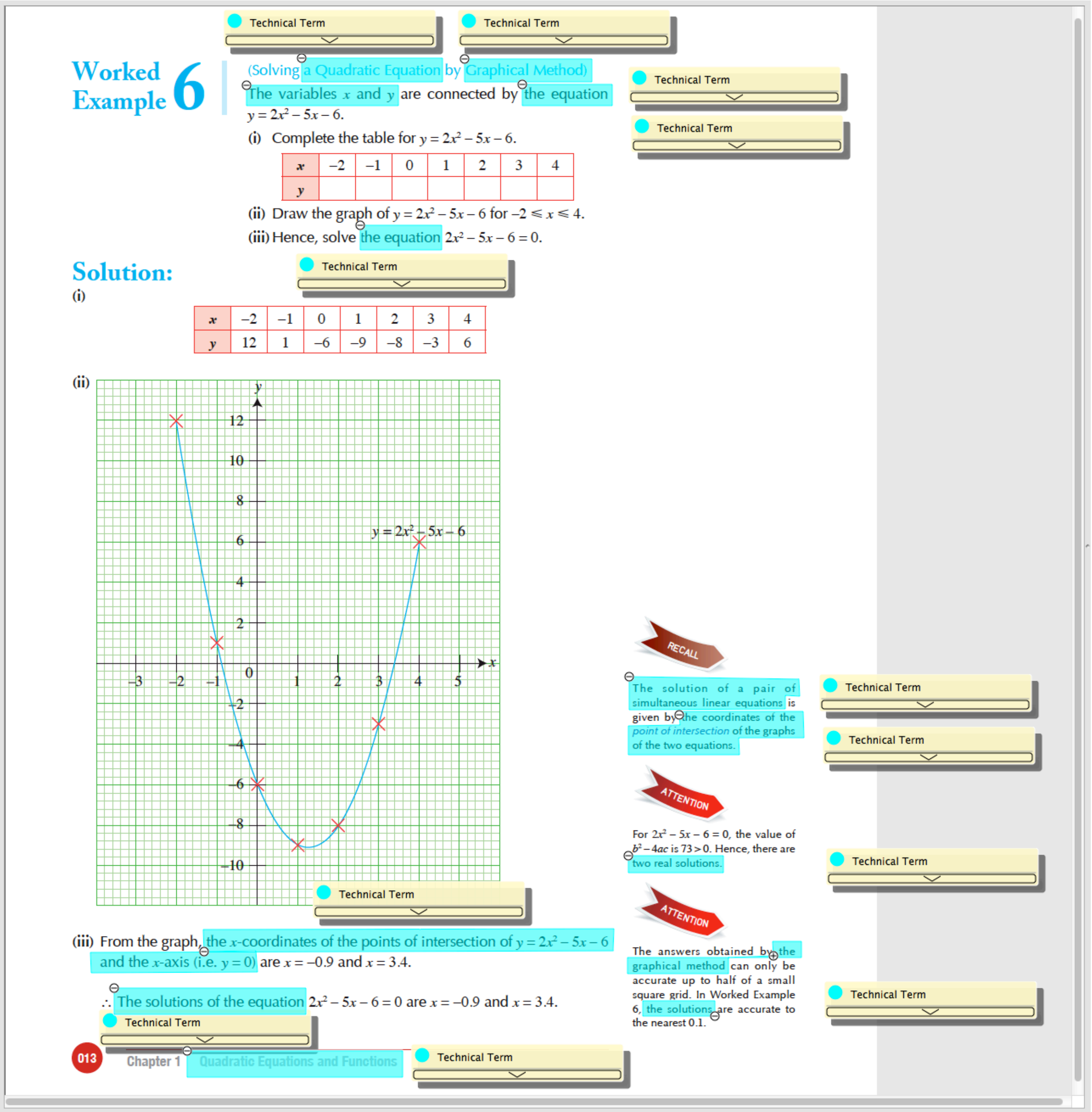


Figure 4(b) Technical Terms (Yeo et al., 2016, p. 13)

Movements between the three resources

The movements between language, symbolism and images are vital, not only because these result in access to different meaning potentials of the three resources, but also because elements from each resource can be 'resemiotised' into another form and thus recontextualised into a new semantic field. This is particularly significant, given that the three resources structure mathematical reality differently, according to the purpose of each resource.

For example, in "Worked Example 6" in Figure 1(b), the students are asked to (i) complete the table for $y = 2x^2 - 5x - 6$; (ii) draw the graph for $y = 2x^2 - 5x - 6$ for $-2 \leq x \leq 4$ and (iii) solve the equation $2x^2 - 5x - 6 = 0$. The instructions contain material (action) processes in the form of commands. That is, students are required to undertake actions in a mathematical world which is largely constructed symbolically and visually. The solution contains (i) the table, (ii) the graph and (iii) statements with the answer: "From the graph, the co-ordinates of the point of intersection of $y = 2x^2 - 5x - 6$ and the x-axis (i.e. $y = 0$) are $x = -0.9$ and $x = 3.4$. ∴ The solutions of the equation $2x^2 - 5x - 6 = 0$ are $x = -0.9$ and $x = 3.4$." In addition to completing the table and drawing the graph, the students need to understand that the solution is given by the intersection of $y = 2x^2 - 5x - 6$ and y = 0 (i.e. the x axis). The solution to the problem involves a series of shifts between language, image and the symbolism. That is, "the equation" (noun) is a complex of symbolic relations ($y = 2x^2 - 5x - 6$ and $2x^2 - 5x - 6 = 0$) which is used to derive a series of mathematical relations (i.e. the coordinates). These relations become points (entities) in the graph. This series of points creates a new representation in the form of a graph, confirming that the sum of the parts is *other than* the whole (Koffka, 1935). The graph is used to locate the intersection of $y = 2x^2 - 5x - 6$ and y = 0. To solve this problem, linguistic, visual and symbolic grammars need to be understood, in particular in relation to each other. That is, the shifts are from language and symbolism, to image to language and symbolism, as listed below. In each case, the shifts of meaning are not equivalent semantically:

- Problem (linguistic, symbolic)
- Mathematical relations (symbolic): $y = 2x^2 - 5x - 6$ and $2x^2 - 5x - 6 = 0$
- Table (symbolic): Relational processes
- Graph (visual): Entities (points)
- Graph (visual): new representation (graph)
- Solution (linguistic and symbolic): The co-ordinates of the point of intersection of $y = 2x^2 - 5x - 6$ and the x-axis (i.e. $y = 0$) are $x = -0.9$ and $x = 3.4$. ∴ The solutions of the equation $2x^2 - 5x - 6 = 0$ are $x = -0.9$ and $x = 3.4$.

Beyond these movements across language, image and symbolism, a large body of mathematical knowledge needs to be drawn upon. This is explained linguistically in the form of relational processes appearing under the banners "RECALL" and "ATTENTION" (see right hand side of Figure 3(c)). However, as discussed above, the writing is very dense and difficult to understand, and moreover, the actual background knowledge is not provided, rather it is referred to. The implications of the hierarchical nature of the various domains of mathematical knowledge are further discussed below.

Problems with teaching and learning mathematics

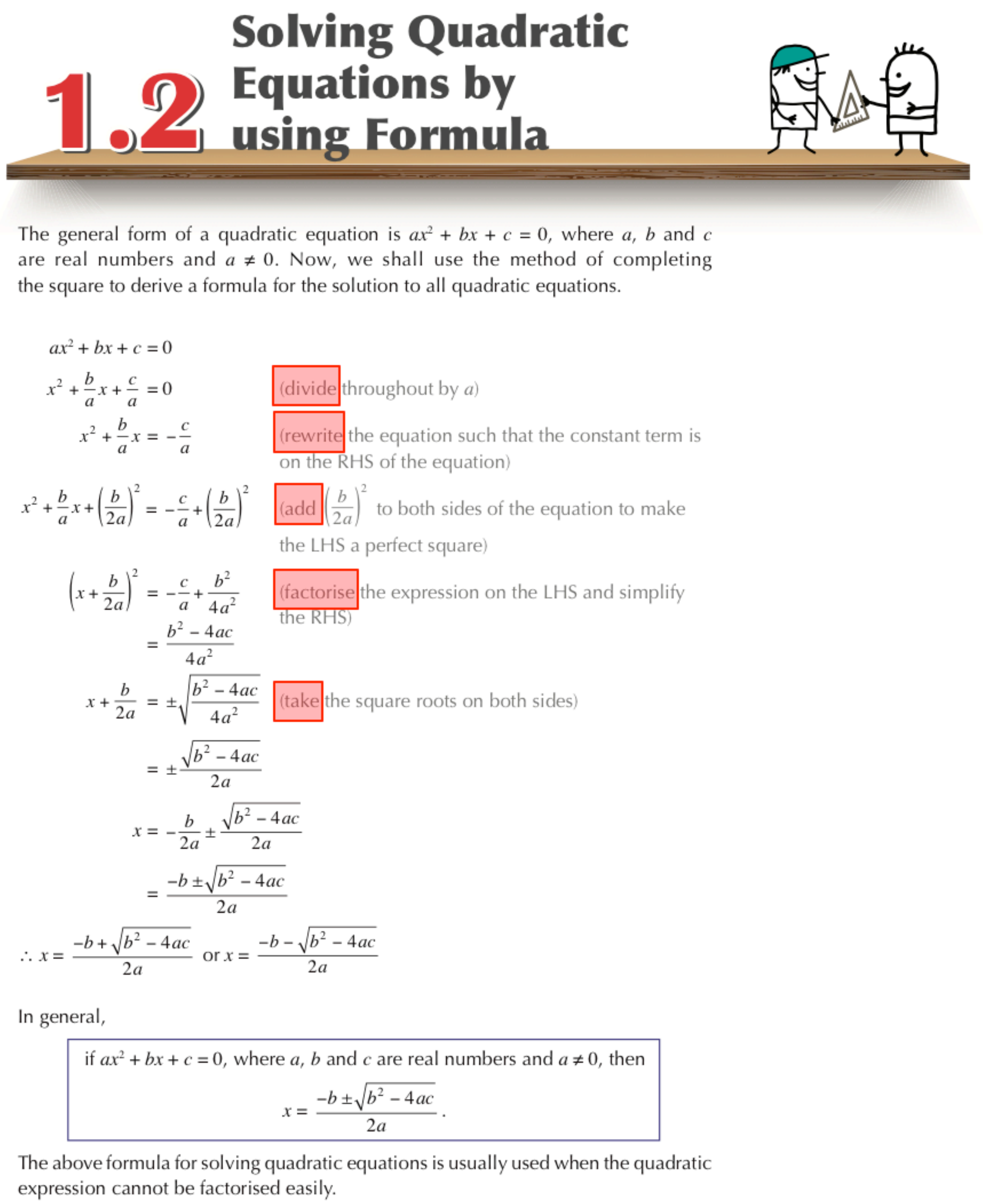
The unique problems which occur in the teaching and learning of mathematics are exemplified in mathematics textbooks. First, mathematics is primarily concerned with the expansion of experiential and logical meaning, and thus interpersonal meaning is toned down to the extent of being formulaic. The interpersonal stance across language, image and symbolism is one of certainty, making it somewhat daunting for learners of mathematics. In addition, the layout and ordering of mathematical content in the mathematics textbook into chapters, sections and individual elements are also standardised, given the importance of organising mathematical knowledge into various domains, and developing content within each topic of those domains.

Mathematical writing (i.e. the linguistic constructions) is difficult to read and understand, given the amount of content which is condensed into long noun groups, the relational processes and operations which are used to configure mathematical entities, and the technical terms which have precise meanings in accordance with different taxonomies. Mathematical symbolism is also difficult to read and understand because it constructs a reality which is different to that found in language. That is, mathematical constructions are streamlined into embedded configurations of mathematical operations using special symbols, ellipsed processes, brackets and rules of order that express the mathematical content in the simplest manner possible so it can be used as a tool for logical thinking. That is, the underlying architecture and systems for language and symbolism are different. Significantly, mathematical relations can be visualised in many cases (though not all), which in turn introduces new mathematical representations for expanding the experiential and logical domains of mathematics. The images have unique strategies of representing mathematical relations as well.

In addition, a key problem is the hierarchical nature of mathematical knowledge where different topics and domains build upon previous knowledge. To counter this problem, textbook writers explicitly refer to the background knowledge which is required, as seen in the examples considered above. Also, attempts are made to explain how to solve mathematical problems, as shown in Figure 5 where readers are instructed to undertake various actions in relation to the symbolic derivation of the solution to the problem. These linguistic commands direct the reader to perform different operations but the main difficulty is that the reader is required to work within mathematical constructions of reality which are symbolic and visual in nature, and thus differ from the types of constructions found in language. Thus, language can be used to describe, explain and issue commands, but ultimately mathematical reality is expressed symbolically and visually.

The hierarchical knowledge structure across the various areas of mathematics, where new knowledge is based upon earlier definitions and results, can be accessed to a greater degree in online mathematics textbooks. For example, as shown in Figure 6, it is possible to link definitions, properties, theorems, examples, solutions and proofs an in new ways in online environments that extend beyond printed textbooks. In Figure 6, Beezer (2017) uses "knowls" referenced within the body of the page of the mathematics textbook "to provide relevant, supplementary information".[1] Unlike a hyperlink which opens up a new web page, a knowl displays the relevant information (e.g. definition, theorem, proof) at the right location which can be opened up and closed with the click of a mouse button. In this way, the mathematical content is linked "in-context", proving immediate access to the information which is required. Today 'design-free' markup language can be used to write mathematics textbooks which can be published in hardcopy, pdf and online versions (O'Halloran et al., 2018), thus providing access to the same content in different formats.

[1] https://aimath.org/knowlepedia/



1.2 Solving Quadratic Equations by using Formula

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$. Now, we shall use the method of completing the square to derive a formula for the solution to all quadratic equations.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{(divide throughout by } a\text{)}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{(rewrite the equation such that the constant term is on the RHS of the equation)}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{(add } \left(\frac{b}{2a}\right)^2 \text{ to both sides of the equation to make the LHS a perfect square)}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{(factorise the expression on the LHS and simplify the RHS)}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{(take the square roots on both sides)}$$

$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

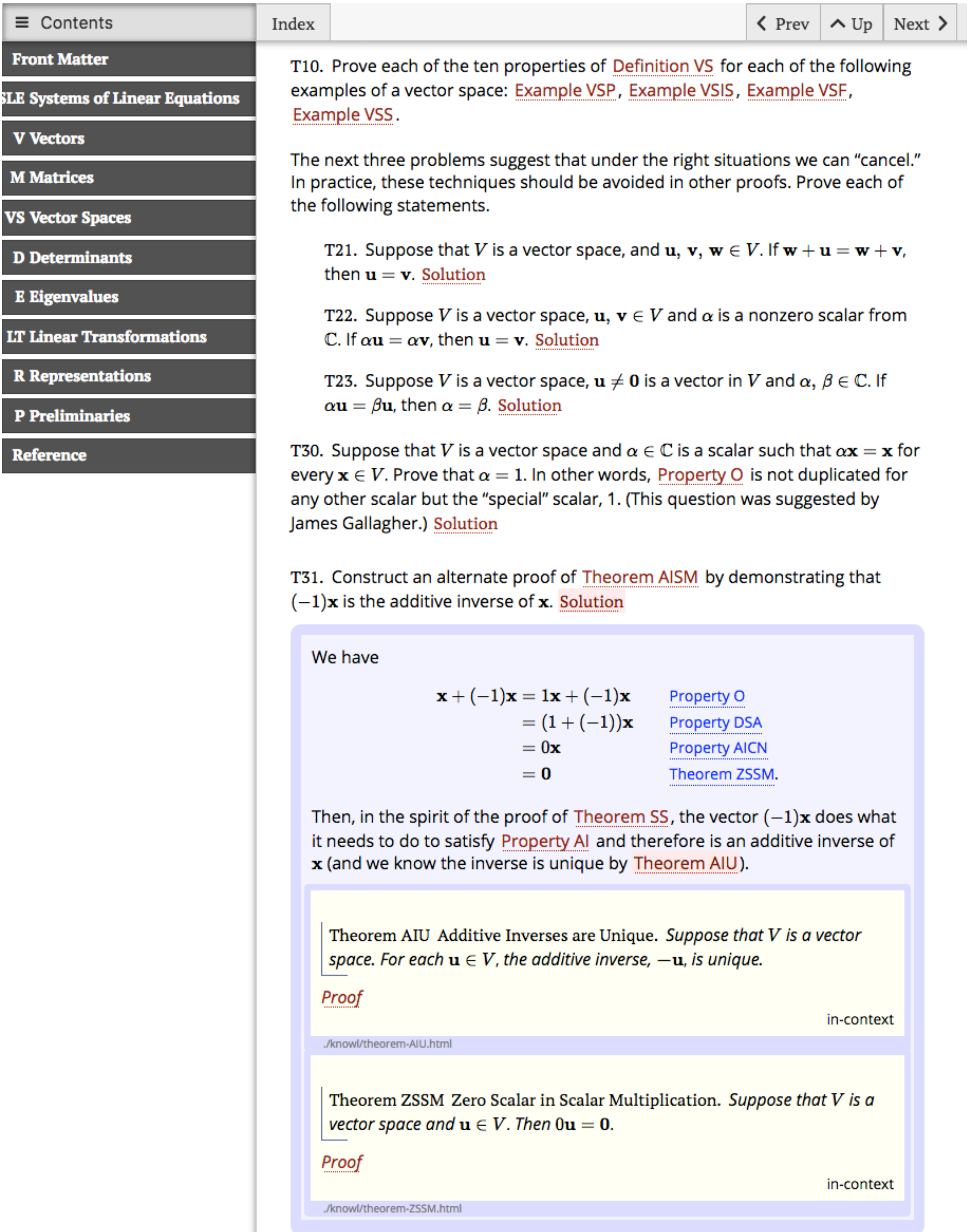
In general,

if $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The above formula for solving quadratic equations is usually used when the quadratic expression cannot be factorised easily.

Figure 5: Linguistic instructions (Yeo et al., 2016, p. 10)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T10. Prove each of the ten properties of Definition VS for each of the following examples of a vector space: Example VSP, Example VSIS, Example VSF, Example VSS.

The next three problems suggest that under the right situations we can "cancel." In practice, these techniques should be avoided in other proofs. Prove each of the following statements.

T21. Suppose that V is a vector space, and $\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V$. If $\mathbf{w}+\mathbf{u}=\mathbf{w}+\mathbf{v}$, then $\mathbf{u}=\mathbf{v}$. Solution

T22. Suppose V is a vector space, $\mathbf{u},\ \mathbf{v} \in V$ and α is a nonzero scalar from $\mathbb{C}$. If $\alpha\mathbf{u}=\alpha\mathbf{v}$, then $\mathbf{u}=\mathbf{v}$. Solution

T23. Suppose V is a vector space, $\mathbf{u} \neq \mathbf{0}$ is a vector in V and $\alpha,\ \beta \in \mathbb{C}$. If $\alpha\mathbf{u}=\beta\mathbf{u}$, then $\alpha=\beta$. Solution

T30. Suppose that V is a vector space and $\alpha \in \mathbb{C}$ is a scalar such that $\alpha\mathbf{x}=\mathbf{x}$ for every $\mathbf{x} \in V$. Prove that $\alpha=1$. In other words, Property O is not duplicated for any other scalar but the "special" scalar, 1. (This question was suggested by James Gallagher.) Solution

T31. Construct an alternate proof of Theorem AISM by demonstrating that $(-1)\mathbf{x}$ is the additive inverse of $\mathbf{x}$. Solution

We have

$$\begin{aligned} \mathbf{x}+(-1)\mathbf{x} &= 1\mathbf{x}+(-1)\mathbf{x} && \text{Property O}\\ &= (1+(-1))\mathbf{x} && \text{Property DSA}\\ &= 0\mathbf{x} && \text{Property AICN}\\ &= \mathbf{0} && \text{Theorem ZSSM.} \end{aligned}$$

Then, in the spirit of the proof of Theorem SS, the vector $(-1)\mathbf{x}$ does what it needs to do to satisfy Property AI and therefore is an additive inverse of $\mathbf{x}$ (and we know the inverse is unique by Theorem AIU).

Theorem AIU Additive Inverses are Unique. *Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.*

Proof

in-context

./knowl/theorem-AIU.html

Theorem ZSSM Zero Scalar in Scalar Multiplication. *Suppose that V is a vector space and $\mathbf{u} \in V$. Then $0\mathbf{u}=\mathbf{0}$.*

Proof

in-context

./knowl/theorem-ZSSM.html

Figure 6: Linking and Accessing Content in Online Mathematics Textbooks (Beezer, 2017)

Conclusion

The multimodal approach to mathematics textbooks, where the functions and organisation of language, image and symbolism are taken into account, offers new opportunities for exploring how mathematical knowledge is constructed and how meaning is expanded through movements across the three resources. Most importantly, the approach offers ways to show how mathematics orders thinking and reality, opening up the possibility of further exploration of the problems associated with teaching and learning mathematics. For example, mathematical reality is constructed symbolically and visually, and even though linguistic explanations and instructions are an essential part of mathematics textbooks (and classroom discourse), the symbolism and the image are ultimate tools through which mathematical reality is created. These tools need to be understood and mastered, along with dense technical writing found in mathematics textbooks. In addition, mathematical knowledge is built up in the hierarchical structures found in mathematical textbooks, hence it is essential to provide links and references to this content in the most accessible form possible. As seen above, online environments provide new opportunities for linking and accessing relevant mathematical knowledge.

In this way, mathematics is positioned as a field which is designed to expand human knowledge in certain directions. The results of this endeavour and the technological innovations which have ensued (including computers) are remarkable, to say the least. Mathematics textbooks provide detailed accounts of mathematical knowledge (including the structure and organisation of this knowledge) and as such are critical resources for teachers and students alike. Mathematics textbook research, particularly a multimodal approach where the language, image and symbolism are taken into account, offers much potential for understanding the nature of mathematics, its functions and organisation, and the associated problems for teaching and learning mathematics.

Acknowledgements:
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TRACES OF ORAL TEACHING IN EUCLID'S *ELEMENTS*—DIAGRAMS, LABELS AND REFERENCES

KEN SAITO

Introduction

Euclid's *Elements* were used as mathematics textbooks until the nineteenth century, and their influence is still visible in today's geometry textbooks. Their use as textbook in Europe began in mediaeval universities. We should rather say that the *Elements* constituted one of the major contents in the curriculum that enabled the emergence and development of universities.

First, I shall show the diagrams in the *Elements*, which are transmitted to us through mediaeval manuscripts. Surprisingly, the diagrams in these sources are quite different from what we see in the printed editions available today.

The geometry diagrams in the *Elements* we see today were practically invented by August, a gymnasium teacher who published the Greek *Elements* (August 1826-29) "for the use of beginners" (these words are contained in its title), without consulting any manuscript. His diagrams were copied in the critical edition of Johan Ludvig Heiberg (1854-1928), the great Danish scholar in the field of Greek mathematics. Heiberg published a new critical edition of the Greek *Elements* in the 1880's—the first five volumes (Heiberg & Menge 1883-1916) of which are still the standard edition used by all the professional scholars of the field, and the source of all current translations.

Thus, the diagrams of Euclid's geometry that we see in translations come not from manuscripts but from August's edition

On the other hand, the diagrams of arithmetic that occupy Books 7 to 9 in today's editions seem to be an invention of Heiberg himself, for I have been able to find neither manuscript nor printed edition before him with diagrams like those in his edition. Though he consulted several manuscripts for his edition of the text, he ignored the diagrams in the manuscript.

In the second part of my lecture, I will examine some stylistic features of the *Elements* that are inconvenient to modern readers, and I will interpret them as traces of the style of mathematical teaching in ancient times that depended heavily on oral communication.

Diagrams in Euclid's *Elements*

I begin with the diagrams. The study of diagrams in Greek mathematical texts is rather a new field. I can identify the work that called scholars' attention to diagrams in Greek mathematics; it is *The Shaping of Deduction in Greek mathematics* of Reviel Netz (1999). This book, with several arguments with novel points of view, shows how manuscript diagrams are different from what we see in today's editions.

The Tradition of Euclid's Elements: from ancient times to printed editions

Before examining the diagrams in various manuscripts and printed editions, a brief exposition on the history of the tradition of Euclid's *Elements* is in order.

The *Elements* are a collection of fundamental theorems and problems of mathematics. For example, their Greek name, *stoicheia*, means the letters of alphabet, and, thus, elementary components. So, this is the title for a book containing basic propositions. The title was applicable to any work of such character, and the first *Elements* are said to have been compiled by Hippocrates of Chios, in the mid-fifth century. This Hippocrates is more or less contemporary to, but to be distinguished from, the famous founder of medicine, Hippocrates of Cos.

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The most important feature of the *Elements* and Greek mathematical works in general is that the propositions are accompanied by demonstration. This was not the case in Egyptian and Babylonian mathematics. We may say that demonstration was an invention of the Greek people.
Hippocrates's *Elements* have not come down to us, and the extant *Elements* are attributed to Euclid of Alexandria; however, about this person practically nothing is known. Many of you have probably heard that Euclid said that there is no royal road to geometry, when he was asked by King Ptolemy (more precisely, Ptolemy I, surnamed Soter) if there was an easier way to learn the *Elements*. King Ptolemy was one of the generals of Alexander the Great, and, after Alexander's death in 323 BCE, held Egypt and declared himself king in 305.
However, Stobaeus attributes a similar conversation to Alexander the great and a mathematician named Menaechmus (Heath 1925, 1:1). Therefore, our anecdote on Euclid tells us only two things: that mathematics was not easy to learn, and that the person who invented this anecdote thought that Euclid was a contemporary of King Ptolemy.
Proclus, a neoplatonist philosopher of the fifth century CE, in his commentary to Euclid's Book I, tries to determine Euclid's date. The anecdote of King Ptolemy and Euclid is one of the sources he cites to this purpose. This suggests that he did not have reliable material concerning Euclid's lifetime. In consequence, we have practically no hope of knowing more about Euclid's date, life and personality.
You may have also heard that Euclid may have been a name of a group of mathematicians. This is the conjecture of a French scholar, Jean Itard (1961, 11); in the preface of his book on Euclid's arithmetic, after pointing out that there is too much variety of style in the thirteen books of Euclid's *Elements* for it to be the work of a single mathematician, he reasonably suggested that the works of his disciples may be included. This is quite possible, for it was common in ancient Greece that people of a school attributed their works to the founder[1]. For example, many medical works come down to us with the name of Hippocrates, and it is not always easy to distinguish the genuine and spurious ones.
However, Itard added the gratuitous conjecture that it is also possible that the mathematician Euclid did not exist, and it was a name of a group. Yet, no evidence suggests that there was any group in ancient Greece, which published their results in the name of an imaginary person. Itard published his book on Euclid's arithmetic in 1961 in the heyday of Bourbaki's activities, so this conjecture is completely unfounded. We should say that Itard has simply projected the Bourbaki of his age into antiquity, without good foundation. Unfortunately, the specter of "Euclid as a group" survives even today, and it is still sometimes heard. To make such a gratuitous comment is easy, and it is hard to eliminate it, if once diffused. Please do not diffuse it.
As Itard has correctly pointed out, the contents of the *Elements* are not uniform. Until the seventies of the twentieth century, scholars were eager to find traces of older mathematics in the *Elements.* Today, we scholars recognize that the heterogeneous character of the *Elements* is rather due to later intervention[2].
Indeed, being the most basic work of Greek mathematics read and studied by everyone, the *Elements* have many traces of later intervention. We know, thanks to various pieces of evidence in extant documents, that Heron (1st century CE) and Theon of Alexandria (4th century CE) edited the *Elements*, and that certain propositions or arguments can be attributed to them. However, many of the apparent later interventions that we recognize in the text by some particular word or style cannot be attributed to someone whose name is known.

[1] However, we have practically no reliable documents about Euclid's disciples and his "school", and the heterogeneity of the different books of the *Elements* may as well be due to other factors.
[2] Knorr (1975) can be seen as the last example of the older approach (the author himself changed his research style after it), and the recent approach is best illustrated in Vitrac (2012).

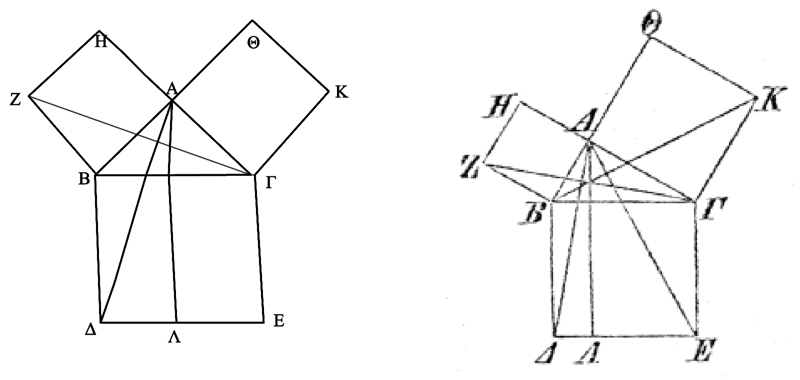
Manuscripts and Printed editions

The oldest extant Greek manuscript goes back to ninth century; and the six manuscripts (in some books more) used in Heiberg's critical editions were copied by the twelfth century.
The traditions in other languages are also important. Western Europe knew the *Elements* in the twelfth century through Latin translations made from Arabic translations. Then, Campanus of Novara in thirteenth century made a heavily edited and commented on Latin edition, which had great success in universities and enjoyed wide circulation, before the invention of printing in the fiftieth century—it was this version that appeared as the first printed edition of the *Elements* (Ratdolt 1482)[3].
In 1505, Zamberti published the Latin *Elements* translated directly from Greek. Then, the first Greek edition, *editio princeps*, was published together with Proclus' commentary (Grynaeus 1533). Commandino's Latin version with accurate translation and adequate comments had great influence (Commandino 1572).
Though there are hundreds of editions of the *Elements*, there are only five editions of the whole text in Greek to date; the second Greek edition was by Gregory (1703), and the third by Peyrard (1814-18). Peyrard had the chance to use an important Vatican manuscript, now named P after him, brought from Rome to Paris by the scholars in Napoleon's expedition.
We have already briefly mentioned the fourth Greek edition (August 1826-29). Though the diagrams in the *Elements* had been modified in previous editions, August made drastic and thoroughgoing changes to the diagrams. It seems that he had pedagogical intention and tried to make the diagrams as general as possible. We will soon see what changes were made.
Heiberg, the editor of today's standard version, traveled very much for this work. In 1881, he visited Rome to consult the Vatican manuscript, and in Florence he compared the manuscript preserved in the Laurenziana library with that of the Archiginnasio library in Bologna (apparently sent to Florence upon his request). In the following year, we find him in Oxford, consulting the manuscript of the Bodleian library. The manuscripts of Paris and Vienna (Heiberg and Menge 1883-1916, VIII-IX) (Heiberg and Stamatis 1969-1977, VIII-IX) were sent to Copenhagen, where he lived (interlibrary loans of manuscripts were possible at that time). Whatever the case, it seems that he loved traveling; I have found his signatures in the registers of the readers of manuscripts at the Biblioteca Marciana in Venice (just in front of Basilica di San Marco). The last of Heiberg's signatures was dated August 1927, only some months before his death in January 1928.
However, all this meticulous work of consulting manuscripts was directed to the text only, and no attention was paid to the diagrams in the manuscripts except in a few cases. Heiberg simply copied the diagrams of August's edition, though he knew very well that they were alien to those in the manuscripts. We do not know why Heiberg was completely indifferent to the diagrams.

Peculiar features of manuscript diagrams

Now let us examine the diagrams[4]. First, I show the diagram of proposition I.47, Pythagoras` theorem[5].

[3] The manuscripts of Campanus's version and this 1482 edition have been edited (Busard 2005).
[4] For a more detailed discussion of the diagrams in the *Elements*, see Saito (2006), Saito-Sidoli (2012), and Saito (2018).
[5] As the image of manuscript is not always convenient for study, for lines and labels are not always clear, I have used reproduced drawings like this one. For the reproduction of the drawing, the software DRaFT was used (http://www.greekmath.org/draft/draft_index.html).



Codex P Heiberg

Fig. 1: Proposition I.47 (Pythagoras` Theorem)

This is the image of a Vatican manuscript page, number 190, to which Heiberg gave the siglum `P' in Peyrard's honor (a siglum is the abbreviated name of each source in one or two letters, used in critical editions).

Let us compare it with the diagram we are accustomed to seeing in today's editions. All of them come from Heiberg's Greek edition, which you see to the right of fig. 1.

The difference is evident; the manuscript draws a rectangular triangle, but it is also an isosceles. Drawing diagrams, we are careful not to introduce specific conditions such as right angle or equal sides when they are not required in the proposition. In Pythagorean theorem, the triangle is rectangular, but not always isosceles, so that we avoid an isosceles triangle.

The manuscripts often add specific conditions: they tend to draw an isosceles or equilateral triangle for a general triangle, and a rectangle or square for a parallelogram. I call this phenomenon ``overspecification''.

All the manuscripts Heiberg used have a diagram of I.47 similar to that of codex P. The only exception is the Bodleian manuscript, which shows an isosceles but not a rectangular triangle.

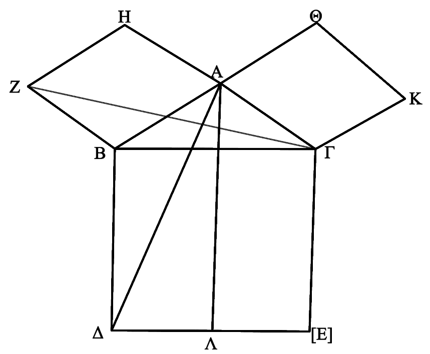
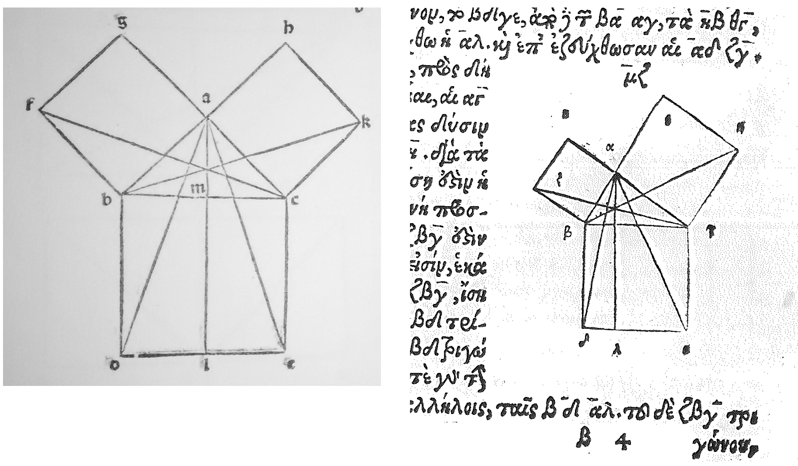


Fig. 2: Proposition I.47 (codex B)

The scribe seems to have drawn too big a square in the limited space for the diagram and constrained to compress the upper part. The scribes were not so much concerned about drawing accurate diagrams.[6] I will return to this feature soon, but for the moment let us see the diagrams of the printed editions.

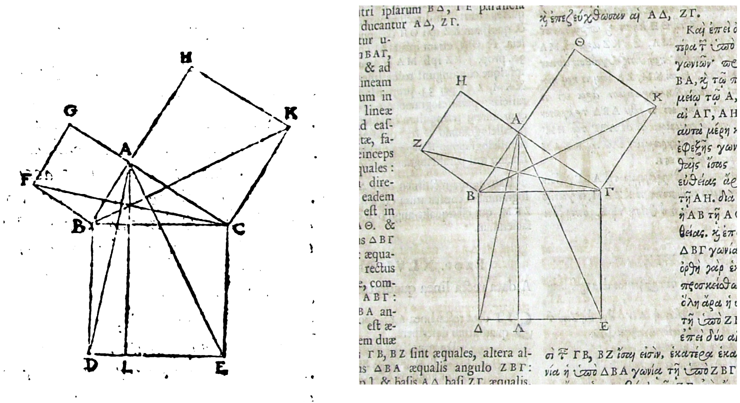
[6] The diagrams were often drawn by someone else after the text was copied, either in the margin or the blank space which the copyist had made for the diagram, by leaving a part or whole of a column without the text.



Ratdolt (1482) Grynaeus (1533)

Fig. 3: Propositions I.47 in early printed editions.

Fig. 3 (left) is the diagram in the first printed edition of the *Elements* of 1482, based on Campanus's version. It has two more oblique lines. All the older manuscripts have only two oblique lines on the left (and not two lines on the right), which are used to show the equality of the small square ABHZ with the parallelogram (rectangle) BΛ, part of the big square on the hypothenuse. The text proves this equality and then says that, AE and BK being joined, it will be proved that the square AΓKΘ is equal to the parallelogram ΓΛ. The diagram of the 1482 edition has visualized these lines, AE and BK mentioned in the text. Mathematically, this is not bad, but these two lines did not exist in the manuscript tradition. Then, in Grynaeus's Greek edition (fig. 3, right), published half a century later, the triangle is no longer isosceles, against the manuscript tradition. The manuscript he used may have had such a diagram. Anyway, all the subsequent editions have rectangular but not isosceles triangles. We show Commandino (1572) and Gregory (1703) in fig. 4.

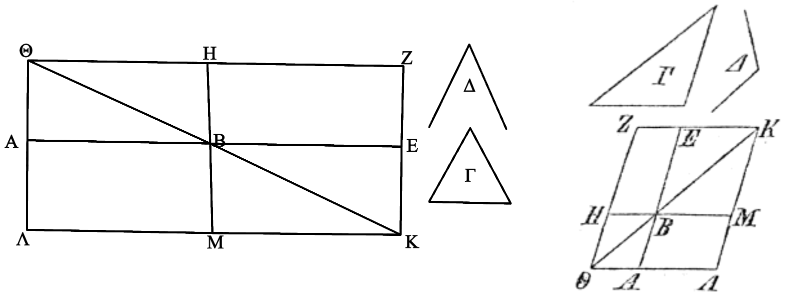


Commandino Gregory

Fig. 4: Proposition I.47

Let us see another example of overspecification in a manuscript diagram: proposition I.35 proves that parallelograms having the same base and between the same parallels are equal to each other.[7] In manuscripts, one of the parallelograms is a rectangle or a square, and Heiberg's diagram, which is a copy of August's, avoids a rectangle, for the parallelogram is not always rectangular (fig. 5).

[7] When Euclid says that two figures are equal, he means that their areas are equal.

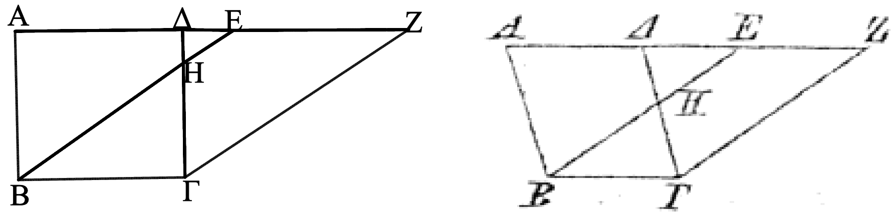


Codex F Heiberg

Fig. 5: Proposition I.35

Now, I go on to the second feature, which I call “metrical inaccuracy”. I show the diagram of proposition I.44 (fig. 6). This is not a theorem, but a problem: given a line AB, an area Γ, and an angle Δ, it is required to construct a parallelogram ABMΛ, having one side AB, with an area equal to Γ, and one angle equal to Δ.

However, it is difficult to imagine from the manuscript diagram that the angle of the parallelogram ABMΛ is equal to the angle Δ, and that its area is equal to the given area Γ. Heiberg’s diagram is accurate in this respect.



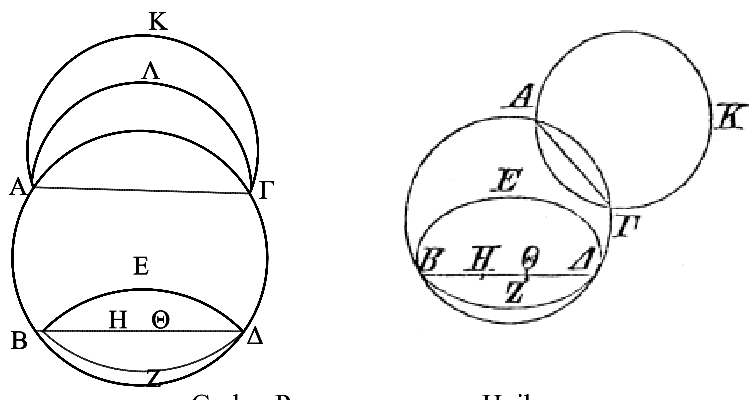
Codex P Heiberg

Fig. 6: Proposition I.44

Generally, it is easy to find diagrams in which metrical relations such as equality or ratio of lines, areas and angles, are not correctly represented. It was the text that showed such metrical relations, not the diagram.

Let us examine the third feature of manuscript diagrams. In a proposition which uses proof by contradiction, it is impossible to draw a correct diagram, for the assumption in such a proposition contains something impossible, so that the diagram necessarily ignores some of the conditions.

In proposition III.13, Euclid shows that two circles cannot touch each other at two points, whether internally or externally. For the proof of the impossibility of touching at two points externally, the diagram draws two circles: ABΔΓ and AKΓΛ (fig. 7)[8].



Codex P Heiberg

[8] Point Λ is never mentioned in the text.

Fig. 7: Proposition III.13

In the manuscript diagram, the second circle is a lunula, not a circle. But this is necessary, and Heiberg's diagram, which draws two circles intersecting each other at points A and Γ, does not represent the assumption of the proposition, causing difficulty in understanding Euclid's arguments. Indeed, the proof is thus,

> (III.13 part of the proof) Let two circles touch each other at two points A and Γ, externally. Therefore, the chord AΓ falls within each circle, for this is demonstrated (III.2).

Then the text says:

> But it fell within the circle ΑΒΓΔ and outside ΑΚΓ.

What does he mean? This is understood as the definition of touching circles. Touching circles are those, which meet but do not cut each other (III. Def. 3). Therefore, if the line AΓ falls inside the first circle, it *must* fall outside the second circle, because these circles touch each other externally. Heiberg's diagram (that is, August's diagram) does not represent this situation. There are two circles cutting each other. This contradicts the definition of touching circles, and Euclid's argument becomes difficult to understand. Consequently, it is important not to avoid drawing an obviously impossible diagram in a proof by contradiction.

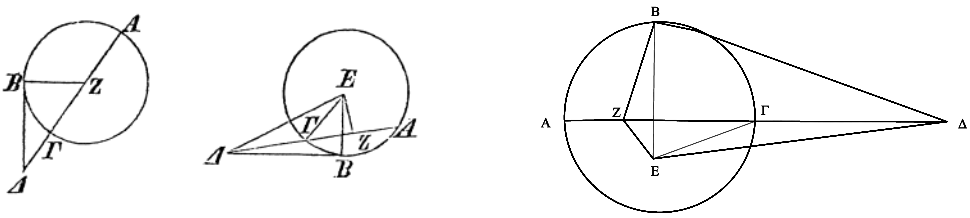
Now we have come to the last of the four features of manuscript diagrams. I call it "one diagram for two cases". Let us look at the diagram of a case of the chord theorem (III.36), at the end of book III. If point Δ is taken outside the circle ΑΒΓ (fig. 8), and from point Δ a cutting line ΔΓΑ and a tangent ΔΒ are drawn, then the rectangle contained by two segments ΑΔ and ΔΓ of the cutting line is equal to the square on the tangent ΔΒ. This relation is often expressed as an algebraic equality:

$$\mathrm{A\Delta} \times \Delta\Gamma = \Delta\mathrm{B}^2$$

However, Euclid never mentions any product nor second power of a line, so I use the following notation.

$$r(\mathrm{A\Delta}, \Delta\Gamma) = sq\,(\Delta\mathrm{B})$$

The proof first treats the case in which ΔΑ goes through the center Z. Then, the more general case is treated in which ΔΑ is not through the center.



Heiberg Codex P

Fig. 8: Proposition III.36

Heiberg gives two separate diagrams (fig. 8, left), but in mediaeval manuscripts we see a strange diagram. The only possible interpretation is that we are supposed to assume that Z is the center of the circle in the first case, and that in the second case, we continue to use the same diagram, assuming that E is now the center, and that EZ is perpendicular to the chord AΓ drawn from the center E. The indifference to metrical accuracy in the manuscripts makes such a flexible double use of the diagram possible[9].

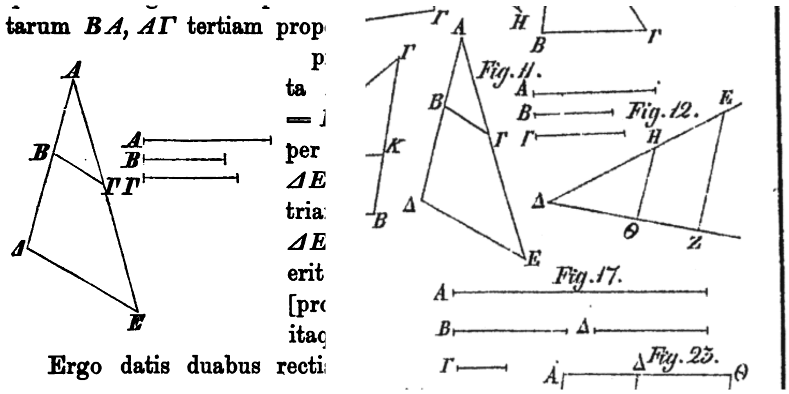
Here, I sum up the features of the diagrams of plane geometry in the *Elements*. (1) Overspecification: the angles tend to be right angles, and the triangles tend to be isosceles, or even equilateral. (2) However, metrical accuracy is not so important, and diagrams are not expected to

[9] I have to add that the treatment of the first special case of AΓ through the center in the text may be a later addition, for the letter Z is assigned before the appearance of the letter E. This is against Euclid's way of assigning labels, which is always in alphabetical order, as we shall see later. But as this is present in all the extant manuscripts, the addition must have been done quite early, certainly before Theon.

represent the equality of lines, angles, and areas. (3) In proof by contradiction, an obviously impossible diagram was drawn without hesitation. (4) Sometimes one diagram can serve two different cases.

The features of diagrams we have seen are strange to us, but they are not so strange nor embarrassing if mathematics was not learned from a written source but taught by a teacher. In other words, the text and the diagram of the *Elements* were not written for readers in distant places who would learn them without a teacher; the diagrams were probably drawn for those who already knew the proposition, and mathematics was not something to be learned from a textbook. In the following part of this article, I develop this idea and try to show that we can find traces of a prevalently oral teaching of mathematics in the text of the *Elements*.

Before closing the examination of the diagrams of geometric books of the *Elements*, I add two brief arguments. First, I show evidence that Heiberg copied the diagrams of August's edition. Then, I show the basic characteristic of the diagrams of arithmetic that is lost in the printed editions of today. There is decisive evidence that Heiberg copied August's diagram.



Heiberg August

Fig. 9: Proposition VI.11 in Heiberg's and August's editions

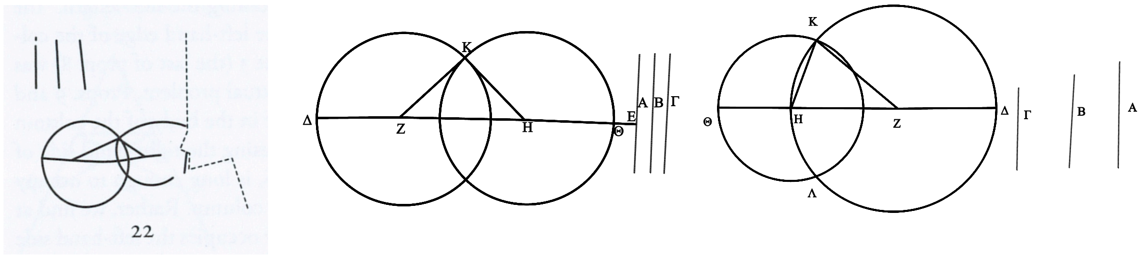
Fig. 9 (left) is the diagram of proposition VI.11 in Heiberg's edition. There appear the labels A, B, and Γ twice. The three horizontal lines A, B, and Γ on the right of the triangle do not belong to this proposition but to the following proposition VI.12. If we look at the diagrams of August's edition (fig. 9, right), we see why this has happened. In this edition, all the diagrams are printed together at the end of the volume, and the diagrams of proposition 11 and 12 are placed one beside the other, and it is evident that the three horizontal lines A, B, and Γ of proposition 12 have been wrongly attached to proposition 11 in Heiberg's edition.

This is decisive evidence that Heiberg copied the diagrams of August. Probably, Heiberg sent the diagram pages of August's edition to the printer and commissioned the task of dividing the page into single diagrams and arranging them.

Second, I argue that the features of the diagrams found in extant manuscripts go back to ancient times. All the diagrams we have examined are found in mediaeval manuscripts, of which the oldest is dated to the ninth century. It is therefore natural to ask whether their features come from ancient diagrams (if not those of Euclid himself), or if we see mediaeval degradation. A recently published papyrus fragment, Oxyrhynchus 5299, suggests an ancient origin for mediaeval diagrams, for it has peculiar features common to mediaeval diagrams[10]. I show only one example (fig. 10). In the diagram of proposition I.22, which constructs a triangle from three given lines—the three given lines appear as parallel and equal vertical lines in this papyrus manuscript, as well as in two ninth-century manuscripts, codex P and codex B (though the latter shows that the diagrams were

[10] Cairncross and Henry (2015). The diagram of proposition I.22 in the papyrus (fig. 10, left) is copied from this article.

not very faithfully copied). This can hardly happen by chance, so that we may suppose the mediaeval diagrams are copied from ancient ones.



Oxyrhunchus 5299 Codex P Codex B

Fig. 10: Diagram of proposition I.22

Diagrams in arithmetic

A brief mention of the arithmetic diagrams is in order. I recently found a situation different from, but no less interesting than, that in geometrical books; Heiberg invented all the diagrams for arithmetic, but the diagrams he invented were a disaster.

As a typical example, let us look at proposition VIII.2. This is a problem of finding numbers in continuous proportion in the given ratio. The ratio given is that of A to B (both are integers, for this is arithmetic). Then, Euclid makes A square, A multiplied to B, and B square, which are named Γ, Δ, and E, respectively. Then A multiplies each of Γ, Δ, and E, making Z, H, and Θ, respectively, and B multiplying E makes K.

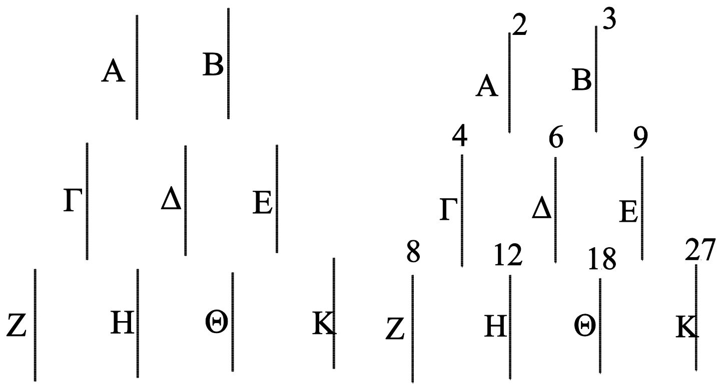
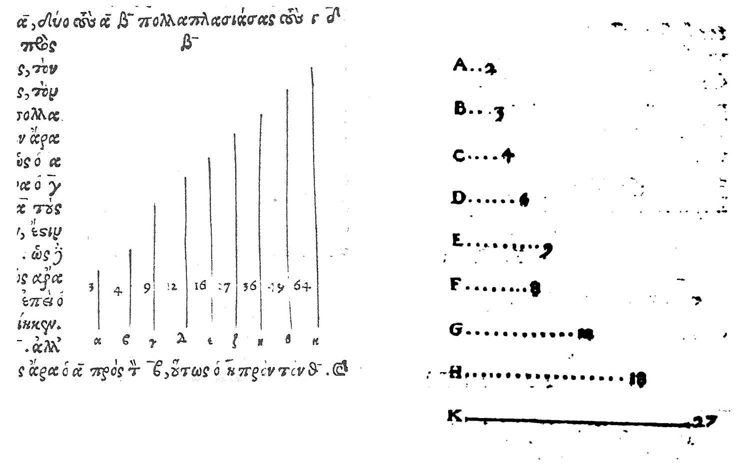
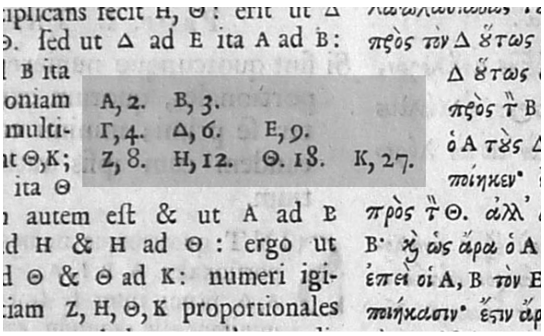


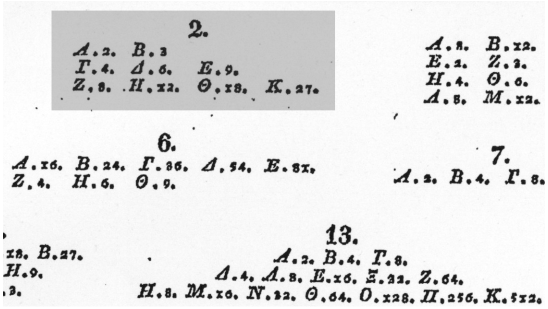
Fig. 11: Proposition VIII.2 (codex P)



Grynaeus (1533) Commandino (1572)

Fig. 12: Proposition VIII.2 in early editions





Gregory August

Fig. 13: Exemplary numbers replacing the diagrams in Proposition VIII.2 (shadows added)

We would represent the three numbers Γ, Δ, and E by A^2, AB, and B^2, and the four numbers from Z to K by A^3, A^2B, AB^2, and B^3.

But Euclid did not have such algebraic symbols, and simply assigned new letters every time a new number was introduced. This makes the reading very difficult, for you are supposed to remember what each of the nine symbols from A to K are. For example, you must remember that H is the product of A and Δ, and Δ, in its turn, is the product of A and B, to understand the proof. Thus, the arrangement of the lines in the diagram is very helpful for readers.

The manuscripts are also accompanied by numbers which serve as examples of the truth of the proposition. They are written either in Greek numerals or Arabic numerals (more precisely, eastern Arabic numerals, those used today with Arabic scripts). The numerals are Greek in the diagrams of older manuscripts, including codex P, whose reproduction is shown in fig. 11 above. I have reproduced the numerals in the right figure in (western) Arabic numerals.

We can say that the numbers in the diagrams did not exist in the original, because (1) these particular numbers never appear in the text of the *Elements*, (2) the numbers in diagrams are often written by a different hand from the main text, and (3) sometimes numbers are different from one manuscript to another.

Let us now see how these diagrams have changed in printed editions. The arrangement of lines was discarded in early editions, while exemplar numbers were retained (fig. 12).

Then, Gregory decided to do away with lines, while keeping the arrangement of lines as that of exemplar numbers (fig. 13), and August followed Gregory[11].

However, Heiberg decided to revive the lines, abolishing the exemplary numbers. Since exemplary particular numbers did not exist in the original in all probability, this decision was right. Heiberg did not use the diagrams in the manuscripts, but made new diagrams himself. I say "himself", for I have not been able to find any diagrams like his before his edition (fig. 14).

[11]In Book VII, Gregory represents numbers by dotted lines (the number of dots represents the exemplar value of the truth of the proposition), and from Book VIII, he adopts the representation without lines, which August uses from Book VII.

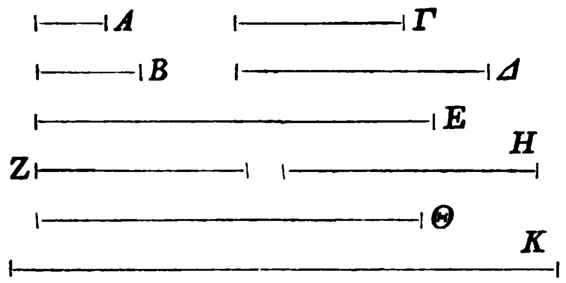


Fig. 14: Proposition VIII.2 (Heiberg)

This was a tragedy; Heiberg's diagram suggests nothing about the relationship between numbers. The merit of the manuscript diagrams is lost. The text is still much more readable with exemplary numbers provided in earlier editions. Since Heiberg, we have been reading Euclid's arithmetic with invented diagrams, which do not help us at all—the manuscript diagrams should be restored[12].

The Traces of Oral Teaching in the Text of the *Elements*

Particular and inconvenient features of the text of the Elements

I now move on to the other topic, the mathematical teaching style of which I believe we can catch a glimpse from the text of the *Elements*. Generally speaking, it is obvious that oral communication was much more important in ancient times than it is today. But how can we prove it in the field of mathematical teaching from the written documents we have?

I point out four particular features of the text of the *Elements* which I think are traces of the mathematical teaching style of ancient Greece.

First, although the *Elements* develop logical deductions using previous propositions in subsequent ones, proposition numbers are never used in the text of the *Elements*. Euclid never says something like "because of proposition III.37". Therefore, we should assume that the proposition number did not exist in Euclid's time, and that those in our manuscripts were added later.

Second, the diagram of each proposition is accompanied by labels, as we have seen. Labels or letters, A, B, Γ, etc., are assigned in alphabetical order, at the first appearance in the text. There is nothing wrong with this manner of assignment within one proposition. This means, however, that the assignment of labels may be different between propositions, even though they may be very similar in content.

For example, proposition III.37 (fig. 15) is the converse of III.36 (fig. 8 above). III.36 shows, as we have seen above, that if ΔB is tangent, for any line ΔΓA cutting the circle, the equality

$$r(\mathrm{A\Delta}, \Delta\Gamma) = sq\,(\Delta \mathrm{B})$$

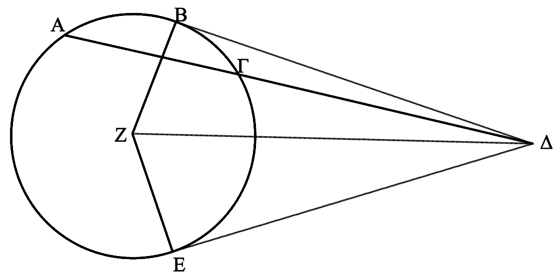


Fig. 15: Proposition III.37 (codex P)

[12] In the new Japanese translation of the *Elements* (Saito 2015), for which I worked with my colleagues, both Heiberg's diagrams and reproduced manuscript diagrams appear.

holds. III.37 says that if this equality between rectangle and square holds, then the line ΔB is tangent to the circle at the point B. However, the assignment of labels is slightly different; while the first four points A, B, Γ, Δ correspond to each other in two propositions, the points E and Z have different roles in these two propositions. E is the center of the circle in III.36, while it is Z in III.37. This is confusing to modern readers.

Why were the labels assigned in such an inconsistent way? The answer should be that somehow this was not inconsistent, and that it was not felt inconvenient to students in Euclid's time. But how was that possible?

For the moment, I go on to the next feature. The labels I have just mentioned are given to the points; lines and angles are indicated by two or three labels, for example, line ΔΓA. However, the order of the points is not fixed. In proposition III.37, the angle of the tangent at point E appears first as ZEΔ, then, later in the same proposition, the same angle is called angle ΔEZ. Such a change in the order of points happens quite often. Though there is no risk of misunderstanding, this is not very convenient for a reader of written text.

Finally, I explain the fourth, last feature. Every proposition of Euclid's *Elements* begins with a so-called "*protasis*", or general enunciation, where the proposition is stated without diagram and without the names of points, but in a general way[13]. For example, the *protasis* of proposition III.13 is thus: "a circle does not touch a circle at more than one point, neither internally nor externally".

This is fine. However, the *protasis* can become very long and complicated, so that no one reading it for the first time can understand it. As an example, let us read the *protasis* of Proposition III.37.

> (III.37 protasis) If a point be taken outside a circle and from the point two straight lines fall on the circle, and if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.

This *protasis* is incomprehensible, at least for someone who reads it for the first time. Indeed, when we read Euclid's *Elements*, we often skip the *protasis* and begin with the *ekthesis*, or setting out, which follows the protasis.

> (III.37 ekthesis) For let a point Δ be taken outside the circle ABΓ; from Δ let the two straight lines ΔΓA, ΔB fall on the circle AΓB; let ΔΓA cut the circle and ΔB fall on it; and let the rectangle AΔ, ΔΓ be equal to the square on ΔB (fig. 15).

Then, Euclid restates the conclusion with the names of points.

> I say that ΔB touches the circle ABΓ.

This affirmation is called the *diorismos*, the English translation of this name would be specification. Some constructions follow if necessary, then comes the demonstration.

Now, if the *ekthesis*, or setting out, with diagram and names of points, is much easier to understand, why does the text of the *Elements* always preserve the *protasis*, which is often skipped by modern readers. Moreover, the *protasis* is repeated at the end of each proposition, as the *sumperasma*, or conclusion. Most propositions of the *Elements* repeat the *protasis* almost literally at the end, as a conclusion, adding only one word '*ara*', i.e., therefore. So the *protasis* occupies considerable space on precious manuscript parchment.

An explanation for particular features of the Elements: traces of oral teaching and communication of mathematics

Now, I respond to all of the four questions. We should first note that all the inconveniences are for those who read the written text. Then, we may hope to find some explanation related to our reasonable assumption that the ancient teachers and students did not use written textbooks.

[13] For the division of a proposition into six parts found in Proclus, see (Mueller 1981, 11).

Let us again try to imagine how mathematics was taught in ancient times; did students possess some written textbook? Obviously not. Printing did not exist, and it is highly improbable that a student had some hand copy of the *Elements* at the beginning of learning. Then, how did the teaching proceed? There was no blackboard. A probable assumption is that the teacher drew diagrams on sand (sometimes on papyrus), and indicated the points and other geometrical objects by finger, explaining the proposition and its demonstration.

As a result, references by proposition number would have made no sense. Even if one had had the written text, the text was written on papyrus role, so that it would not have been so easy to open the place of a proposition, especially when it is one of the last propositions of one volume. To find it, you would have had to open and extend almost the whole of one papyrus role. Therefore, any reference to previous propositions must have had another form, which I will explain later.

The teacher probably drew diagrams and indicated their points in the presence of students. And the diagram was naturally erased when the next proposition was treated. The assignment of labels to points in the diagram had to be done anew, and there was no inconvenience, even if the assignment was different from that of preceding proposition, for in front of the teacher and students, there was always only one diagram[14].

We have a textbook in codex which consists of hundreds of papers, one sheet put over another and bound up forming a block—this is the etymology of the word codex. So, we can look up the diagram of another proposition, and the inconsistency of label assignment leaps to the eye. But it was not the case in ancient times.

The assumption that the diagram was drawn in front of disciples when a proposition is taught can explain the third particular feature, the inconsistency in indicating lines and angles by two or more labels. As the teacher speaks, indicating the labeled points, a line may as well be called BA, even if it was called AB before[15].

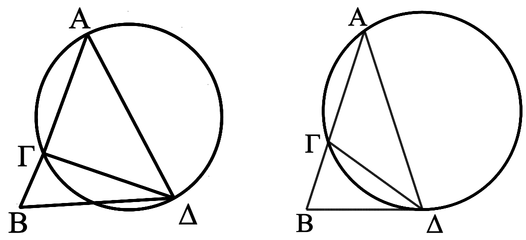
Now, I respond to the last point, the long and incomprehensive *protasis*. My suggestion is that the *protasis* was the format for memory. I said that the diagram of a proposition was erased when next proposition had to be treated, and the reference to a previous proposition was not done by proposition numbers. Then, there must have been some way of memorizing and referring to the propositions. I argue that this is exactly the role of the *protasis.*

Though some *protases* are long and difficult to understand, it is possible to memorize them *after* one has learned and understood the proposition. Let us go back to the *protasis* of III.37 and look at it. Imagine that you have already learned the proposition, know what it purports, and have in your memory the diagram like fig. 14 (the line BE is not necessary for it serves only for the demonstration)—then, this *protasis* is quite clear.

Moreover, I suggest that this is a quotable format, and was indeed quoted, not literally, but in accordance with the context. Let us see how the quotation is done. I pick up Proposition IV.10, which makes an auxiliary construction for the construction of a regular pentagon. In fact, it constructs an isosceles triangle whose angles at the base are double of the angle at the vertex (the size of three angles are 36, 72, 72 degrees). With this triangle, one has three of the five vertices of the regular pentagon.

[14] Thus the labels are like local variables in computer programming language. Local variables have their "scope" where they are valid, and outside their scope, a variable has another meaning (or has no meaning). The scope of a label is the proposition in which it is assigned.

[15] As for such switching of names, Netz (1999, 74ff.) develops illuminating arguments.



Codex P   Codex P (revised)

Fig. 16: Proposition IV.10[16]

Euclid starts with any line AB (fig. 16), and after some construction, he has point Γ on AB, and another point Δ, with the particular condition that the rectangle contained by AB, BΓ is equal to the square constructed on BΔ[17].This is exactly the premise of III.37, and now Euclid wants to show, using III.37, that the straight line BΔ touches the circle ΑΓΔ.

Euclid could not say "because of proposition III.37" (the first peculiar feature of the *Elements*: no use of a proposition number). Instead he says the following.

(IV.10 part of proof) Since point B has been taken outside the circle ΑΓΔ, and from B the two straight lines BA, BΔ have fallen on the circle ΑΓΔ,

We see at once that this can be obtained by transforming the *protasis* of III.37:

(III.37 beginning of protasis) If a point be taken outside a circle and from the point two straight lines fall on the circle, and if one of them cut the circle, and the other fall on it,

Indeed, replacing the `if' that introduces a conditional sentence by `since', for the points are already taken in IV.10, and replacing the references to points and lines in the *protasis* of III.37 by their labels in IV.10—"a point outside a circle" by B, "the circle" by ΑΓΔ, and "the two straight lines falling on the circle" by BA and BΔ—one can obtain the text of IV.10.

Here, I show this and the following parts of the *protasis* of III.37, and the demonstration of IV.10, giving the same numbers to the corresponding phrases.

(III.37 protasis)

(1) If a point be taken outside a circle and from the point two straight lines fall on the circle,

(2) and if one of them cut the circle, and the other fall on it,

(3) and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference

(4) be equal to the square on the straight line which falls on the circle,

(5) the straight line which falls on it will touch the circle.

(IV.10, the argument where III.37 is applied)

(1) Since a point B has been taken outside the circle ΑΓΔ, and from B the two straight lines BA, BΔ have fallen on the circle ΑΓΔ,

(2) and one of them cuts it, while the other falls on it,

---

[16] I have omitted from the diagram the circle with center A passing through B and Δ, with which our argument is not concerned. The line BΔ cuts the circle ΑΓΔ in the diagram of codex P, though it must be a tangent. I have provided a revised diagram, too.

[17] There is another condition that BΔ is equal to AΓ, but this is not used in the part of the demonstration we are interested in here.

(3) and the rectangle AB, BΓ

(4) is equal to the square on BΔ,

(5) therefore BΔ touches the circle AΓΔ.

This comparison shows how *protasis* of a proposition in memory was used when that proposition was applied later. One remembers (or recites) the *protasis*, and replace the references to geometrical objects (points, lines, circles etc.) in a general form with the labels of the objects one has in the diagram on which one is working.

I think that this was the role of the *protasis*, which is embarrassing for us; *Protasis* was a format for memory and application of a proposition. Therefore, it was the most important part of the proposition, and we understand why the text of *Elements* does not omit it in spite of its difficulty (or unreadableness) to those who first read it, and the space it occupies.

I hope to have shown the traces of practice—prevalently oral—of ancient mathematics in the written text we possess. The style of the text of the *Elements* was not invented for transmitting mathematical ideas to a person living far away, and writing text was not probably the main task of a mathematician.

To conclude, I refer to a passage in the introduction of Apollonius' *Conics* to support this view. Apollonius explains the motive to write this gigantic and difficult work of 8 books, of which we possess Greek text of the first four and Arabic translation of first seven. The last book was lost.

According to Apollonius, Naucrates, a friend of his, visited him in Alexandria, and when he left, he wanted to carry Apollonius' theory of *Conics* with him, so Apollonius wrote the *Conics* in a hurry. In the preface of the text we possess, Apollonius says that he has revised the first version, and now he is sending the revised version.

From this preface, we can gather that even for Apollonius, who was active one century after Euclid, to write down a work did not have priority in his activities. We should be grateful to his friend Naucrates who gave the occasion to write down the *Conics*, which have come down to us.

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[18] *Pisauri* is locative of *Pisaurum* (Pesaro).

SYMPOSIUM CONTRIBUTIONS

SYMPOSIUM A

TEXTBOOK USE BY TEACHERS AND STUDENTS – RESULTS AND METHODS

organised by Sebastian Rezat and Rudolf Sträßer

SYMPOSIUM A: TEXTBOOK USE BY TEACHERS AND STUDENTS – RESULTS AND METHODS

SEBASTIAN REZAT and RUDOLF STRÄßER

Teachers and students are regarded to be the main users of textbooks. Textbooks offer opportunities to learn (for both up students and teachers), which on the one hand have to be understood and taken by teachers and on the other hand have to be implemented in classrooms. In most recent studies the textbook is regarded to be the mediating artefact, which links the goals, the knowledge and the beliefs of teachers and students (Brown 2009; Remillard 2005; Rezat & Sträßer 2012). Besides these teacher- and student-related variables, textbook use is also influenced by additional social and cultural influences, such as institutions, especially the school, the influence of peers, family and tutors as well as the one from those interested in mathematics education in general (the "noosphere" sensu Chevallard) and finally the influence of conventions, norms and the public image of mathematics (Rezat & Sträßer 2012). Usually, the textbook is not the only artefact that teachers and students use for teaching and learning, but one resource among several others (Gueudet & Trouche 2009).

The aim of the symposium was to collect research on the use of mathematics textbooks in order to develop a deeper understanding of textbook use by teachers and students and to give an outline of the state of the art. Furthermore, the symposium aimed at discussing three methodological questions:

1) What are appropriate methods to investigate teachers' and/or students' use of mathematics textbooks?
2) What about additional challenges and potentials when analysing the use of interactive/digital textbooks? What are appropriate ways to investigate interactions and interrelations of students' and teachers' use of mathematics textbooks?

Among the contributions five were selected to be presented at the symposium because of their originality and contribution to the field:

1) Margot Berger: Reading mathematics textbooks: different reading styles;
2) Kristina Reiss, Stefan Hoch, Frank Reinhold, Bernhard Werner, Jürgen Richter-Gebert: Analyzing classroom work: students' use of electronic textbooks
3) Vilma Mesa, Angeliki Mali, and the UTMOST Team: Uses of dynamic textbooks in undergraduate mathematics classrooms;
4) Elena, Naftaliev: Pedagogical functions of interactive texts;
5) Shai Olsher, Michal Yerushalmy, and Jason Cooper: Developing categories of curricular metadata: lenses for studying relationships between teachers and digital textbooks

Two page summaries of these presentations follow this introduction and general overview of the Symposium. The references of all contributions are collected at the end of this presentation of the symposium.

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Before we summarize the contributions of the presentations in the symposium to questions 1 – 3, we will give an overview in terms of (a) the relation between research on traditional and digital textbooks, (b) the user in focus, and (c) the different meanings of "use".

Looking at the contributions to this symposium, there seems to be a growing interest in the use of digital textbooks compared to the interest in the use of traditional print textbooks. Only one paper in this symposium focuses on traditional print textbooks while the other four focus on digital textbooks or parts of them. However, it appears that the variety among digital textbooks seems to be much bigger than among their traditional predecessors. Without taking the technical and technological differences into account, the variations seem to mainly relate to the amount and nature of possibilities of interaction and collaboration.

While research related to traditional print textbooks mainly focuses on teachers as the main users the contributions to this symposium might indicate that the student as a user seems to attract more attention in research related to digital textbooks: Three papers focus solely on the student, one focuses solely on the teacher, and one takes both the teachers' and students' use into account.

Depending on the particular activity the textbook is involved in, the very meaning of 'use', might differ. This is exemplified by the different presentations in the symposium. 'Use' might refer to the actual reading of the content and to the question how readers actually make use of the contents of a text. Berger differentiates five different "reading styles" that vary in terms of extent of reading, focus, connections made, and the quality of solutions to exercises. Thus, reading styles vary from "close reading with strong connections" through "scanning" and "skimming" to "avoiding". 'Use' might also refer to the understanding that users develop of the opportunities to learn provided by textbooks and which pedagogical potential they see in these. Naftaliev also aims at understanding the activity of students interacting with texts. However, she focuses on one particular representation within interactive textbooks, namely interactive diagrams. She finds that interactive diagrams with different organizational functions have different effects on the activity of the students with these diagrams and thus on students' learning. Reiss, Hoch, Reinhold, Werner, and Richter-Gebert also analyse students use of interactive diagrams. They analyse students' solution strategies related to different visualizations of fractions (bar, circle) and draw inferences regarding affordances and constraints of the two visualizations. They also hint at the possibility to draw inferences from students' solution strategies regarding their conceptual understanding.

Olsher, Yerushalmy, and Cooper, as well as Mali and Mesa, focus on teachers as the user. Olsher et al. analyse the didactical categories that teachers apply in order to make sense of opportunities to learn in textbooks in terms of their correlation with the authors' intentions. In contrast to Berger, who aims to understand the process of reading and thus the interaction of the reader and the text itself, Olsher et al. focus on the understanding and interpretation of the opportunities to learn by the use. Accordingly, they term the set of categories that they get from the teachers' coding activities "didactic metadata". While Naftaliev focuses on interactive diagrams as one particular aspect of interactive texts, Mali and Mesa aim at understanding the interaction between teachers and a whole set of resources in terms of instrumental genesis.

In terms of appropriate methods in order to analyse the use of textbooks (question 1), case studies within the qualitative research paradigm seem to be the preferred method. Only the study by Reiss et al. presents a mixed methods design. This might have different reasons: On the one hand, the understanding of the relevant factors and mechanisms related to the use of textbooks is still in phase of exploration. Generalizable patterns that describe the interplay of different factors and related consequences on textbook use are difficult to unveil. On the other hand, textbook use seems to be influenced by many and very individual factors and therefore is only adequately accessible, when the methodology takes this individuality into account.

All contributions to this symposium have in common that they understand 'use' as an interaction of artefact and user. Especially in terms of traditional textbooks the focus is on the user. However, the role of the artefact within the interaction is not always clear. For example, Berger analyses reading styles, which seem to be a characteristic of the user. It is not clear how the readings styles are affected by properties (affordances and constraints) of the artefact. It seems to be a particular potential of digital artefacts (question 2) that the role of the artefact within the interaction comes more to the fore related to digital artefacts. For example, Naftaliev suggests that characteristics of the activity with interactive diagrams are dependent on their organizational function. At this stage, it seems as if the understanding of the particular contribution of each actor – user and artefact – within the activity is more like to develop if the artefact at hand is very specific. This seems to be the case in the analysis of interactions of students with visualizations of fractions by Reiss et al. as well as the analysis of the interaction of students with interactive diagrams by Naftaliev. As soon as the artefact is a whole textbook or even a set of resources the insights seem to be much more general and not specifically linked to the properties (affordances and constraints) of the artefact.

From a methodological perspective, there might be two reasons why the role of the artefact within the activity is more apparent in terms of digital artefacts: First, interactive digital artefacts react upon actions of the user and thus change within the interaction. This is the very meaning of interaction. The analysis of mutual related actions of artefact and user seems to be more easily related to properties (affordances and constraints) of the artefact than in the case of analogous artefacts, in which the artefact appears to be mostly unchanged in the interaction. Second, digital artefacts facilitate the collection of user data. While with analogous artefacts the data that is collected – mainly video recordings and interviews – stems from the user, digital artefacts allow for the collection of user related data in combination with data on the changes of the artefact.

Among the contributions to the symposium, the UTMOST project (Mesa et al.) is the only one that takes teachers' and students' use of a digital textbook into account. Therefore, it is the only one capable to make a contribution to question 3. However, at this stage only data on teachers' use was analysed.

The analysis of interactions between teachers' and students' use of textbooks still seems to be a challenge in the field. In the end, it is neither the teacher's nor the student's use of textbooks by itself, but it is the interaction between both that is crucial for the quality of the learning situation.

READING MATHEMATICS TEXTBOOKS: DIFFERENT READING STYLES

MARGOT BERGER

Introduction

I develop a broad typology for categorising the ways in which students read a section of a chapter in a mathematics textbook. The categories of the typology derive from those found in informal literature around academic reading skills, such as skimming, scanning and intensive reading (BBC 2011). These categories are inductively refined and elaborated to the mathematics reading context using observations and video transcripts of five specially chosen students studying out loud from a prescribed mathematics textbook.

Research on the way in which learners read mathematics textbooks is scant. Exceptions to this are Rezat (for example, 2013). Berger (2016) developed a framework, specific to mathematics discourse, for exploring the relationship between enacted discourse (the student's way of reading the text) and the written discourse (the text). I develop the latter framework so as to categorise different approaches to reading mathematics by different learners. Reading is understood as a transaction (enacted curriculum) between text (written curriculum) and reader.

Context

The five students were enrolled in a self-study course for in-service or pre-service high school mathematics teachers at a South African university. These teachers are conceptualised as mathematics learners both within the course and within the research; this is because their prior knowledge of mathematics is often weak and not mathematically rigorous. In the pre-calculus course, students were expected to study (read the text, do worked examples, exercises, and so on) from the prescribed textbook (Sullivan 2012) prior to the lecture on their own. The precise sections for self-study were designated by the course designer (myself).

Methodology

Five students were chosen by the researcher (myself) as participants in the research. They were chosen according to my informal observations of their different learning styles in class. For the research study, these learners were individually video-taped while reading and studying out loud from a sub-chapter, 'Properties of Logarithms' (Sullivan 2012, pp. 296–304), as they would in preparation for class. They were also given a set of exercises at the back of this sub-chapter, as was the case for their weekly sessions. This sub-chapter addresses operations with logarithms and change of bases. It was chosen because it focuses on mathematical ideas with which the students are familiar, but from a more advanced perspective. For example, they know procedures for working with logarithms but are not familiar with proofs of theorems around logarithms.

Aside from the transcripts of each video session, the interviewer (myself) wrote a set of notes during each video session, noting points of interest. In addition, the writings and solutions to exercises of each student were photocopied after the video session. These field notes and student solutions were used in the interpretation of the transcripts.

Analysis and results

I used the constant comparative method (Glaser and Strauss 1967) as my method of analysis. The analysis of data consisted of four major iterative steps. First, there was the descriptive level: I read through the transcript of each video session, together with field notes and the student's photocopied

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solutions. This allowed me to make notes on the transcripts describing what was happening. (For example, "Student A writes Property 1 ($a^{\log_a M} = M$) explicitly in Proof of Property 6; Student B does Exercise X quickly and correctly.)

During the descriptive level, I generated four broad analytic categories regarding the student's reading of the textbook sub-chapter. These four categories were 'Extent of reading', 'Focus', 'Connections', 'Quality of solutions to exercises'. 'Extent of reading' relates to the *how* of reading. It is loosely measured by the extent of paraphrasing or explaining of the text. This category derives from the literature on academic reading skills where a close or intensive reading of the text is distinguished from skimming or scanning the text. 'Focus' relates to *what* the student was reading. It refers to the component of the text (proof, worked example, alternate representation, etc.) that the student is paying attention to. When reading a mathematics textbook, the issue of whether the student pays attention to theory, procedures, or alternate representations, and so on, is an important criterion in describing the actual reading. 'Connections' refers to which component of the text, if any, the student is making connections to. In mathematics education literature (see, for example, Watson and Mason, 2006) the quality of connections between different mathematical ideas, representations, examples and so on is fundamental to understanding. Correctness of solution refers to the correctness and quality of justification and explanation in the execution of the exercises. This in itself is one measure of the quality of reading.

I then re-read the transcripts, applying the analytic categories to interpret the data (for example, **focuses** on proof of Property 5; **connects** solutions of exercise to property 2). I then generated short narrative descriptions of each student's activities while reading the textbook; concurrently I produced a table containing the names of the five students against the analytic categories. All these steps were repeated several times so as to refine the descriptions, the analytic categories, the summary accounts and the table. After this, a typology of different styles of reading was generated for different patterns revealed during categorization.

This analysis resulted in five different styles of reading: close reading with strong connections; close reading with some connections; scanning, skimming and avoiding. 'Close reading with strong connections' is characterised by a comprehensive reading of all the text (evidenced by paraphrasing and explanations) as well as the making of explicit connections to prior knowledge or to other parts of the text while reading the text and doing exercises. It resembles the category of intensive reading in academic skills reading. 'Close reading with some connections' is similar to 'close reading with strong connections'. The difference is that the reader does not make explicit connections to the text when doing the exercises. However, the reader does make explicit connections to prior knowledge or components in the text when reading the text and she is able to conceptually justify her solutions to exercises when asked to do so. Skimming and scanning relate to skimming and scanning respectively in academic reading skills. Specifically, 'scanning' is characterised by the reader looking for keywords or information in the text and using this information productively. In contrast, 'skimming' is characterised by the reader noticing specific keywords or information in the text but not finding the appropriate components (e.g. worked examples, properties) to support the reading in the text, or not being able to use the appropriate component in a productive way. 'Avoiding' is peculiar to mathematics reading: theory and proofs are mostly avoided and most attention is focused on procedures. Further investigations with different students and different texts should yield further elaboration of these reading styles and, possibly, other different mathematics reading styles.

ANALYZING CLASSROOM WORK: STUDENTS' USE OF DIGITAL TEXTBOOKS

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Introduction

In a time where students are considered to be *digital natives* (Prensky 2001), the change from traditional printed textbooks to digital tools seems inevitable. While digital textbooks may offer the same content as a paper-based version, they may provide other methods of presenting the content and – maybe more importantly – offer new ways of experiencing content, due to their digital nature: Features such as automatic correction and automatic, immediate feedback that is programmed to support and not to judge students (cf. Hattie and Timperley 2007) offer valuable add-ons to traditional textbooks. Moreover, interactive multimedia exercises give new opportunities in textbook design. ALICE:fractions (Adaptive Learning in an Interactive Computer-supported Environment) makes use of these advantages. As an interactive digital textbook, it offers a learning environment aiming at assisting students' work with fractions. The environment is intended to be used on tablets, allowing for a natural way of input which has been found to be beneficial for the acquisition of certain mathematical concepts (cf. Black et al. 2012).

In addition, digital textbooks can support teachers in diagnosing students' learning processes. Since the content is displayed in a computer-supported environment, the recording of *process data* is possible. The data gathered by ALICE:fractions allows for a closer look into how students use the learning environment, providing access to the amount of tasks they solve and how long it takes them to solve each task. Moreover, the analysis of students' answers can include data like their finger movements on the touchscreen, allowing their *way to the answer* to be examined. Accordingly, it is possible to get insights in students' understanding or misunderstanding of fraction concepts.

In our study, we concentrate on the analysis of visualization tasks that are supposed to foster students' understanding of the magnitude of fractions. These tasks can be solved in various ways according to the strategies used by the students. For example, students may see a solution immediately or may use other and simpler fractions for orientation. These different strategies result in different ways of representing fractions on a touchscreen and in different finger movements.

Method and Sample

During a four-week intervention, 28 students (one grade six class) worked with two visualization exercises during their first lesson on fractions (one iPad per student; see Figure 1 for examples). The tasks were randomly generated on each iPad; denominators varied from 2 to 12. During the

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work on the iPads, students' finger movements on the touchscreen were recorded as lists of coordinates plus a timestamp. This allows the solving process to be replayed by researchers. The total of 578 solutions (346 circle, 232 bar) were individually screened and grouped with respect to recurring patterns.

The visualization exercises were adapted from a hybrid representation model for rational numbers (Carraher 1993). Students were asked to find a given fraction in a bar or a circle by filling the correct portion of the object either by dragging their finger over the touchscreen or by tapping at the correct part of the object. Both tasks are continuous in the sense that no division was predefined; a solution could not be found by counting, but was only possible based on understanding.

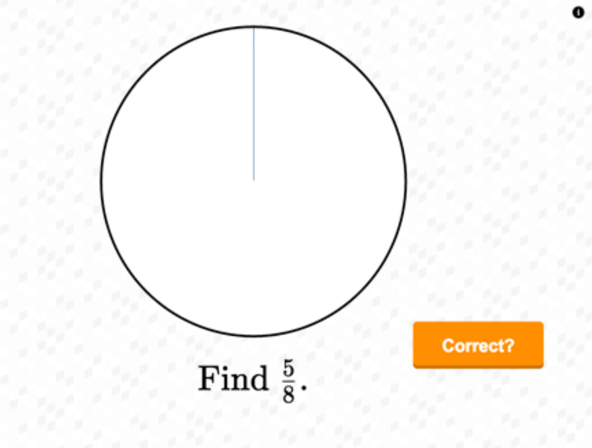

Figure 1. Visualization exercises in the digital textbook.

Results and Discussion

Students' finger movements provided evidence for typical patterns in their ways of solving the problems. In particular, one can differ between correcting and immediate processes: when using a correcting pattern, the solving process showed clear correction movements at the end of the process. In the other case, a student let her first input be evaluated. Furthermore, one can distinguish patterns that contain visible pauses. These pauses may appear at certain benchmarks (like e. g. ½) or depict a dividing process into equal shares.

We observed that the different shapes (bar, circle) might lead to different strategies for a solution: students showed lower rates of correcting errors when working with circles and no student separated a circle into equal shares (thus probably felt more comfortable with well-known objects). Furthermore, students started more often with separating into halves (falling back to well-known fractions) when working with bars. These findings suggest that identifying fractions on a continuous bar is more difficult for students with low experience on working with fractions – a fact mirrored by the observed solution rates (0.45 vs 0.61; deviations up to 3% from the exact solution were accepted as correct).

Finger movements could be linked to typical solution strategies: Pauses at ½ in the solving process might indicate an intrinsic comparison to ½, while separating into equal shares facilitates counting strategies that can also be applied to discrete tasks.

Our observations indicate new possibilities of research in mathematics education. However, further research is needed relating solving patterns to solution rates and the development of the usage of specific strategies over time. Teachers should benefit from this data because certain patterns (e. g. separating into equal shares) might indicate the lack of a holistic concept of fractions calling for more classroom work.

These findings are also available in German (Hoch et al., forthcoming).

USES OF DYNAMIC TEXTBOOKS IN UNDERGRADUATE MATHEMATICS CLASSROOMS

ANGELIKI MALI, VILMA MESA, UTMOST TEAM

Introduction

We present preliminary findings from a pilot study conducted within the context of the *Undergraduate Teaching and Learning in Mathematics with Open Software and Textbooks* (UTMOST) project. The project lays the ground work for understanding the affordances and challenges of developing and using open source learning platforms in the teaching and learning of linear algebra and abstract algebra. We pursue two foundational questions: (1) How do students and instructors use textbooks? and (2) How can we develop textbooks that will improve teaching and learning? We seek to develop data collection instruments and test analytical processes. The products are instruments for data collection for a larger study that will also investigate correlations of resource use and student learning.

Context

The UTMOST project focuses on the development and use of open source computational resources in the teaching and learning of mathematics at the undergraduate level. The project continues work in the development of four interconnected suites of technological resources, including the Collaborative Calculation in the Cloud (CoCalc), a web application that provides a scientific computing environment for collaboration among groups of people. The research component investigates how textbooks in the open source platform are used by instructors and students in Linear Algebra and Abstract Algebra. Our overarching aims of the research component are: (1a) To identify the instructor and students' use of textbooks that are either in the open source platform or as an identical PDF, (1b) To contrast the uses of these resources (same content, different platforms), and (2) To propose measures of student learning that would potentially identify the impact of these resources.

Theoretical and Analytical Underpinnings

Following Gueudet and Trouche (2009), we seek to investigate two documentation processes, *instrumentation*, the influences on the user of the affordances and constraints of a set of available resources, and *instrumentalization*, the influences on the resources that are a consequence of the user's use of those resources (see Figure 1).

A document is seen as a set of resources together with the schemes of utilization. The set of resources include three distinct components, a material component (e.g., the physical textbook, the software available), the mathematical component (e.g., the definitions available in the resources that can be different from canonical definitions because of the availability of computational resources), and the didactical component (e.g., the process of designing assignments). We focus on lesson planning and lesson enactment, seeking to identify operational invariants, instructors' beliefs that

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shape the design and use of resources (e.g., beliefs about ways with which students better understand definitions). Because of the exploratory nature of the work, we are not, at this point concerned with institutional influences, the exploration of different contexts, nor with the evolution of the documents over time. We also seek to identify students' utilization schemes, possibly adapting some of the theorization proposed by the documentational approach, because the research of students' use of their textbooks in undergraduate settings is in its infancy.

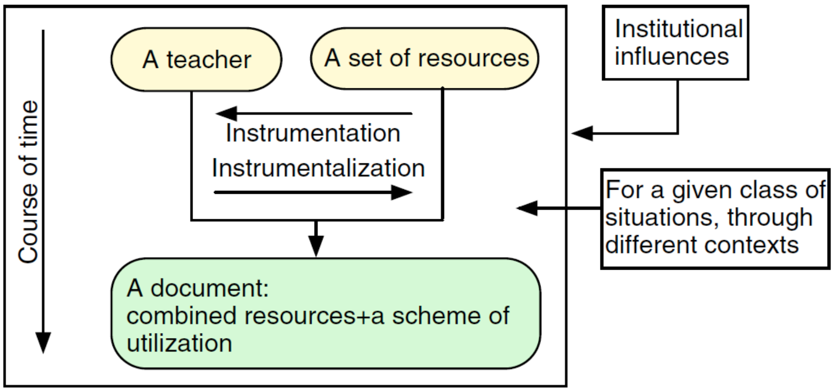

Figure 1: The documentational approach (Gueudet and Trouche 2009).

Methods

In the pilot phase, we collected in-depth data from four instructors, three of whom were using the dynamic version of the textbooks, via surveys, video recordings of lessons and of planning sessions, interviews, and bi-weekly self-reports of textbook use. In addition, we collected student surveys, bi-weekly self-reports of textbook use, focus groups, and tests of knowledge. We followed the methodologies proposed by Gueudet and colleagues (2012) and by Rezat (2012). Importantly, the nature of the dynamic textbooks allows for capturing computer generated data of student and instructors' use, which think of using to complement the instructor and students' self-reports.

Findings

Our first finding pertains to wide differences both in the network of resources mentioned by the four instructors, and in their awareness of that network. They consulted departmental archives; their own prior lecture notes, lesson plans, and exams; the course textbook; secondary textbooks (e.g., used by other instructors, or as students); colleagues; and listservs. In addition, they used Sage and LaTeX, and considered as resources students' questions and difficulties with the mathematics or, in the case of the instructors using the CoCalc, students' productions in that environment. Instructors mentioned many resources without labeling them as such. We identified them as they described their processes of planning and teaching their lessons. In spite of the variation in material components, we identified several common mathematical and didactical components in regards to the textbook: specifically, the mathematical component included the available mathematical topics (e.g., orthogonality, basis), examples, applications, and Sage (in the case of the dynamic textbooks) for which the didactical component included: selecting, rewriting, and summarizing topics, examples and applications, and design of assignments for the dynamic textbook. We also noted variation in the use of dynamic features, which seems to be related to instructors' knowledge, understanding, and familiarity of those features and their location within the textbook. Finally, we identified the *lecture notes* as a key document that was created by all instructors and used differently both in planning and lesson enactment. This will be the focus of further data collection. Regarding our methods, we realized that lesson planning occurs over extended periods of time, so one week of site visits is not always enough to capture that aspect. Also, we received limited student responses of their textbook use in bi-weekly logs, so we now consider including questions about their use of resources and creation of documents.

PEDAGOGICAL FUNCTIONS OF INTERACTIVE TEXTS

ELENA NAFTALIEV

Technological development has caused changes in learning environments in general and textbooks in particular. The affordances and constraints of the presentational media were always part of the mathematics culture. Describing the communication of Greek mathematicians, Netz (1999) suggests that the limitations of the media available (wetted sand, dusted surface or wax tablets) were essentially similar to those of modern books: "Diagrams, as a rule, were not drawn on site. The limitations of the media available suggest rather, the preparation of the diagram prior to the communicative act - a consequence of the inability to erase" (p.16). An interactive diagram (ID), is a relatively small unit of an interactive text (in e-textbook or another material). The ID`s components are the given example, its representations (verbal, visual and other) and interactive tools. A static diagram that presents specific information presents a point of view thus implicitly engaging the viewer in meaningful interpretations. An ID presents information and explicitly requires the viewer to take action and change the text within given limitations.
"Visions of the future of the textbook raise questions about the pedagogical functions of this educational form: What is it that textbooks provide pedagogically and epistemologically, besides a reminder of the weight of the past? How might they change in the future, and how could such changes serve the interests of publishers, authors, students, and educators?" (Friesen, 2013, p.2). Interactive textbooks are envisioned as allowing the learner and teacher to approach texts in an exploratory mode, rather than simply receiving it in a fixed, prepackaged form (Naftaliev, 2018). There are profound differences between the traditional page in math textbooks that appears on paper and the new page that derives its principles of design and organization from the screen and the affordances of technology. This issue requires scholars to develop lenses for analyzing pedagogical design and teaching-learning processes with interactive texts. Therefore, we developed and elaborated semiotic framework for pedagogical functionality of ID that would allow an orderly discussion of the subject (Naftaliev & Yerushalmy, 2017). There are three functions of ID in the framework: the presentational function, the orientational function and the organizational function (Table 1).

Table 1. The semiotic framework: Three types defining the functionality of IDs

Presentational function	Orientational function	Organizational function
Specific	Sketchy	Illustrating
Random	Accurate	Elaborating
Generic	Sketchy and accurate	Guiding

The *presentational function* focuses on what and how is being illustrated by the diagram. Three types of examples are widely used: specific, random, and generic examples. Specific examples present the exact data of the activity of which they are part of. They serve as a dynamic illustration that helps analyze the situation without being able to change the information. Random examples are specific examples generated within given constraints, presenting different information at various times and for different users. In a generic example, the diagram is structured to be representative; it presents a situation that can be part of the given task, but it is not intended to present the specific

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data of the activity but to help learners become acquainted with the generic views of the example through a process of inquiry. The tone in which the text addresses the learner is subject to design decisions having to do with the *orientational* function. “Sketchiness” vs. “rigorousness” of diagrams is an important factor in reader orientation. An accurate ID has richness of detail, but completeness of detail in sketch means that the user has to work in order to see through the whole, to make contact with and examine details. For example, the sketchy ID in Figure 1b could serve an accurate ID, by providing the values of ordered pairs for any point on the plane according to the user's choose. In our research (e.g., Naftaliev & Yerushalmy 2011) the activity was first illustrated by a paper diagram (Figure 1a) and then by an ID (Figure 1b). With the paper diagram Roni found the coordinates of the marked points but was not able to write the symbolic expression of the function. With the ID she perceived the given graph as a sketch; the description was of a line with a positive slope that intersects "somewhere below." At the same time, dynamics of mouse tracing in the ID sketch that accompany the changes in coordinate values helped her start to consider the idea of rate. She followed the changes of the coordinates along the line, tracked the coordinates on the graph, organized values of consecutive integers in a table and calculated the differences between the values in the table and the ratio between the differences to find the slope. The ID’s design made it possible to address the given graphs as a sketch, but at the same time the sketch can be interactively unfolded into a detailed accurate diagram, which causes students to change their focus from data testing to choosing the necessary data.

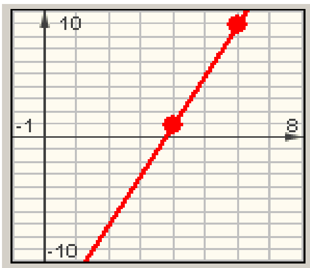
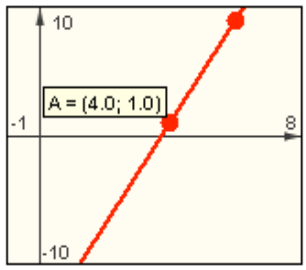
Write an expression describing the given line graph containing the two points marked on the diagram.	 (a)	 (b)

Figure 1: “Sketchiness” vs. “rigorousness”

The organizational function looks at the system of relations defining wholes and parts and specifically at how the elements of text combine together. IDs can be designed to function in three different ways: Illustrating, Elaborating, Guiding. *Illustrating IDs* are simply-operated, unsophisticated representations. They are intended to orient the student’s thinking to the structure and objectives of the activity by usually offering a single representation and relatively simple actions. The important components in the design of the *Elaborating IDs* are rich tools and linked representations that enable various directions in the search for a solution. We use the term *Guiding ID* referring to guided inquiry. This kind of ID provides the means for students to explore new ideas. In addition to providing resources that promote inquiry, they also set the boundaries and provide a framework for the process of working with the task.

Using the framework as an analytical lens, we were able to examine the characteristics of activity consisted of reading and solving tasks which are presented as IDs and to analyze how do the characterizations of processes vary in accordance to the three designed organizational functions of IDs: illustrating, elaborating, and narrating? We analyzed the work of 13-14 year-old students in task-based interviews, focused on three major fields in the school-algebra curriculum (Modeling, Formulating mathematical phenomena, Manipulating). The chosen sequences intended to reduce the effect of specific algebra content on research conclusions. Each series includes a preliminary task and three comparable tasks; each was designed upon a semiotic framework.

Across our studies, we had found that similar tasks with different IDs should be considered as different learning settings. We found that even the minimal interaction designed in the *illustrating*

ID can be helpful in consolidating relevant knowledge that is not adequately structured yet. Students who worked with the ID looked for ways to bypass the designed constrains: they changed the representation of the data in the given example and expanded the given representations or built new ones. Regarding *guiding* IDs, we found that it can be a form of instruction toward development of new mathematical ideas. The guiding IDs' design limits the student's action and at the same time provides an open space for student's ideas. The various linking tools and representations in the *elaborating* IDs lead to different problem-solving processes and a variety of solutions. The differences between methods were manifest in the variety of the significant items in the examples, in the representations students chose to work with, in the order of preference of the various representations and in the choice to use or not to use the included tools.

DEVELOPING CATEGORIES OF CURRICULAR METADATA: LENSES FOR STUDYING RELATIONSHIPS BETWEEN TEACHERS AND DIGITAL TEXTBOOKS

SHAI OLSHER, MICHAL YERUSHALMY, & JASON COOPER

Digital textbooks and other curricular resources can provide many opportunities for teacher engagement in the form of organizing content and generating innovative mathematical experiences for their learners. When attempting to study the interactions of teachers with digital curricular materials, we explore ways to support the teacher's potential role as a cartographer (Remillard, 2016) - having to generate a representation of a domain of curricular materials, and also to set paths that would help learners get better acquainted with the mathematical terrain.

Our object of mapping is a digital textbook, for which we use the broad definition of Pepin et. al (2015) for an evolving e-textbook: an evolving structured set of digital resources, dedicated to teaching, initially designed by different types of authors, but open for re-design by teachers, both individually and collectively. While the evolving content (tasks, tools, learning objects) is an important aspect of the textbook quality, we view the properties of the “collection” - structure, balance, and sequencing - as crucial to the coherence and quality of the book. Considering that textbooks include a representation of their content, e. g. table of content, or a site map, teachers, when given the opportunity, may still suggest to modify textbooks to make them more accessible (Olsher & Even 2014).

Research on textbooks includes two foci – the intended curriculum constituted in the textbook, and teachers’ enactment, often investigated in small-scale case studies. Analysis of textbook usage tends to be theory-driven; researchers decide what aspects of textbooks and their use to analyse – from the number of pages taught in a particular topic to nuances of common core standards evident in tasks, which are or are not enacted. We propose a methodological approach that addresses both intended and enacted aspects, while making room for the teachers' perspective alongside the researcher's. This is achieved through teachers’ tagging of curricular material, using a tool implemented as a browser extension; web-based tasks, tagged by predefined categories of metadata, create a collective dataset of curricular material for teaching. Patterns of individual teachers’ tagging provide insight on their interaction with the textbook, while collective tagging of a single textbook by a diverse group of teachers (averaging over all taggers) provides a more objective view of the textbook itself.

Our work includes design-based research of the tagging tool and its categories of didactic metadata. One of the considerations in the initial selection of coding categories was the ability to describe technology-related characteristics of interactive tasks. This provided a literature-based starting point, from which we listen to and learn from teachers. We have modified the categories based on teachers’ perceptions of relevance for representing curricular material. As proof of concept for our categories of metadata for characterizing the design of a textbook, we gathered 9 practitioners -

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teachers, mathematics education graduate students, and mathematics education researchers – to jointly tag 74 tasks (3 chapters) in the pre-calculus Visual Math textbook "Analysis – Computer supported inquiry activities for high school" (Yerushalmy et al. 1996). The tasks were randomly distributed among the taggers, and the tagged chapters were represented visually using a Keshif-configured dashboard (Yalcin, Elmqvist & Bederson 2016).

Initial analysis of this representation of the textbook contents revealed 17 "insights" regarding the data set, in the form of correlations among the categories of metadata. The textbook designer was then invited to comment on these insights, and was subsequently interviewed. Insights were classified as: 1) Intentional correlation: Insights that are consistent with the author's didactic intentions; 2) Tacit correlation: Insights that the author acknowledged, but had not incorporated intentionally; 3) Not relevant: Insights that were deemed not relevant to the author's intentions. We now present examples of each of these categories, along with the designer's reaction. Metadata categories appear in *italics*.

1) Intentional correlation: Nearly all the tasks that were perceived as suitable for *opening a topic* did not include an explicit *symbolic representation* (i.e. algebraic) of a function. Author's reaction: "This is the definition, generally speaking, of an opening task – to arrive at the symbolic from sensing, complex problems which one can think about, non-mathematical information etc". 2) Tacit correlation: In the "derivative" chapter, students are rarely expected to provide *non-technological justifications* for their answers. Author's reaction: "This is a logical implication of two other principles: derivative requires a lot of symbolic work, and when working symbolically you cannot rely solely on technology". 3) Not relevant: *Symbolic representation* is especially common in tasks that invite students to *draw conclusions*. The author's reaction: "I am not sure why this phenomenon shows up in the data… Maybe this attests to the type of conclusions [the taggers] aspire to. For example – what does a conclusion from a *graphic representation* without *symbolic representation* look like? That [kind of activity] is probably not represented in this collection."

We have focused on how tagging didactic metadata can contribute to research on textbook analysis, and have shown that the categories of didactic metadata can assist in revealing characteristics of datasets of curricular materials. Furthermore, these insights can serve as an object for discussion with designers to elicit their didactic intentions, and with teachers to elicit their interpretation of the material. Investigation of individual teachers' interactions with textbooks are ongoing and will be reported elsewhere. We note that there are limitations to the method we have employed: taggers and designers might not share meanings for metadata keywords, small scale tagging may be tagger-dependant, and the represented characteristics are limited to what can be tagged.

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SYMPOSIUM B

DEDUCTIVE REASONING, ARGUING AND PROOF IN TEXTBOOKS

ORGANISED BY LUISA RODRIGUEZ DOERING AND CYDARA CAVEDON RIPOLL

THE EUCLIDEAN DIVISION IN THE EARLY GRADES

LUISA RODRÍGUEZ DOERING, JANETE JACINTA CARRER SOPPELSA, CYDARA CAVEDON RIPOLL

Abstract

In this article we aim to highlight the importance of the remainder in the Euclidean division starting at the first grades of elementary school. We present questions that help the reader to reflect on the meaning of the Euclidean division, in which the remainder plays a fundamental role. Taking into account these questions, the guidelines of the official documents and the analysis of textbooks of the initial grades, we report on the topics of reasoning, discussion and proof in textbooks, suggested in Symposium B (Deductive Reasoning, Argument and Proof in Textbooks) and we propose activities that we believe will help to promote the understanding of the Euclidean division in its entire scope.

Key words: Division in the first school grades. Euclidean division. Remainder.

Introduction

The ideas of division between natural numbers appear in the early grades of elementary school. At this level, division is a complex operation for the student, since, "compared to other elementary operations, division with natural numbers is different in the following sense. While in addition, subtraction and multiplication we have two input values and obtain only one third output value, which is the result of the operation, the division with naturals involves two values as a result, namely, the quotient and the remainder." (Ripoll, Rangel & Giraldo 2016, p. 104)

This first consideration shows that division is a complex operation for the student in the early grades and points to the particular attention that the teacher should devote to the introduction of the Euclidean division, as well as to its follow-ups in the 6th grade, and taking into account aspects that are often ignored in the classroom, as well as in textbooks.

In this article we invite the teacher to reflect on the Euclidean division and the important role of the remainder, considering the essential aspects of this operation, as follows.

- Is it natural to begin the discussion concerning the division of natural numbers exclusively with dividends that are multiples of the divisor?
- Are the zero remainder situations really the divisions that happen most frequently in the student's daily life?
- Does not the division in **N** (the natural numbers) have a life that is "independent" of multiplication, unlike what happens between addition and subtraction?

Based upon this reflection and the analysis of some textbooks, we present some suggestions for activities which, we believe, could help to promote the understanding of the division operation.

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Thus, in the following,

- we present our reflections on the orientations in official documents concerning the division operation;
- we describe how the Euclidean division is introduced in some textbooks of the early grades, reporting the situation on the theme concerning reasoning, arguing and proof in the textbooks --- suggested in Symposium B (Deductive Reasoning, Arguing and Proof in Textbooks) --- and make some comments on it;
- aiming to contribute to the student's understanding of such an operation in its entirety, we suggest some activities to approach the Euclidean division in the early grades.

What the official documents say about the division of natural numbers

The proposals presented in the *Parâmetros Curriculares Nacionais*[1](PCN) are open and flexible, not being an homogeneous and imposing curricular model.

The idea of sharing is present in children's experiences from an early age on, for example, in sharing candies with their siblings. The PCN themselves recognize that students bring knowledge and ideas built upon their daily experiences into the school and get to the classroom with some knowledge about the division. This document suggests that the teacher should take advantage of such experiences and make use of manipulative resources such as chips, sticks, grains, etc. to introduce and deepen the study of this operation (Brasil 1, 1997, p. 25). Regarding operations, PCN states that the work to be done should focus on the understanding of the different meanings of each one of them, on the relations between them and on the reflexive study of the calculus, contemplating different types - approximate or not, mental and written and the objectives for the first cycle reinforce the resolution of problem situations in order to give the student the perception of the meaning of the fundamental operations, besides recognizing that the same operation is related to different problems and that the same problem can be solved by using different operations. (Brasil 1, 1997, p. 39)

The document emphasizes that the addition and subtraction operations should be emphasized in the first cycle of Elementary Education and that multiplication and division calculations should be carried out through personal strategies, not offering more details about the latter operations.

In the PCN we find that "as in the case of addition and subtraction, it is important to work together on problems that exploit multiplication and division, since there are close connections between the situations that involve them and the need to work upon these operations (...)" (Brasil 1, 1997, p. 72). It is suggested that, starting with multiplication situations, it is possible to formulate situations that involve division, reinforcing the close relationship between the two.

The *Base Nacional Comum Curricular*[2] (BNCC) contains the objects of knowledge and the skills intended for the children and young people in each stage of Basic Education. In the version currently available (http://basenacionalcomum.mec.gov.br/images/BNCCpublicacao.pdf), BNCC agrees with the PCN considering important that the teaching and learning of operations in the initial grades of primary education should be supported in situations of interest of the students, associating questions of reality to those that involve the world of fantasy, play or games that justify the realization of some calculation. Also in this document the expectation, with respect to natural numbers, is that students, at the end of the initial stage, "solve problems with natural numbers involving the different meanings of operations, argue and justify the procedures used for the resolution and assess the plausibility of the results found. Regarding the calculations, students are

[1] *Parâmetros Curriculares Nacionais* (PCN) are the Brazilian standards for the first nine grades of Elementary School.

[2] *Base Nacional Comum Curricular* establishes the minimum curriculum that will soon guide every Brazilian school.

expected to develop different strategies for obtaining the results, especially by estimation and mental calculation, as well as algorithms and calculators.” (Brasil 2, p. 224)

Specifically, about working with the division operation, it is suggested that it should start in the 2nd grade, having as expected ability to “solve and elaborate problems involving double, half, triple and third part, with the support of images or manipulative material, using personal strategies” (p. 239). For the 3rd grade it is suggested that the different meanings of the division be worked out (division into equal parts and measurement), seeking to attain the ability “to solve and to elaborate problems of division of a natural number by another one (up to 10), with remainder zero and with remainder different from zero, with the meanings of equitable distribution and measurement, through strategies and personal registers.” Here we notice an allusion to the Euclidean division, although it was not highlighted among the Objects of Knowledge of the 3rd grade. When suggesting for the 4th grade to reach the ability “to solve and to elaborate problems of division whose divisor has a maximum of two figures, involving the meanings of equitable distribution and of measurement, using several strategies, like estimate by calculation, mental calculation and algorithms” (p. 247) no mention is made of the problems involving exclusively an Euclidean division in its resolution, but whose response is neither the quotient nor the remainder, much less its approach is suggested. For the 4th grade, the document also requires that the student should “recognize, through investigations, that there are groups of natural numbers for which divisions by a given number result in equal remainders, identifying regularities” (p. 246), and that “the relations between multiplication and division are worked out in the 4th grade with the purpose of leading the student to recognize, through investigations, using the calculator when necessary, the inverse relations between addition and subtraction operations and multiplication and division, to apply them in problem solving.” (p. 247). It certainly seems that only the particular case of Euclidean division with remainder equal to zero is suggested, thus allowing multiplication and division to be interpreted by the student as inverse operations.

What do the textbooks of the early grades say about division with natural numbers

We analyzed six collections of textbooks of the initial grades of Elementary School (Table 1), all approved in the *Programa Nacional do Livro Didático*[3] (PNLD). In this analysis we observed how the division is introduced and how the remainder of a Euclidean division is treated, as well as the type of activities proposed.

We noticed that, in all the analyzed collections, the division is introduced in the second grade, after multiplication, with both meanings: splitting in equal parts and measure.

Collection	PNLD
Projeto Buriti: Matemática	2013
A Conquista da Matemática	2013
Matemática: Pode contar comigo	2010
Coleção Ápis – Matemática	2013
Coleção Aprender - Muito Prazer	2013
Coleção Construindo o Conhecimento	2007

Table 1: Analysed collections

With the exception of one of the analyzed collections, the division always appears with a dividend that is multiple of the divisor, with no mention to the remainder in the first examples. Although it is

[3] The *Programa Nacional do Livro Didático* (PNLD) is Brazil´s textbook assessment program, which includes mathematics and selects the textbooks that are freely distributed by the Brazilian Ministry of Education.

clear that there is a close and undeniable relation between multiplication and division, the student may be led to think that the division also has only one output, the quotient.
It is visible in these textbooks the absence of a convention for the word “division” with regard to “splitting in equal parts”. It should be noted that, for a 2nd grade student, the splitting in equal parts is not always natural, so at the very least the term “division” requires a reflection, as in the situation suggested by Figure 1.

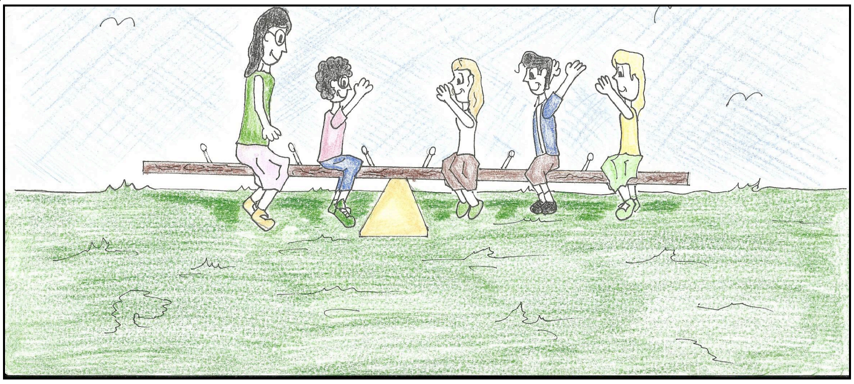

Figure 1. Source: archives of the authors

We reassure, still regarding the little emphasis given to the convention of the term division, that not only the term implies “equal parts” but also implies “remainder < divisor”, an essential condition to be considered in the construction of algorithms for division. In none of the reviewed books we found reference to the fact that, for example, although both equalities $17 = 5 \times 2 + 7$ and $17 = 5 \times 3 + 2$ are true, only the second one comes from what is defined as division in **N** (Euclidean division). In the analyses we could find only one textbook, which emphasized that the quotient is the largest multiple of the divisor that is smaller than the dividend. In all other textbooks the necessary conditions for a process of splitting to be called division are not emphasized, and therefore a crucial element of mathematical thinking is neglected.
Many authors of the analyzed textbooks, right after introducing the division and giving examples involving exclusively zero remainder, end up exploiting multiplication and division as inverse operations, some even including a section entitled “Multiplication and Division: Inverse Operations” and emphasizing that “it happens with multiplication and division the same that happens with addition and subtraction.” In addition to the fact that this statement is not true in the numerical universe **N**, it goes against the orientation of the official documents cited in this text concerning the preference that should be given at this level to contextualized examples. Moreover this phrase suggests to the student that in all divisions in **N**, the remainder is always equal to zero.
Figure 2 shows a situation that illustrates that the non-zero remainder in a Euclidean division is part of the daily life of the child. In other words, **the Euclidean division is the only one that makes sense in the numerical universe N, and very often the remainder has a preponderant role in the problem being considered.**

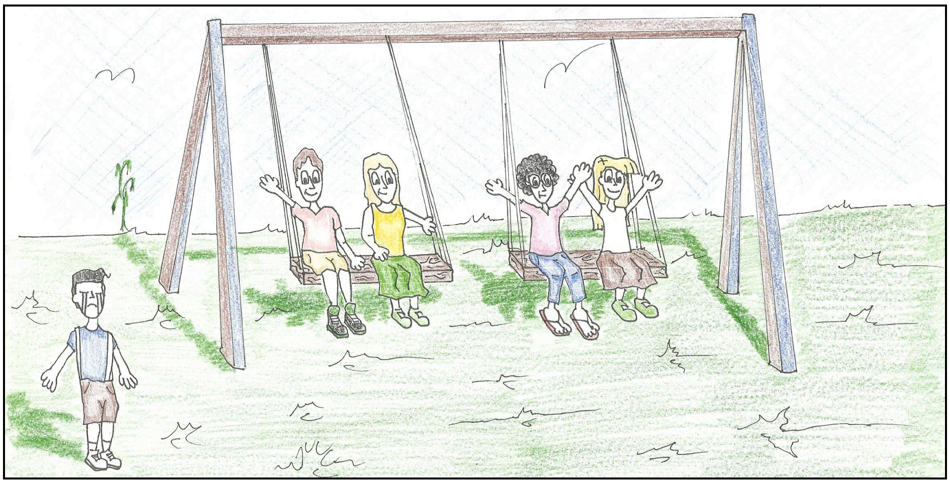

Figure 2 . Source: archives of the authors

Soppelsa (Soppelsa, 2016) in her analysis of the textbooks of the 6th grade remarks that, in most of them, the meaning of the remainder in an Euclidean division has been given no importance or emphasis, receiving only the status of leftover. Also, most of the exercises require only direct computations, focusing almost exclusively on the quotient.

In our analysis of textbooks, we could reaffirm Sooppelsa's findings, and in only one of the analyzed collections we could find some exercises that are solved making use of a division but for which the answer is neither the quotient nor the remainder, but the interpretation of this result.

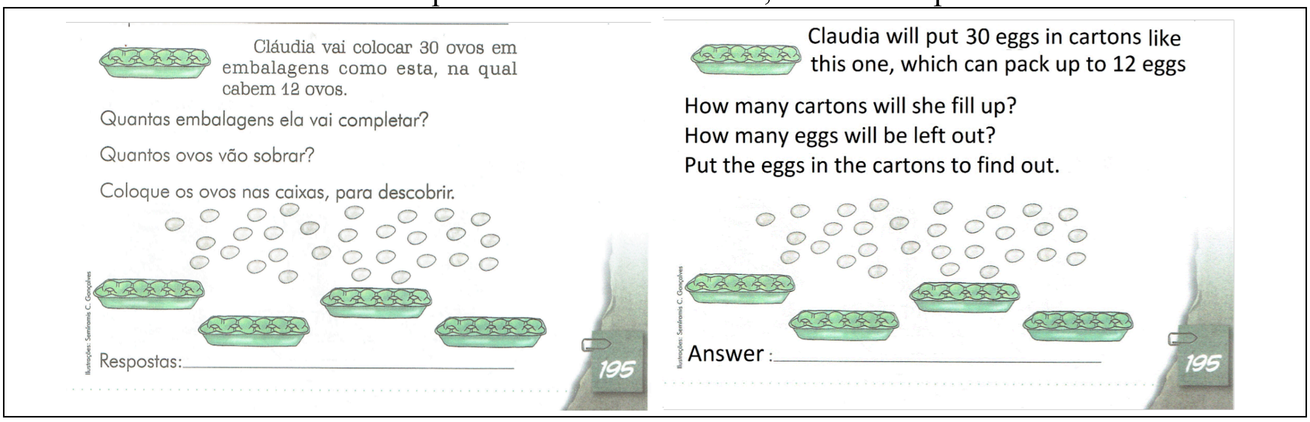

Figure 3: on the left, an excerpt from Isolani et al. (2005a), p. 195 and its translation on the right

We could also find little emphasis on pictorial representation in the calculation of divisions with nonzero remainder. It should be noted that such representation can really help the student, both in the visualization of the calculation (see, for example, Figure 3) and in the establishment of the inverse process, which not only allows to verify if the operation was performed correctly but also generalizes generic thinking, leading the student to perceive the relationship between the terms of division,

dividend = divisor × quotient + remainder,

and to justify its validity with words. Moreover, the above relationship suggests that division in **N** has an "independent" life of multiplication, unlike what happens with addition and subtraction, which are inverse operations of one another.

Some suggestions for the teaching of Euclidean division

Taking into account the issues previously discussed and the textbooks analysis carried out, and also intending to stimulate the mathematical thinking of first grades' students, we suggest that teachers offer their students opportunities to explore:

- awareness of the need to establish a convention for the ideas associated with division, including

i) the terms “splitting”, “repartition” and “distribution” do not imply “equal parts” (as shown in Figure 1, with respect to the number of persons);
ii) splitting in equal parts in **N** does not imply remainder equal to zero (as in Figure 2);
iii) splitting in equal parts does not imply leftover smaller than the divisor (as in Figure 4 and in Activity B);

- sensibilization to the universal convention for the expression “division in **N**” or Euclidean division, including

iv) equal parts (as in Figure 1, with regard to the distribution of the weight of the persons);
v) remainder smaller than the divisor (as in Activity B, below, including questions (a) and (b), aiming to emphasize this condition), including a discussion of the equivalence between “smaller than the divisor” and “the greatest quotient possible”;
vi) the fundamental relation dividend = divisor × quotient + remainder (inspired by, for example, Figure 3 and Activity B);
vii) uniqueness of the quotient and the remainder satisfying the necessary conditions for a process of splitting to be called division (as in Activity B).

A suggestion of activities

In the following we suggest some activities to approach the Euclidean division in the early grades, aiming to contribute to the student's global understanding of this operation.

Activity A: Observe Figure 4 and answer: how many children will have to wait for the next trip on the train?

Figure 4 . Source: archive of the authors

In order to motivate the definition of the Euclidean division as the splitting in equal parts that generates the largest quotient and the smallest remainder, we suggest the following activity, inspired and adapted from exercise 2, p.103 of Isolani et al (2005b).

Activity B: Peter has 38 candies. He would like to offer an equal amount of candies to each of his 7 friends. Complete the table below to assist him with all the possibilities of distribution.

Candies distribution	Number of distributed candies	Amount of leftovers
1 candy for each of the 7 friends		
2 candies for each of the 7 friends		
3 candies for each of the 7 friends		
4 candies for each of the 7 friends		
5 candies for each of the 7		

friends		
6 candies for each of the 7 friends		
7 candies for each of the 7 friends		
8 candies for each of the 7 friends		

Table 2: Activity B

This activity allows a comparison between the various distributions in 7 equal parts. The teacher can encourage the students to record in the second and third columns, in addition to the values, the operations performed to obtain them (see Table 3).

Candies distribution	Number of distributed candies	Amount of leftovers
1 candy for each of the 7 friends	$7 \times 1 = 7$	$31 = 38 - 7 = 38 - 7 \times 1$

Table 3: Analysis activity B

We propose next some questions that could guide the ideas of the Euclidean division and which contemplate the items listed in the previous section.

(a) What is the largest number of candies that each friend can receive? Is there any leftover in this distribution? If yes, how many candies is the amount of leftovers?

(b) Which distribution corresponds to the smallest amount of leftovers?

(c) Is it possible to distribute 6 candies to each of the 7 friends? Why?

Item (a) allows the recognition of the distribution that guarantees the largest number of candies per friend (5 candies for each friend); item (b) allows the observation that this distribution is also the one that has the smallest amount of leftovers (3 candies) and that is the only one in which the number of leftovers is smaller than the number of friends. Item (c) seeks to emphasize that, in order to distribute one more candy to each friend, we should have at least $42 = 7 \times 6$ candies, which exceed the 38 candies that Peter has.

The discussion of the above items should lead the students to conclude that the distribution with largest quotient and smaller remainder is unique, thus receiving a special denomination: (Euclidean) division of 38 by 7, the value 5 being called the quotient, 3 the remainder, 7 the divisor and 38 the dividend.

This activity also allows retrieving the total number of candies from each of the rows of the table, as shown in the following table.

Candies distribution	Number of distributed candies	Amount of left overs	Recovering the total amount of candies
4 candies for each of the 7 friends	$7 \times 4 = 28$	$38 - 7 \times 4 = 38 - 28 = 10$	$7 \times 4 + 10 = 38$

Table 4: Recovering the total amount of candies

The complete table shows six ways to retrieve the total number of candies as a multiple of 7 plus one leftover. However, only the equality $38 = 7 \times 5 + 3$, coming from the fifth line of the complete table, corresponds to the Euclidean division of 38 by 7, equality that we suggest to be called the *real proof of the Euclidean division operation* (instead of "the fundamental relation of the division", found in some of the analyzed textbooks).

The following activity contemplates, in the same context and with the same division, situations in which the solution is sometimes the quotient, sometimes the remainder, and sometimes it is neither the quotient nor the remainder. One should also notice the non-unique answer to item (b).

Activity C: There will be a field trip for the 123 students of the 4^{th} grade students and their 7 teachers. The school intends to rent buses with capacity for 55 passengers.
(a) If everyone confirms to take part in the trip, how many buses should be rented?
(b) How many unconfirmed persons are necessary for all buses to be full?
(c) In this case, how many buses should be rented?
The following activity contemplates, in the same problem, different strategies (Euclidean division or multiplication), and can be solved with different operations as well as aiming to contemplate one of the abilities of BNCC already mentioned.
Activity D: Teacher Carlota has 123 marbles to be distributed among 6 groups of students.
(a) Is it possible for the groups to get the same amount of marbles? How can we find it out?
(b) How many more marbles should the teacher have so that all of them were distributed among the groups?
The following activity contemplates the meaning of measure of the division and also a situation for which the result is neither the quotient nor the remainder of the Euclidean division and can serve to stimulate mental strategy.
Activity E: Juliano gave R$ 35,00 to his 9 nephews to buy a snack, but everyone wanted a popsicle. Since each popsicle costs R$ 3,00, one wonders
(a) Is the money enough for each nephew to buy a popsicle?
In the affirmative case:
(b) Was there any money left?
(c) How much was left?
(d) Would the money be enough as well to buy popsicles for Uncle Juliano and Aunt Mary?

Final considerations

In this work we reflect on the Euclidean division in the early grades of Elementary School, discussing this theme from the official documents and the analysis of six collections of textbooks that were approved in PNLD Program.
The analysis of the textbooks showed that, in many instances, the guidelines of the official documents were not followed. It also showed that the content "division in **N**", although naturally appearing in the students' daily life, is not treated in this way in textbooks, and the convention for the term "division" is not properly emphasized, namely, splitting in equal parts with as little leftover as possible.
Believing that the nonzero remainder must be approached since the first contacts of the student with the concept of division, we present proposals of activities that contemplate and emphasize the different roles of the remainder and encourage the argumentation and mathematical thinking, aspects which are also suggested in the official documents.

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AREA FORMULA DEDUCTIONS FOR PLANE FIGURES IN TEXTBOOKS

FRANCIELE MARCIANE MEINERZ AND LUISA RODRÍGUEZ DOERING

ABSTRACT

In this paper we present an analysis on the deduction of formulas for calculating areas of plane figures in some selected textbooks. Our book analysis is guided by the questions suggested by Symposium B: Deductive Reasoning, Arguing and Proof in Textbooks. The analysis showed that some books only present the formulas for calculating areas of plane geometric figures, immediately after some numerical example, without any kind of justification, while the others present inconsistencies and incomplete arguments for the proofs. We offer some suggestions of activities and complements to the analyzed texts that may contribute to the development of the deductions presented by the authors, and an excerpt of a successful teaching experiment, where students developed arguments to deduce the formulas for calculating areas of plane figures using composition and decomposition of simpler plane figures.

KEYWORDS: Mathematical Argumentation; Textbook; Area of plane figures.

Introduction

In the International Congress on Mathematics Education ICME 13 there were reports that in many countries an emphasis on proof is reappearing in mathematics curricula.

> "There is international recognition of the importance of reasoning and proof in students' learning of mathematics at all levels of education, and of the difficulties met by students and teachers in this area. Indeed, many students face difficulties with reasoning about mathematical ideas and constructing or understanding mathematical arguments that meet the standard of proof. Teachers also face difficulties with reasoning and proof, and existing curriculum materials tend to offer inadequate support for classroom work in this area." (Thematic Study Group 18 Reasoning and Proof in Mathematics Education – ICME 13, July 2016)

Therefore, it is important that textbooks support this tendency. Stylianides (2009) remind us the relevance of textbooks and their analysis in order to develop activities of reasoning-and-proving (RP) in the classroom.

> "The studies that examined how mathematics textbooks influence mathematics instruction used different methodological techniques and offered different kinds of evidence, but the bottom line of these studies was that mathematics textbooks have significant influence on students' opportunities to learn mathematics in many classrooms [...] Although mathematics textbooks can play an important role in the opportunities that students have to engage in RP, to date, we lack knowledge of how RP is promoted in contemporary mathematics textbook series."

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Since 1996, Brazil has the *Programa Nacional do Livro Didático*[1] (PNLD) that selects and distributes, each year, textbooks to public schools; therefore, frequently, public school teachers have in their classroom a mathematics textbook that has probably not been chosen by them, but it is the only one available to their students. So it is also important to verify if the books approved by the PNLD include adequate support for the development of the mathematical argumentation and reasoning.

According to the *Parâmetros Curriculares Nacionais*[2] (PCN), geometry "is a fertile field of problem situations that favors the development of the capacity to argue and construct demonstrations" (Brasil 1998); consequently, we investigate how textbooks deal with one of the first "proofs" of basic geometry: the deductions of the formulas for calculating the area of some plane figures. We analyze only textbooks accepted by PNLD. In the books we analyze, we are interested in answering some of the questions suggested in Symposium B (*Deductive Reasoning, Arguing and Proof in Textbooks*): Do textbooks include contexts, contents, results that meet the standard of proof? If yes, is the arguing appropriate for students at the proposed level? Is it correct?

We begin this article presenting and commenting briefly on how the argumentation and teaching of geometry appear in the official documents, as well as justifying our choice of this theme through other researches. In the sequence we present an analysis of how the deductions of the formulas for the calculation of area of plane figures appear in some textbooks accepted by the PNLD, permeated by suggestions of activities, or complements to the text that could contribute to the development of the deductions presented in these books. Afterwards, we present an excerpt of the Meinerz (2015) undergraduate final paper, which was successful in a teaching experiment where students developed arguments to deduce the formulas for calculating areas of plane figures using composition and decomposition of some plane figures.

Geometry, argumentation and official documents

In recent years, the importance of mathematical argumentation in the classroom has gained prominence in Brazil, being present in important documents about mathematics teaching, such as the PCN and the *Base Nacional Comum Curricular*[3] (BNCC). It is one of the general objectives for teaching mathematics in Elementary Education that students are presented with situations that favour the process of "accurately describing, representing and presenting results and arguing about their conjectures, using oral language and establishing relations between it and different mathematical representations" (Brasil 1997). Also stating the importance of mathematical argumentation in the classroom, BNCC defines that mathematics teaching should propose an understanding of the world and social practices,

> "qualifying the insertion in the world of work, which needs to be sustained by the capacity for argumentation, security to deal with problems and challenges of diverse origins. Therefore, it is fundamental that teaching be contextualized and interdisciplinary, while at the same time pursuing the development of the capacity to abstract, to perceive what can be generalized to other contexts, to use the capacity for imagination" (Brasil 2015).

Moreover, the PCN mentions that mathematics has the role of arousing curiosity and instigating the ability to generalize, project, predict and abstract, favouring the development of logical reasoning. The importance given to the development of reasoning is directly connected with justification and argumentation, and such concepts are related to the idea of learning mathematics with understanding, and not simply by memorizing formulas or methods.

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[1] The *Programa Nacional do Livro Didático* (PNLD) is Brazil´s textbooks assessment program, which includes mathematics and selects the textbooks that are freely distributed by the Brazilian Ministry of Education.

[2] Parâmetros Curriculares Nacionais are the Brazilian standards for the first nine years of Elementary School.

[3] *Base Nacional Comum Curricular* establishes the minimum curriculum that will soon guide all Brazilian schools.

Despite all the importance given to the mathematical argumentation in the classroom in the official documents, studies show that the involvement of students in activities related to the development of argumentation are not usual in classrooms (Nunes & Almoloud 2013). In fact, in addition to the students' not being involved in activities related to argumentation, students also often do not have the opportunity to be involved in the teaching of geometry, which boils down to the teaching of measures, however "it is a fact that geometric questions tend to arouse the interest of adolescents and young people in a natural and spontaneous way. In addition, it is a fertile field of problem situations that favors the development of the ability to argue and build demonstrations." (Brasil 1998). According to the PCN

> "Geometry has had little prominence in mathematics classes, and its teaching is often confused with that of measures. In spite of its abandonment, it plays a fundamental role in the curriculum, in that it enables the student to develop a particular type of thought to understand, describe and represent, in an organized way, the world in which he lives. It is also a fact that geometric questions tend to arouse the interest of adolescents and young people in a natural and spontaneous way. In addition, it is a fertile field of problem situations that favors the development of the ability to argue and build demonstrations." (Brasil 1998)

Teachers sometimes fail to use classroom demonstrations and argumentation because they do not feel comfortable with formal mathematical language, but it is important to emphasize that in school, when it comes to developing mathematical thinking with the student, often the most important is not the demonstration itself, but rather the process of recognizing that an argument made for a particular case can be generalized, and then making use of generic thinking to express such a generalization, i.e., making use of a representative of a generic element of a set, and not of particular elements or cases, all in oral or written language, using mathematical symbology or not. (Hanna 1995). We think it is important for students to be encouraged to routinely argue in math classes, turning it into a natural process and that wherever possible, the teacher should choose the so-called demonstrations that explain rather than just demonstrate. (Hanna 1995). The rigor of mathematical representation (formal mathematical language) must be gradually worked out with the student over the years (D'Amore 2007).

According to BNCC, one of the objectives of geometry knowledge in elementary schools is for students to be able to observe the equivalence of area of plane figures, calculating "areas of figures that can be decomposed into others, whose areas can be easily determined, such as triangles and quadrilaterals." (Brasil, 2017). Still, it is said that developing the idea of equivalence helps students in the development of mathematical thinking.

Geometry is not prioritized in basic education. Problems involving figures or physical space "tend to be approached in the numerical or algebraic pathways, with the abandonment of procedures more proper to geometric thinking. The teaching of geometry, when it occurs, is reduced to the calculation of angles, lengths, areas and volumes through the application of formulas that are not discovered or verified, and to the algebraic representation of the locus in the Cartesian plane" (Búrigo 2005).

Meinerz (2015) carried out a mathematical investigation in the classroom, in which the students were invited to develop arguments about the formulas for calculating areas of plane figures using the composition and decomposition of these figures. In this work it was possible to notice an evolution in the students' argumentations. Initially the students presented very little elaborated justifications, and usually orally. As students were being encouraged to justify their statements, they began to present written arguments and to form fairly complete arguments.

Textbooks analysis

In this section we analyze how the formulas for the area calculation of some plane figures are presented in some textbooks, restricting our attention to the classic figures that appear in all textbooks: rectangles, squares, triangles, parallelograms, trapezoids and lozenges. We observe that this content appears in books of different levels, namely, of the 6th, 7th, and 8th years. We then selected 5 textbooks (Table 1) approved in the 2011 and 2017 PNLD, among them Matemática Bianchini and Projeto Teláris, which are in the list of the PNLD most distributed books in Brazil. Our interest is to answer some questions suggested by symposium B, mentioned in the introduction, and for this we look at how these books present the formulas for calculating the area of the selected plane figures: whether there is any kind of deduction, or whether there are only given examples, followed by the formulas; if there happens to be a deduction, we verify if it is sufficient, if it covers all cases, if it is correct, and if the language used is adequate and coherent. Throughout our analysis we present complements for some arguments, aiming to complete and correct them. In addition, we offer suggestions for activities that illustrate the need for certain assumptions in some of these deductions.

Titles	Authors	PNLD
Matemática Bianchini 6	Bianchini	2017
Matemática Bianchini 7	Bianchini	2017
Projeto Teláris Matemática 6	Dante	2017
Projeto Araribá: Matemática 8º ano	Leonardo	2017
Matemática e Realidade 7	Iezzi, Dolce e Machado	2011

Table 1: Books approved by PNLD and analyzed in the research.

In our analysis we present the formulas for geometrical figures in the following order: rectangle, square, triangle, parallelogram, trapezoid and lozenge, which is the order in almost all analyzed books.

Rectangle and square

Luiz Roberto Dante, in his 6th year book of the Teláris Collection, shows a rectangle with 5 cm of base and 3 cm of height divided into squares of 1 cm² and states that if we count the units of area we will have 15 squares, thus an area of 15 cm². He still makes a relation with multiplication, asking the reader to note that 5 × 3 = 15. However, the author does not explicit that we have five columns, with three squares in each one, which would facilitate the generalization of the formula. The author generalizes directly, just stating that it is possible to calculate the area of any rectangle by multiplying the measures of the base and the height.

Bianchini, in his 6th year book, presents a rectangle of dimensions 7 cm by 2 cm, makes the relation with the rows and columns, as can be seen in Figure 1 and generalizes directly. It is important to emphasize that we verified the corresponding teacher's manual, where we also could not find any justification for the generalization of the formula of the area of the rectangle. We note that in the decomposition of the rectangle, different nomenclatures appear for the same definitions: the author names the dimensions of "rectangular region length" and "width of that region", but when he writes the rectangular region area formula, he uses the terms "base" and "height measurement", which had not appeared previously at any point in the development of the example - and confuse the students.

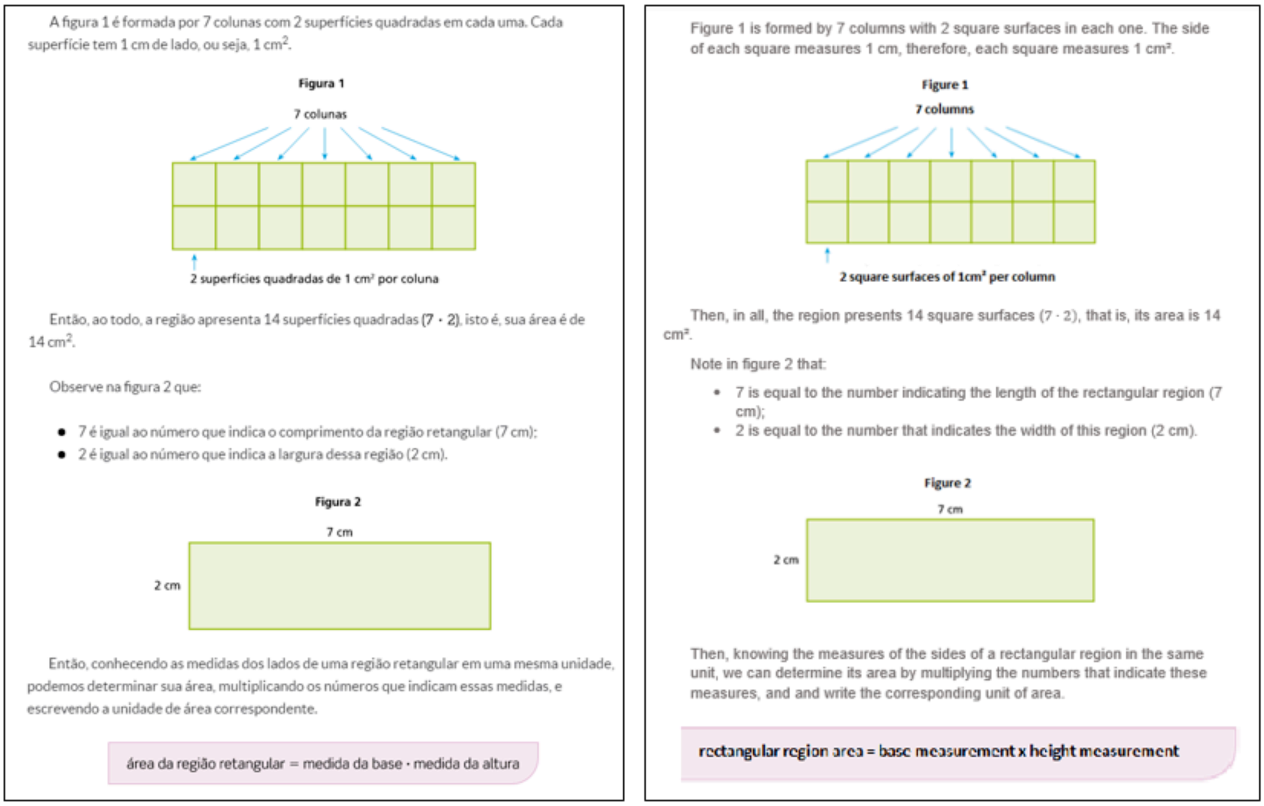

A figura 1 é formada por 7 colunas com 2 superfícies quadradas em cada uma. Cada superfície tem 1 cm de lado, ou seja, 1 cm^2.

Figura 1

Então, ao todo, a região apresenta 14 superfícies quadradas (7 · 2), isto é, sua área é de 14 cm^2.

Observe na figura 2 que:

- 7 é igual ao número que indica o comprimento da região retangular (7 cm);
- 2 é igual ao número que indica a largura dessa região (2 cm).

Figura 2

Então, conhecendo as medidas dos lados de uma região retangular em uma mesma unidade, podemos determinar sua área, multiplicando os números que indicam essas medidas, e escrevendo a unidade de área correspondente.

área da região retangular = medida da base · medida da altura

Figure 1 is formed by 7 columns with 2 square surfaces in each one. The side of each square measures 1 cm, therefore, each square measures 1 cm².

Figure 1

Then, in all, the region presents 14 square surfaces (7 · 2), that is, its area is 14 cm².

Note in figure 2 that:

- 7 is equal to the number indicating the length of the rectangular region (7 cm);
- 2 is equal to the number that indicates the width of this region (2 cm).

Figure 2

Then, knowing the measures of the sides of a rectangular region in the same unit, we can determine its area by multiplying the numbers that indicate these measures, and and write the corresponding unit of area.

rectangular region area = base measurement x height measurement

Figure 1: on the left, an excerpt from Bianchini- 6ºano (p. 296) and its translation on the right

In the book of Projeto Araribá for the 8th year, the author presents an example of a rectangle, measuring its area by counting squares, without explicitly explaining the relation of multiplication with rows and columns. We verified that in the 6th year book, where the content of area of rectangles and squares is introduced, there is also no mention of the relation of the multiplication of rows and columns with the area calculation. This book only gives an example and states that one can calculate the area of a rectangle multiplying the measure of the base by the height measurement. In the book *Matemática e Realidade* for the 7th year, the author first recalls how we can calculate the area of a rectangle and a square, stating that this subject was already worked out in the 6th year. It shows a rectangle with a grid pattern, with base of 4 cm and height measuring 3 cm, and states that its area is 12 cm², because of the multiplication (3 cm x 4 cm = 12 cm²), but does not address the counting of squares. In the book of sixth year, the author addresses the counting of squares, but does not explicitly state the columns and lines to justify multiplication. After the example, it mentions that "For any rectangle, the area is the product of the measure of the base by the measurement of the height".

We believe that the authors could offer activities (at least in the teacher's manuals) to allow students and teachers to do the deductions together in the classroom, and then justify that a rectangle based on *b* units and height measuring *a* units can be divided into *b* columns, with one unit width, and in *a* rows, also with unit width. Thus, the rectangle would be formed by *b* columns, with *a* squares in each, and therefore by a total of *b* multiplied by *a* squares. Since this is an argument that refers to the idea of multiplication, a concept supposedly appropriated by the students at this stage, and fundamental for the deduction of the formula, it can't be omitted. In omitting this argument the authors lose the opportunity to develop the generalization of students' thinking. This argument is valid only for the integers and would be a good preface to later extend the formulas to rational and real measures, making use of the concept of these numbers.

All authors declare the square to be a particular case of the rectangle, all sides having the same measure, and the area is calculated by multiplying the measures of its sides. In the books of the

Projeto Araribá and Teláris Collection, the formula of the area is presented as a power, since both sides have the same measure: $l \times l = l^2$.

Parallelogram

The four authors transform the parallelogram into a rectangle through decomposition and composition of plane figures. In Figure 2, we can observe the decomposition made by Dante in the book of the Teláris Collection; in this case, it is interesting to observe that the author is concerned with emphasizing the right angle symbol in the picture to argue that we will indeed have a rectangle after the composition. We think that, in addition, it would be important to make it explicit in the text that the cut is perpendicular, so that students can understand the argument easier. Another important point we note is that the author is careful to write that when we translate a part of a plane region, the area of the figure does not change, that is, we get an equivalent figure. Bianchini presents an example of a parallelogram with numerical measures for its sides, and decomposes it, using a square grid. After that he generalizes, transforming a parallelogram with generic measures into a rectangle, showing the right angle in the triangle cut-out to be transported to the formation of the rectangle. The author argues that the two figures are equivalent and that it is always possible to transform a parallelogram into a rectangle of the same base and the same height, so one can calculate its area multiplying the measure of the base by the height measurement.

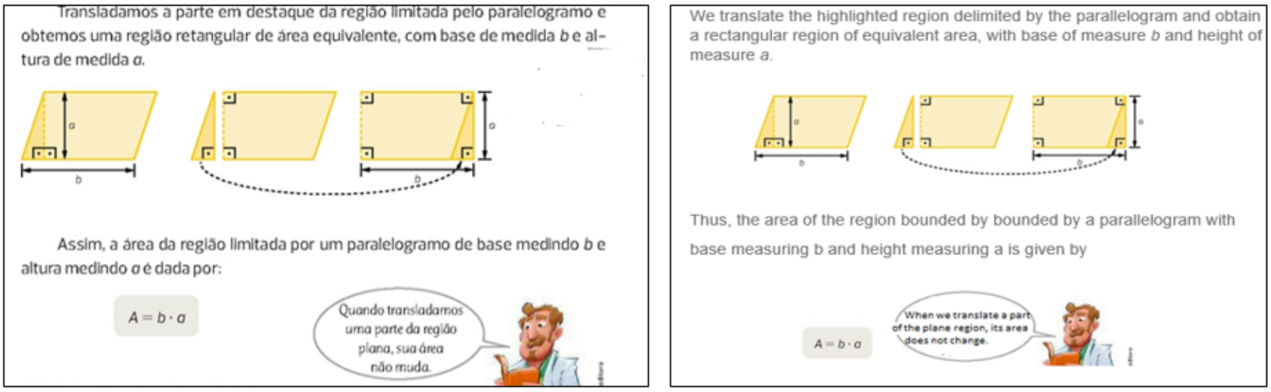


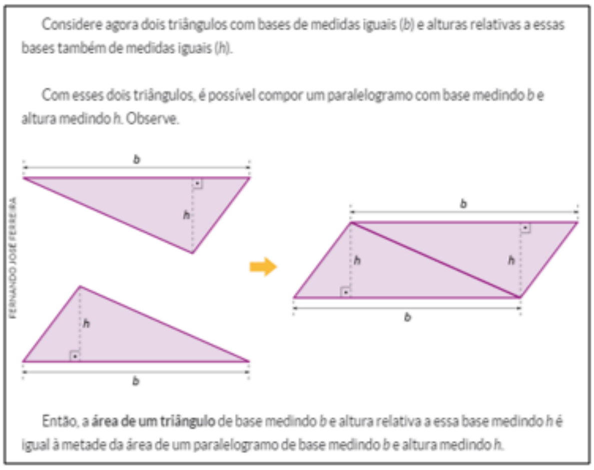
Figure 2: on the left, an excerpt from Teláris Collection- 6ºano (p.269) and its translation on the right

Unlike Dante and Bianchini, in the collections Matemática e Realidade and Projeto Araribá, although the figure is decomposed, it is not mentioned that the cut must be made perpendicular to the base and also the right angle symbol is not presented in the pictures, so it is not clear that the formed figure is, in fact, a rectangle. In addition, in both books, only numerical examples appear, followed by formulas, generalizing directly. We had access to the teacher's manual in the Mathematics and Reality collection, where no suggestion of deduction is made to the teacher.

Triangle

In calculating the area of a triangular region, Dante uses an acute triangle to form a parallelogram with measurements of the same base and the same height as the triangle. He states that the triangle has half the area of this parallelogram without arguing that this is due to the fact that the parallelogram is formed by two identical triangles, and therefore the area of one of them is obtained multiplying base by height, and then dividing by two. The author only presents the figure and formula for a right triangle and for an obtuse triangle. It is important to note that in the teacher's manual, the author only states the same as in the student's book: that the triangle has half the area of the parallelogram.

Bianchini presents the deduction for the formula for calculating the area of the triangle (Figure 3), starting from two triangles with equal base and height measures, relative to these bases, also with equal measures, that is, two equivalent triangles. But it uses implicitly, in the sketch, as well as in the argument, that they are congruent triangles. And he concludes that with these two triangles "it is possible to compose a parallelogram". This may lead the students to think that two triangles with the same base and the same height are always congruent, which is not true. Also, if the triangles are not congruent, they do not form a parallelogram. In the teacher's manual there is no additional comment besides what is presented in the student textbook.



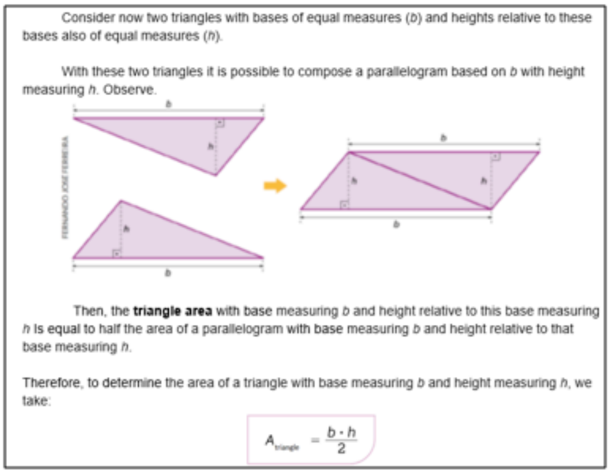


Figure 3: on the left, an excerpt from *Bianchini- 7ºano (p. 307)* and its translation on the right

In the Matemática e Realidade book, the authors present an interesting example in which they assemble a parallelogram duplicating a triangle with base measuring 5 cm and height measuring 3 cm, and argue that the parallelogram is formed by two equal triangles, and therefore, the area of one of the triangles is obtained by dividing the area of the parallelogram in two. Again, the authors do not include a generic figure, they merely state that the area of the triangle can always be calculated through the product of the base by height, divided by two. Thus, the authors state, by presenting only one example, that the same thing always happens, but they do not justify it. In the Araribá Collection, the authors present a deduction that is easy for students to understand, with generic dimensions and good argumentation, but above a square grid, which may compromise the generality of the argumentation.

In order to complement these deductions, we suggest two activities that can be developed with the students and that lead to the justification of the deduction of the formula for the calculation of the area of the triangle. The first one is directed to an observation we made at our Bianchini analysis: to form a parallelogram using two triangles, it is necessary for these two triangles be congruent, that is, it is not enough for them to have the same measure of the base and the same height measurement. Thus, we suggest that teachers invite students to form parallelograms using two triangles of the same base and same height, but not congruent. In Figure 4, we present several examples of triangles of the same base and height, which are not congruent.

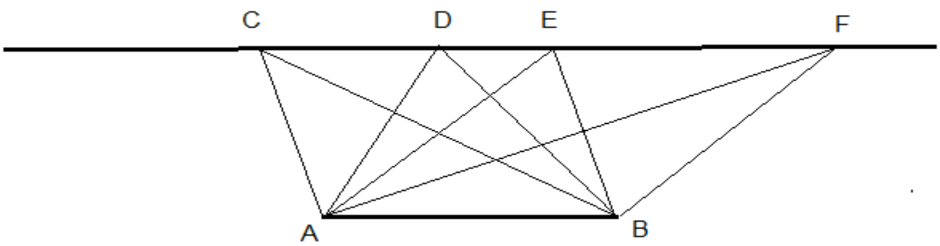


Figure 3: Examples of triangles of the same base and height, which are not congruent.

Once we have verified that triangles of the same base and height, but not congruent, do not necessarily form a parallelogram, the students should be invited to test the composition of a parallelogram with two congruent triangles. Students will realize that they will always get a parallelogram, regardless of the congruent side chosen. In addition, when using two congruent scalene triangles, it is possible to observe that we can form three parallelograms, with different bases and heights, but of the same area.

Trapezoid

Dante duplicates the trapezoid and states that whenever we duplicate it, we may use the two identical trapezoids to form a parallelogram with base measuring $B + b$ (larger base added to the smaller base). By multiplying the trapezoid's base by its height, the area of the parallelogram is obtained, and because they are two equal regions, the area of the trapezoid is obtained by dividing the area of the parallelogram by two: [illegible]. Bianchini and the authors of the Araribá Project argue in the same way as Dante.

In the book *Matemática e Realidade* the authors deduce the formula using a numerical example: decomposition of the trapezoid into two triangles through a cut along one of its diagonals. One of the triangles obtained has as a measure of the base the largest base of the trapezoid and the other the smaller base of the trapezoid and both have the same height. To conclude the calculation they state that the trapezoid's area is obtained by summing the areas of the two triangles. It is important to point out that there is a typo or revision error in the example: when the authors make an observation about the "major" and "minor" bases of the trapezoid it is written "diagonal maior" and "diagonal menor", however these two measures were not used to set the example. Also, in the conclusion, which was made without argument, the authors mention the arithmetic mean, which has not been mentioned before, and which could cause difficulties for the students, besides being an unnecessary use of this nomenclature in the case described.

Lozenge

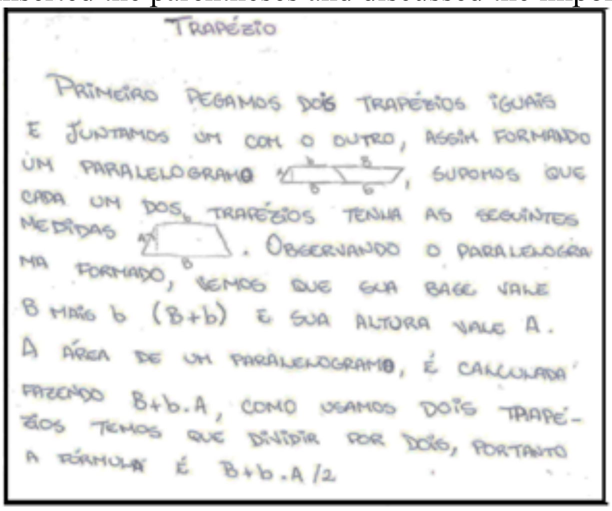
The last plane figure analyzed is the lozenge. The argument presented in the Telaris Collection (Dante), consists of decomposing a lozenge into four triangles, obtained with cuts along the diagonals, and joining the triangles with another identical lozenge to form a rectangle with base measuring the same as smaller diagonal and height measuring the same as larger diagonal. Thus, since two lozenges were used, the area of one of the lozenges is obtained multiplying the greater diagonal (height of the formed rectangle) by the smaller diagonal (base of the formed rectangle) and dividing by two. An important point that the author does not clarify, neither through the right angle symbol nor through the explanation in the text, is that the triangles obtained are right triangles, which is fundamental to obtain the rectangle that leads to the area of the lozenge.

Unlike Dante, Bianchini makes explicit both in the text and in the drawing that the triangles are right. The rest of the argument is similar to that of Dante, and encourages the generalization of his deduction. In the book *Matemática e Realidade*, again there appears only a numerical example, which consists of dividing the lozenge into four equal right triangles, and defines that the area of the lozenge is the sum of the areas of the four triangles. Again the generalization is made without any argument. In the Projeto Araribá, the authors decompose the lozenge and transform it into a rectangle. The rectangle they form is based on the smaller diagonal of the lozenge and its height is the larger diagonal, divided in two. Therefore, it is possible to note that the area of the lozenge is obtained multiplying the greater diagonal by the smaller diagonal, and then dividing by two. The author arguments with generic measures but, again, on a square grid.

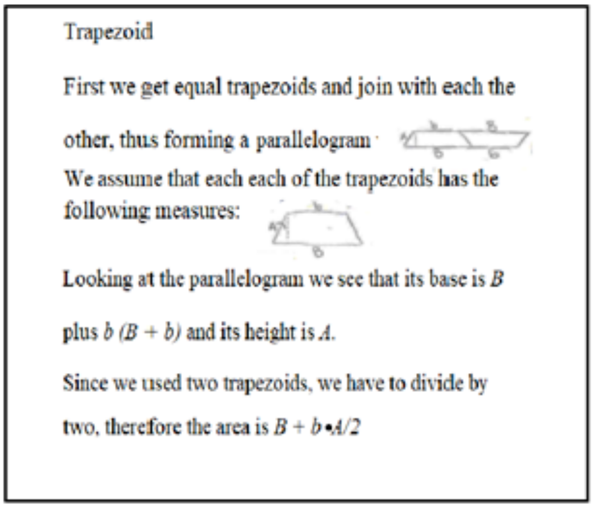
Example of an activity

To illustrate the use of argumentation in the classroom we present a description of a teaching experience with students of the 8th grade of a public school in Porto Alegre conduct by first author in her undergraduate final paper Meinerz (2015), where twenty-nine students were divided into trios or duos. During this research students were encouraged to create conjectures and argue about them using the composition and the decomposition of plane geometric figures to obtain formulas for the calculation of their areas. The practice was carried out in four meetings: in the first meeting the researcher applied an initial questionnaire to evaluate the students' previous knowledge. On the second and third meeting, the groups were invited to conjecture and argue about their conjectures. Each group received cut-outs of the figures to be worked on coloured paper (rectangle, triangle, parallelogram, trapezoid and lozenge) and they were invited to compose these figures with other equal figures, or decompose and compose another figure whose area formula they already knew; the activity was finished with an argument on the deduction of the new formula obtained. In the final meeting, the researcher proposed an activity in which the groups should formulate deductions and argue about them, and then present them to their colleagues with a poster. Each group was responsible for a plane geometric figure.

The argumentation presented by one of the groups participating in the research was developed in a succinct and organized way in the document delivered to the researcher as well as in the poster presented to the colleagues. We observed that the group presented their arguments using natural language, of easy comprehension for colleagues (Figure 5). As the students' algebraic language was still somewhat precarious, we observed that the students did not use the parentheses to indicate the multiplication of $(B + b)$ A. Their attention was drawn to this, and for the presentation they inserted the parentheses and discussed the importance of its use.


TRAPÉZIO

PRIMEIRO PEGAMOS DOIS TRAPÉZIOS IGUAIS E JUNTAMOS UM COM O OUTRO, ASSIM FORMANDO UM PARALELOGRAMO, SUPOMOS QUE CADA UM DOS TRAPÉZIOS TENHA AS SEGUINTES MEDIDAS. OBSERVANDO O PARALELOGRAMA FORMADO, VEMOS QUE SUA BASE VALE B MAIS b (B+b) E SUA ALTURA VALE A. A ÁREA DE UM PARALELOGRAMO, É CALCULADA FAZENDO B+b.A, COMO USAMOS DOIS TRAPÉZIOS TEMOS QUE DIVIDIR POR DOIS, PORTANTO A FÓRMULA É B+b.A/2


Trapezoid

First we get equal trapezoids and join with each the other, thus forming a parallelogram

We assume that each each of the trapezoids has the following measures:

Looking at the parallelogram we see that its base is B plus b $(B + b)$ and its height is A.

Since we used two trapezoids, we have to divide by two, therefore the area is $B + b \bullet A/2$

Figure 4: Argument presented by the students on the deduction of the formula for calculating the trapezoid area and its translation.

Final Considerations

In this work we show that the official documents (PCN and BNCC), as well as several researchers in the area, point to the importance of the teaching of geometry and the development of argumentation in the classroom. We have also pointed out that geometry is a fertile and propitious field for the introduction of argumentation in the classroom. Within plane figures content the argumentation can be facilitated by the use of composition and decomposition of figures, allowing students the opportunity to conjecture, test examples and generalize more easily, since the study can initially be concrete, rather than abstract.

Through the analysis of textbooks approved by the PNLD, we find that some books only present the formulas for calculating areas of plane geometric figures immediately after some numerical example, without any kind of justification, persuasion or argumentation. In this way, they lose the opportunities to generalize the ideas used in examples, or in particular cases, that could lead to the deduction of the formulas. We observe that books that develop arguments to present area formulas often present inconsistencies or inaccuracies, such as incomplete arguments, different nomenclatures, and language beyond the reach of the students, which can confuse the students. We also note that in the teacher's manuals there are no proposals to encourage students in the classroom to develop their own justifications and deductions.

During our analysis we presented suggestions to complement the arguments, using justifications that we think to be appropriate for the students and that validate the argumentation. We believe that in order to encourage classroom argumentation, we need textbooks that present proposals that involve students and encourage them to conjecture, to justify their conjectures, and to deduce them. We also believe that the teacher's manual should present deductions of formulas for the calculation of areas of plane figures in a more complete way, and without inconsistencies, with careful writing and with the suggestion of activities that complement the deductions in the teacher's manual.
We conclude by recalling that "the most important challenge to mathematics educators in the context of proof is to enhance its role in the classroom by finding more effective ways of using it as a vehicle to promote mathematical understanding" (Hanna 1995). We stress that the demonstrations can be carried out without the use of mathematical symbolism, so that there is a greater understanding and learning of the students.

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TEACHER VIEWS ABOUT ARGUMENTATION AND MATHEMATICAL PROOF IN SCHOOL

LILIAN NASSER AND CARLOS AUGUSTO AGUILAR JÚNIOR

Abstract

This article relates a research about argumentation and proof in the mathematics classroom and the teacher views on the challenge of developing in students the ability to argue and prove in mathematics. The study, conducted in two stages, intended to find out how teachers appreciate and accept the various levels and types of argumentation presented by secondary school students. First, a form with tasks, requiring logical-deductive reasoning, was given to students. In the second stage, some selected responses from this form have been categorized according to the types of proof established by Balacheff (1988) and by Harel and Sowder (1998) to assemble the form applied to teachers, following a methodology undertaken by Hoyles (1997), in U.K. In this form, teachers evaluated the selected answers, justifying the marks given. Results show that teachers have great inclination for arguments approaching to the formal proof, rejecting incipient and ingenuous attempts of argumentation. We also bring to the discussion the importance of textbooks for the development of a culture of argumentation and mathematical proof in the classroom, considering that this teaching material is one of the main technologies used in class, if not the only one available.

Introduction

This research has been carried through in the scope of a master degree in mathematics teaching, whose object of research and analysis was the approach of argumentation and proof in the mathematics classroom at grades 8^{th} and 9^{th}. We intended to promote a discussion about the importance and the challenge of developing, in the classroom, activities that stimulated in the pupils the construction of abilities in arguing and proving in mathematics. Such construction dialogues with the proposal to develop the deductive reasoning at school, printed in the Brazilian curricular parameters (PCN, in Portuguese): “mathematics intervenes strongly with the formation of intellectual capacities, the structuring of the thought and the development of pupil’s deductive reasoning” (Brasil 1997, p. 15).

We believe that this proposal only becomes viable and successful if teachers are, in fact, engaged in this kind of approach, which needs to be constructed from diversified activities that stimulate students’ deductive reasoning. This must be reached through the resolution of problems that mobilize deductive thinking, going beyond the mere mechanist work of problem solving and answering questions just aiming to the setting of results (theorems, proposals, properties and formulas), by means of repetitive exercises.

Nowadays, we live in Brazil a process of curricular centralization, which intends to unify, in national scope, the curricular contents to be taught in the whole school network. Although the criticism, which is out of the target of this article, on the process of construction of the National Common Curricular Basis (BNCC), foreseen to invigorate from 2019, we detach the quarrel

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contemplated in the proposal of the BNCC on the need to develop the deductive reasoning in the mathematics teaching-learning process, when establishing that

> the student must be motivated, in his schooling lifetime, to question, to formulate, to test and to validate hypotheses, to search counter examples, to model situations, to verify the adequacy of the answer given to a problem, to develop languages and, as a consequence, to construct ways of thinking that lead him to reflect and to act in a critical way about the questions faced daily. (Brasil 2016, p. 131)

Moved by these ideas, and also considering the central role of the teacher in the development of abilities and competencies in the field of mathematical reasoning, argumentation and proof, the inquiry was undertaken in two distinct, but complementary steps. The investigation intended to determine how teachers appreciate and accept the various levels and types of argumentation presented by secondary school students (Fundamental Level II - final series and high school) to questions that explored logical-deductive reasoning (Aguilar Júnior 2012; Aguilar Júnior & Nasser 2014).

At the first stage of the inquiry, a form has been designed, with questions demanding from the participant students a deeper logical-deductive reasoning, through argumentation and justification. The questions went beyond the simple application of known results and calculations and have been applied in three public schools for 124 students of 8th and 9th years of secondary school, from 12 to 17 years of age. The conclusion was that the level of argumentation of this group is still naive and informal. The great majority of the given answers presented empirical character, since the verification of the truth was based on examples, what Balacheff (1988) calls naive empiricism (which was freely translated as natural empiricism in the research report).

For the second stage, another form has been constructed, directed to the teachers. It was composed by selected answers presented by the students at the first stage, which had been analyzed according to the models and types of proof defined by Balacheff (1988) and Harel & Sowder (1998). This second form was submitted to the evaluation of 59 teachers of secondary school, following a methodology undertaken by Hoyles (1997), in U.K.

In this questionnaire, each teacher participant of the inquiry evaluated the answers given by the pupils, attributing grades from 0 to 10 and justifying the marks given. In the analyses carried through, it could be verified that teachers have great inclination for the arguments approaching to formal proof. Incipient and ingenuous proposals of argumentation had not been so well valued, and the pragmatic answers (Balacheff 1988) had been considered unacceptable as proofs, under the point of view of the mathematical rigor. Data analysis, that also took into account 10 undergraduate students, who took part of the application of the form in a workshop, indicates that the teachers in this group, in general, are not inclined to promote the development of activities in the classroom, in order to construct abilities to argument and to prove in mathematics. Their preference, in terms of evaluation of student answers, is for arguments closer to the Balacheff (1988) model of proof, concerning the rigorous and formal proof, supported, accepted and practiced in the Academy.

The research pointed that pupils better undertake their arguments and succeed in proving the mathematical proposals more efficiently when they are faced with activities, curriculum and teacher prepared for the construction of this ability/competency. This way, it is relevant that the teacher stimulates the contact with this kind of activities.

Within the scope of public policies focused on Brazilian secondary school, the National Textbook Plan (PNLD, in Portuguese) allocates textbooks of the various curricular components in the public schools of the municipalities, states and the Union. These textbooks are previously selected by a group of specialists, and the teachers have the power of choice. Every student at the public network receives the textbooks chosen for free. Although there may be criticisms about formulations in some textbooks, these are the main tools for the teachers work in the classroom, important reference material for students and essential for building national identity, as already pointed out in the literature (Lajolo 1996; Ribeiro 2003; Megid & Fracalanza 2003; Horikawa & Jardilino 2010)

Beyond bringing this discussion to the core of the teacher's role in the construction of pedagogical practices and didactic sequences that allow the foment and the development of the students logical-deductive thinking, we understand that an important debate to be carried through is about the role of the textbook in this proposal, since it acts as the main support material for the teacher in the preparation of the lessons.

Textbooks can help the teachers, since they act as guides for the educational practice. But, even in the cases where the reasoning and the argumentation are emphasized in mathematics textbooks, we do not believe that the majority of the Brazilian teachers adopt this type of activity in his/her practice.

In this article, we focus on the issues carried through in the scope of the research of Aguilar Júnior (2012), on contributions from literature about the role of the textbook in the construction of a pedagogical proposal that foments the development of the logical-deductive reasoning through argumentation and proof and on some considerations about the textbook.

The research with teachers and students about mathematical argumentation and proof

The research, as argued in the introduction, aimed to understand how the mathematics teacher at secondary school evaluates the students reasoning, when submitted to questions that demand argumentation and the construction of demonstrations (proofs) of mathematical statements. For this, we carried out a literature revision, to define a method to analyze the students' answers. The work is, then, based on the theoretical framework proposed by Balacheff (1988) and by Harel and Sowder (1998).

Sowder and Harel (1998, pp. 671-673), in a research carried through in the United States, describe the kinds of proof presented by pupils in a test, classified as: externally based proof schemes, empirical proof schemes and analytical proof schemes. The externally based proof scheme considers "*both what convinces the student and what the student would offer to persuade others reside in some outside source"*. On the other hand, the empirical proof scheme is described as one where *"justifications are made solely on the basis of examples*". Concerning the analytical proof scheme, the researchers remark that Mathematics teachers "*probably regard the analytic proof schemes as giving the ultimate types of justifications in mathematics"*.

Balacheff (1988), in the scope of his PhD thesis, carried through a research with French students, identifying two basic types of proof: the pragmatic type and the conceptual type. For Balacheff (1988, p. 217), a pragmatic proof appeals to validity tests, search for regularities, examples or drawings to justify one statement, called by the author as "action resources", whereas the conceptual proof does not appeal to such resources at the moment to formulate the involved properties and the possible relations between them. According to Pietropaolo (2005, p. 94), the cognitive structures, the knowledge and the language used establish the differences between these kinds of proof.

In order to better design his research, Balacheff (1988) distinguish four modalities of proof: naive empiricism, crucial experiment, generic example, and thought experiment. These four unfoldings originate from the existing movements between the kinds of proof: the naive empiricism and the crucial experiment rest in the field of the pragmatic proof, and the thought experiment inhabits in the field of the conceptual proof. The generic example transits between the two types, given that the generic example "consists in the explanation of the reasons that validate a property that shows a generality, making use of a particular representative" (Gravina 2001, p. 67).

After the reflection on the correlate literature, we could design the research, which was structuralized as follows:

Phase 1: elaboration, application and analysis of forms with questions requiring argumentation and proof, directed for the pupils of grades 8th and 9th;

Phase 2: elaboration, application and analysis of forms for the teachers, based on the results collected in phase 1;

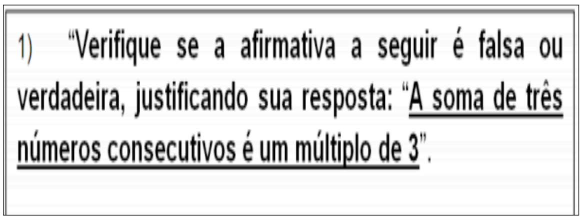
Phase 3: analysis and interpretation of the data.

In phase 1, reiterating what was said in the introduction, we undertook the application of the questionnaires in 3 schools, being two municipal schools (nominated by EM1 and EM2) and a federal one (called EF). The following table illustrates the distribution of the samples in each school unit, which had their names preserved.

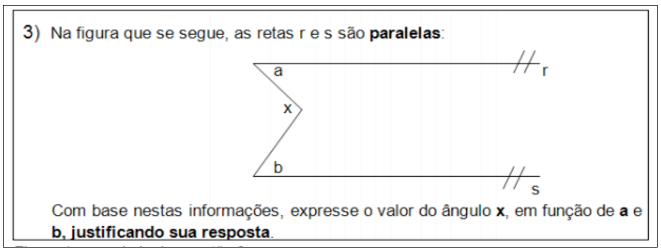
Table 1: distribution of the students in Phase 1 of the research

Grade / School	EM1	EM2	EF
8th grade	-	30 students (12 to 16 years old)	-
9th grade	38 students (14 to 16 years old)	28 students (14 to 17 years old)	28 students (14 to 15 years old)

In general, pupils concentrate in presenting answers based on examples or experiments, being attached to the question of testing some examples to verify the validity of the affirmation, inhabiting, thus, in the pragmatic type of proof, in a naïve empiricism (or ingenuous). However, some answers had been selected to compose the form destined to the teachers (Phase 2).


1) "Verifique se a afirmativa a seguir é falsa ou verdadeira, justificando sua resposta: "A soma de três números consecutivos é um múltiplo de 3".

(Translation: Verify if the following statement is TRUE or FALSE, justifying your answer: "the sum of three consecutive numbers is a multiple of 3.")



(Translation: In the following picture, r and s are parallel lines. Based on these informations, express the value of angle x, in terms of a and b, justifying your answer")

Figure 1: Examples of questions in Phase 1

Figure 1 shows two questions that had presented the most interesting answers for our debate, with a variety of kinds of proof. The first item explored the knowledge on arithmetical properties of the whole numbers (sum of three consecutive whole numbers results in a multiple of 3 - arithmetical context). Question 3 explored the knowledge about the parallel lines theorem (geometric context). The following pictures illustrate some of the answers included in the teachers form.

1) Verdadeiro, pois sempre que somamos três números consecutivos, se subtrairmos 1 do maior número e somarmos no menor, teremos três números iguais multiplicados por três.

ex: 1, 2, 3. $\boxed{1}\ \boxed{2}\ \boxed{3} = \boxed{2}+\boxed{2}+\boxed{2} = 3\times 2$

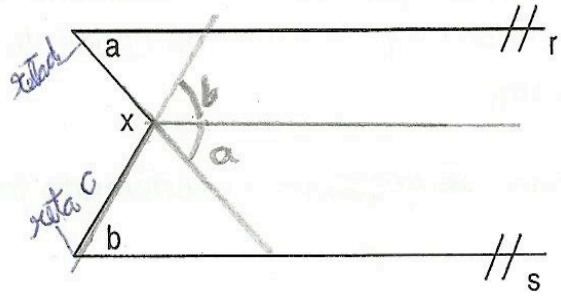
+1 −1

(Translation: "true, because whenever we add three consecutive numbers, if we subtract 1 from the biggest number and add it to the minor, we will have three equal numbers multiplied by three").

1) Verdadeiro. Por exemplo o nº 345 a soma de seus algarismos é um múltiplo de 3; 3+4+5 = 12 é múltiplo de 3

(Translation: "true. For example, the number 345, the addition of its algorisms is a multiple of 2, 3+4+5 = 12 is multiple of 3").

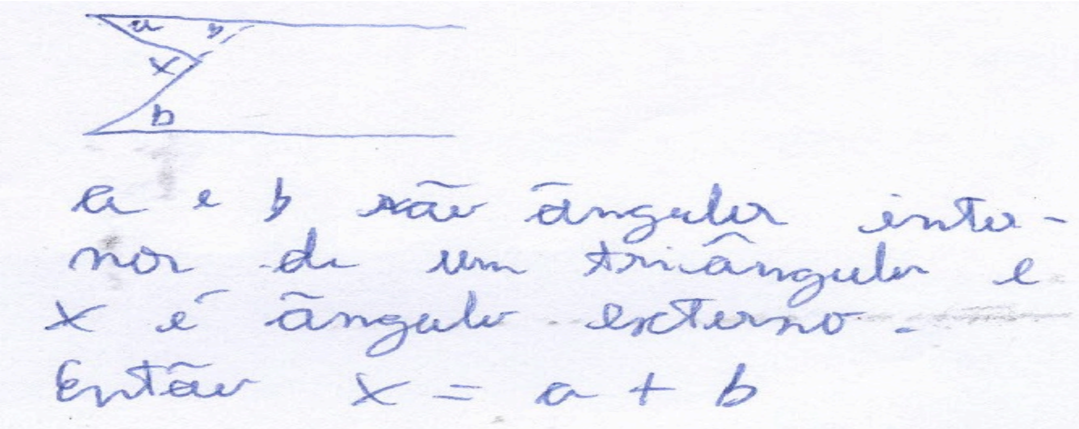
Figure 2: Students' answers to question 1 (arithmetic context)



prolongando as retas c e d e criando uma 3ª reta paralela a r e s, conseguimos transportar as medidas dos ângulos de modo que fiquem opostos pelo vértice a x, portanto $\boxed{x=a+b}$

Translation: "drawing out straight lines c and d and creating one 3rd parallel straight line to r and s, we can carry the measures of the angles such that they became opposite to x, then x = a + b").

Figure 3: Student's answer to question 3 (geometric context)



Translation: "a and b are internal angles of triangle and x is an external angle. Then x = a + b."

Figure 4: Student's answer to question 3 (geometric context)

In a first impression of the data collected, we evidence that the level of argumentation of this sample is still very ingenuous and informal, considering that a good parcel of the presented answers could be included in a category of empiricist character, where the search of the truth was given by means of examples.

We understand that the development of the ability to argue and to prove in Mathematics requires an intended work, that is, the lesson and the teacher must be prepared for the conduction of this task. By the results obtained, we can speculate that, concerning the pupils investigated in this phase of the research, this ability is not being developed in classroom, since great part of the pupils had presented only examples as arguments and justifications.

On the other hand, in the forms of Phase 2, we identified that the teachers had applied quite rigorous evaluations, although we assume they do not have the habit to work under the perspective of the development of proof and argumentation with their pupils. This occurred even concerning those answers that, besides not being a mathematical proof under the point of view of the rigorous deductive model of argumentation and demonstration in mathematics, presented a considerable level of argument and reasoning, as in the answers shown in figures 2 and 3.

We reaffirm that the work with argumentation and mathematical proof requires a change in the paradigm concerning the teacher posture in the mathematics classroom and, for this, beyond the need of contact between the teacher and this approach in his formation (initial and continued), the main material for the great majority of the teachers – the textbook – must be constituted of a repertoire that allows such work. In the following section, we will make a brief discussion of the literature referring to this subject.

Argumentation and proof in mathematics textbooks

The importance of the textbook is understood in the educational debates as a means to support the teaching-learning process of the knowledge systematized throughout the years. Although the speeches pointing that the textbook functions as a kind of "crutch" to carry the teaching work - passing a prejudiced and distorted idea, leading to the impression of a precarious and "limping" formation of the teacher - we understand that the textbook is an important tool for the socialization of the knowledge, historical and culturally discussed and developed in school curricula.

Some authors emphasize the importance and the wealth of the textbook for the contribution in the teaching-learning process and as an important mediator of the teaching work, as well as its good use on the part of the teacher. According to Nuñez et al (2003, p. 1), the textbook is seen as "*an alive source of wisdom, able to guide the development processes of the integral personality of the children.*" Although these researchers refer to science textbooks, we can also understand that the textbook, in general, is considered as a facilitator for the pupils learning, contributing in the access to the studied content. Lopes (2007) affirms that the textbook functions as a pedagogical material,

essential in the process of construction of the knowledge, being a cultural product, plain of ideological values, beyond its specific pedagogical content of each discipline (Lopes 2007, p. 205).
Teachers are responsible for the adequate use of textbooks, in order not to feed biased arguments, the book functioning as a crutch or a cane of the teacher. The protagonist of the education process is the teacher, and not the material/technology used in the lesson. Romanatto (1987, p. 8) affirms that the didactic book still has a marked presence in the classroom and, many times, as a substitute of the teacher, when it should be one more of the elements to support the teaching work. The contents and methods used by the teacher in classroom would be in the dependence of the contents and methods considered in the adopted textbook. Many factors have contributed for this role of the textbook as a protagonist in the classroom. A book that promises everything ready, everything detailed, being enough to order the pupil to open the page and to solve exercises, is an irresistible attraction. The textbook is neither a mere instrument as any another in the classroom, nor it is disappearing faced to the modern media. What is being questioned is its quality.
Lopes (2007) still recognizes the dependence of the teaching in relation to the textbook, stating that good books are a basic part of the quality of the education. In fact, reiterating that the adequate use of the book competes to the teacher, the textbook can bring, in terms of technology, important pedagogical functions for the process of schooling and of teaching and learning.
The debate that we raise aims at placing the textbook in a relevant position for the teaching-learning process. It is an educative technology, which the teacher adopts and manipulates with the objective of mediating the students learning. If we take, then, the argumentation and the mathematical proof as goals of education and learning, it would be important that the textbooks came constructed with activities that stimulated this type of work in our classrooms. This debate agrees with what is already pointed by the Textbook National Program with respect to high school - PNLEM -, when affirming that

> the textbook must value the various resources of the mathematical thought, as the imagination, the intuition, the inductive reasoning and the logical-deductive reasoning, the distinction between mathematical validation and empirical validation and foster the gradual construction of the deductive method in mathematics. Regarding the deductive method, it is advisable to warn frequent deviations to be moved away. The first one is to formulate a generalization as a proven fact, on the basis of the verification of examples - many times one or two only. Others are to present very complicated proofs of some theorems, which can be postponed for posterior studies, or to display difficult demonstrations for intuitively evident facts. Many times, such demonstrations can be skipped without damage of the understanding. (Brasil 2006, p. 75)

Lima and Freitas (2008), in a research undertaken to investigate how high school students dealt with some conjectures involving the set of the whole numbers, verified in textbooks recommended by the PNLEM the presence of activities stimulating this kind of work and reflection with the pupils, subsidizing the elaboration of didactic sequences towards the proposal of adopting the argumentation and mathematical proof at high school.
On the other hand, Nacarato et al. (2012) discuss the role of the mathematical proof, considering that the character of the textbook concerning the theme argumentation and mathematical proof is approaching to the debate in the field of mathematics education about the importance of this thematic. But the classroom practice is still attached to the idea of that to know mathematics is to know how to make calculations and to solve problems that just require the application of mathematical known results (theorems, formulas, proposals and properties), affirming that “the classroom is still attached to the culture of the exercise, with little room for discussion, exchanges and negotiations of meanings, raising of conjectures and their validation” (Nacarato et al. 2012, p. 73).
Given the deductive character of the Euclidean geometry, many studies on textbook analysis (Martins and Mandarino 2014; Nacarato et al. 2012; Rosa et al. 2013) centers the research in the geometry activities. Martins and Mandarino (2014) had analyzed the Brazilian Textbook Guide of

2011, with focus in the chapters related to geometry. They concluded that the 11 collections evaluated from the guide, in general, had presented activities and explanations that mix pragmatic and intellectual justifications, emphasizing the absence of activities exploring the inquiry process, essential and basic in the construction of the mathematical knowledge. In the same way, the research of Rosa et al. (2013) also focuses on the way that the thematic of argumentation and mathematical proofs is presented in the chapters of geometry of the evaluated books. In this inquiry, three books of 8th grade had been evaluated, evidencing, in all the volumes, variations of arguments, going from mere justifications using examples, appealing to the visualization through figures, to more formal constructions, under the point of view of the rigorous mathematical proof (deductive demonstration).

Some remarks

As already pointed in Aguilar Júnior (2012), we understand the importance of providing the students with the contact to activities that stimulate the development of the ability to argue and to prove in mathematics, creating contexts and problem-situations where they are led to conjecture, to refute, to test hypotheses, carrying through a movement of transition from the pragmatic to the conceptual proof, in the terms of Balacheff (1988).

The examples of answers obtained in the research with the pupils disclose great potentialities to work with the exploration of activities that lead pupils to think on the proposed questions, without necessarily having to appeal to calculation procedures to solve problems, but using deductive reasoning. This way, agreeing to Lima and Freitas (2008, 9) this pedagogical proposal must "become the classic demonstration with arguments that respect the individual epistemology reasonable", that is, it respects the previous knowledge of these students and the possible difficulties in language terms, mainly.

Despite the necessity of the school curriculum and the formation of teachers programs valorize the work with argumentation and mathematical proof, as long as the textbook is, even not the main, but an important educative technology, it is necessary that the textbook contemplates activities that allow teachers to stimulate the inquiry and the development of didactic sequences favorable to this intention.

Although the recommendations present in the PCN, PNLD and PNLEM and the proposal of the BNCC point the importance of the work directed to the development of the logical-deductive reasoning, necessary for the full exercise of the citizenship, we verify a lack of proposals of didactic sequences aiming at the work with argumentation and mathematical proof in the adopted textbooks in our classrooms.

The discussion in the field of the Mathematics Education has been fundamental to rethink the use of the textbook and its production. To deepen the research on this subject - and to understand argumentation and mathematical proof as curricular contents to be explored at school - is a powerful way to create the culture of the argumentation in our classrooms, essential for a full understanding of mathematics as a language that models the reality and that provides to students the full exercise of the citizenship.

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THE INTRODUCTION TO ALGEBRA IN TEXTBOOKS

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ABSTRACT

In this article, we analyze the introduction to algebraic expressions in Brazilian textbooks and in textbooks from other countries, taking into account the guidance on this theme in Brazilian official documents and focusing on what is suggested concerning reasoning, arguing and proof. Considering the guiding question of the Deductive Reasoning, Arguing and Proof in Textbooks Symposium, namely, "Do textbooks include contexts, contents, results that meet the standard of proof?", we conclude that the answer is in the negative. We present, nevertheless, an activity about algebraic expressions that can be used in an 8th grade class which can help the development of generic thinking and also naturally motivate the need to prove that two algebraic expressions are equivalent.

KEYWORDS: Textbook comparison. Mathematical thinking. Generic reasoning. Introduction to algebra. Generalized arithmetic.

Introduction

Algebra in elementary school is closely related to the development of students' abstraction ability, and, hence, also to the ability of constructing simple proofs in mathematics. For example, in making use of symbols that represent numbers, it is possible to establish arithmetical results and justify them by making use of generic reasoning, that is, by means of arguments that make use of generic elements of some set.

It is here emphasized that a student's first contact with algebraic expressions should be through symbols representing numbers (which is called *generalized arithmetic* by Usiskin (1999)), agreeing with Vergnaud (1997) and Brazilian official documents as well as Wu (2016), who wrote in the first page of his book *Teaching School Mathematics – Algebra:* "The purpose of this chapter [Symbolic Expressions] is to demonstrate how one can do algebra by taking *x* to be just a number and turn at least the introductory part of school algebra into *generalized arithmetic,* literally."

We analyse the introduction of algebraic expressions in Brazilian textbooks, as well as in textbooks from other countries, considering guidance about this theme in official Brazilian documents. We also pay attention to the question suggested in Symposium B (Deductive Reasoning, Arguing and Proof in Textbooks) of the II International Conference on Mathematics Textbook Research and Development: *Does the textbook help in the development of students' abstraction abilities?*

Finally, we describe an activity about algebraic expressions which aimed at the development of students' abstraction ability.

THE INTRODUCTION TO ALGEBRA IN OFFICIAL BRAZILIAN DOCUMENTS FOR ELEMENTARY SCHOOL

The official document that currently guides the first nine years of primary education in Brazil (elementary school) regarding contents and methodologies is the *Parâmetros Curriculares Nacionais* (PCN) from 1998. In 2020, another document, the *Base Nacional Comum Curricular*

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(BNCC), will become effective in all Brazilian elementary schools. Unlike the PCN, which only guides the teaching and learning process, the BNCC defines the set of knowledge and skills that all Brazilian students should develop in each year of the first nine school years.

With respect to the teaching of algebra, in both the PCN (Brasil 1, 1998) and the BNCC (Brasil 2, 2017), the orientation is clear: letters in algebraic expressions should always represent numbers. The following excerpts confirm this:

> "Although in the initial grades some ideas of algebra may already be developed, it is especially in the final grades of elementary school that the first algebraic activities will appear. By exploring problems, the student should recognize different functions of algebra (generalizing arithmetic patterns, establishing relationships between two quantities, modelling, solving arithmetically difficult problems) and solve problems by making use of equations and inequalities (differentiating the concepts of parameters, variables and unknowns, and coming into contact with formulas)." (Brasil 1, 1998, p. 50-51, authors' translation).

> "At this level, students must understand the different meanings of numerical variables in an expression, establish a generalization of a property, investigate the regularity of a numerical sequence, indicate an unknown value in an algebraic sentence and establish the variation between two quantities." (Brasil 2, 2017, p. 226, authors' translation).

The emphasis given by the authors on this guidance will be clear in this article.

Vergnaud also agrees that the use of symbols that represent numbers should be a student's first contact with algebra (a level called *generalized arithmetic* by Usiskin 1999). In fact, Vergnaud (1997) comments that, with the study of algebra, student's arithmetic understanding is improved and that algebra in *collège* should be treated as a "numerical algebra" (*algèbre du numérique*). He also points out that the understanding of the main additive and multiplicative relationships that can be found in elementary arithmetic problems is not yet complete for students at the beginning of the *Lycée* (Vergnaud 1997, p. 122).

Algebra in elementary school has a strong relationship with the development of mathematical thinking. With respect to algebraic thinking, one can use symbols representing elements of some numerical set to establish or express the results of arithmetic and justify them by means of a generalizing reasoning; later, one can use symbols to express generalizations of arguments, reasoning with generic elements of a given number set, and not of particular cases, which, in this text, we call *generic reasoning*.

> "Algebra allows us to describe concise and coherent general relations (...). It is an essential tool for proving properties, describing patterns and solving problems. (...) For this, abstraction is necessary. It frees us from contexts and particular cases." (Martinez 2013, p.16, authors' translation)

The relation between algebra and the development of students' abstraction capacity can be found both in the PCN and the BNCC:

> "The study of algebra is a very significant theme for students to develop and exercise their capacity for abstraction and generalization, as well as to enable them to acquire a powerful tool for problem solving." (Brasil 1, 1998, p. 115, authors' translation).

> "Teenagers develop their abstract thinking skills in a significant way when they are given diverse experiences involving algebraic notions in the early stages, in an informal setting, and in work that is coupled with notions of Arithmetic." (Brasil 1, 1998, p.117, authors' translation)

> "The deduction of some properties and the validation of conjectures from others can be stimulated, especially at the end of elementary school." (Brasil 2, 2017, p. 221authors' translation).

> "... this field [algebra] must emphasize language development, the establishment of generalizations, the analysis of the interdependence of quantities, and problem solving through equations or inequalities." (Brasil 2, 2017, p. 226, authors' translation).

In the previous excerpts, one can notice the emphasis on mathematical thinking, which involves generalizing, decontextualizing, conjecturing, proving and abstract thinking.

THE INTRODUCTION TO ALGEBRA IN BRAZILIAN TEXTBOOKS FOR ELEMENTARY SCHOOL

Carvalho (2010) analyses nine 8th grade textbooks approved in the Programa Nacional do Livro Didático[41] (PNLD, Brasil 3, 2017) from the years of 2002 and 2008 on the theme "Introduction to algebraic expressions" (Table 1).

Table 1: Analysis of nine textbooks in Carvalho (2010)

Textbook (TB) → / Problem ↓	TB1	TB2	TB3	TB4	TB5	TB6	TB7	TB8	TB9
Absence of definition						X	X		X
Imprecise definition	X	X	X	X	X			X	
Biggest universe of the coefficients is $\mathbb{Z}$		X	X			X			X
Biggest universe of the coefficients is $\mathbb{Q}$	X			X	X		X	X	
Only polynomial algebraic expressions are used	X	X	X	X		X	X	X	X
No information about the measure units in polygonal geometric figures	X	X	X	X	X	X	X	X	X
The universe of the variables is not clear	X	X	X	X	X	X	X	X	X
Examples and exercises that do not contribute to learning		X	X		X		X	X	

In an analogous way, we analyse eight collections approved in the PNLD in 2017 (seven books of the 8th grade and one book of the 7th grade) (Table 2).

Table 2: Analysis of eight textbooks approved in PNLD 2017

Textbook (TB) → / Problem ↓	TB10	TB11	TB12	TB13	TB14	TB15	TB16	TB17
Absence of definition	X							X
Imprecise definition		X	X	X	X	X	X	

[41]The Programa Nacional do Livro Didático (PNLD) is Brazil´s textbook-assessment program, which includes mathematics and selects the textbooks that are freely distributed by the Brazilian Ministry of Education.

Biggest universe of the coefficients is $\mathbb{Z}$								
Biggest universe of the coefficients is $\mathbb{Q}$	X	X	X	X	X	X	X	X
Only polynomial algebraic expressions are used		X		X		X		
No information about the measure units in polygonal geometric figures	X	X	X	X	X	X	X	X
The universe of the variables is not clear	X		X	X		X	X	
Examples and exercises that do not contribute to learning		X	X	X		X	X	
Inadequate nomenclature “polynomial”	X	X	X	X	X	X	X	X
Unique factorization/complete factorization		X	X	X	X	X	X	X
Euclidean division of polynomials			X					
Unnecessary rules for computing with algebraic expressions			X	X	X	X	X	X

In both analyses, we made some remarks, among other things, on conflicts with some of the PCN guidelines, for example, regarding the development of students’ capacity for abstraction and generalization. More details of the analyses performed are given in the following paragraphs.

The definition of an algebraic expression

In the 2010 analysis (Table 1), three textbooks present no definition of *algebraic expression* at all. The same occurs in two textbooks analysed in 2017 (Table 2). In these books, only a few examples are given, followed by the phrase “expressions of these types are algebraic expressions”. All other analysed textbooks include imprecise definitions, such as, “Expressions which include mathematical operations and contain letters and numbers are called algebraic or literal expressions” (Figure 1, authors’ translation). We observe that this assertion allows the expression 2^x, $\ln(x^2)$, $\cos(xy)$ to be called an algebraic expression. Also, in collection number 17, an imprecise definition is presented in the seventh-grade textbook.

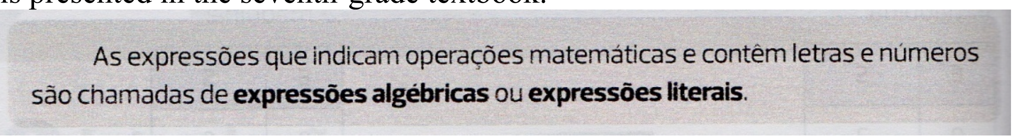

As expressões que indicam operações matemáticas e contêm letras e números são chamadas de **expressões algébricas** ou **expressões literais**.

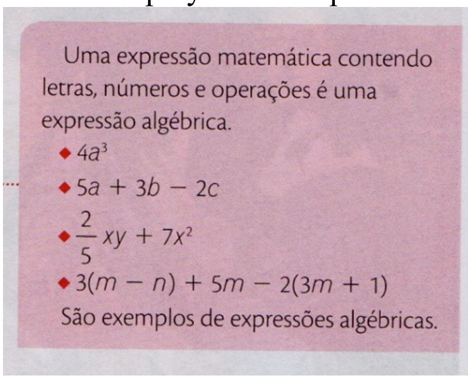
Figure 1: A definition of algebraic expressions, which was found in a textbook (PNLD 2017) and allows 2^x, $\ln(x^2)$, $\cos(xy)$ to be an algebraic expression.

First examples and exercises with algebraic expressions

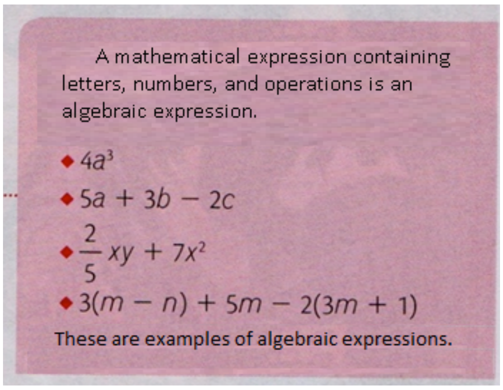
In the 2010 analysis (Table 1), four textbooks use only integer coefficients, even though the rational numbers have already been presented to the students in previous chapters; the other textbooks use rational coefficients, even though the set of real numbers has already been presented in a previous chapter. In the 2017 analysis (Table 2), none of the textbooks are limited to integer coefficients in the first examples or in the exercises dealing with algebraic expressions, but only three of them remain coherent, in the sense that they use only rational coefficients because the set of real numbers appears later on in the collection.

In the 2010 analysis (Table 1), eight of the nine textbooks mention only polynomial expressions in the first examples of algebraic expressions; the same happens with only three of the eight textbooks in the 2017 analysis (Table 2). It should be noted that this change from 2010 to 2017 in the percentage related to varied examples of algebraic expressions may be due to the approach adopted in the 2017 analysis, in which the first exercises were considered together with the first examples of algebraic expressions.

The excerpt in Figure 2 confirms the limitation of the universe of the coefficients as well as the first examples restricted to polynomial expressions.


Uma expressão matemática contendo letras, números e operações é uma expressão algébrica.
- $4a^3$
- $5a + 3b - 2c$
- $\frac{2}{5}xy + 7x^2$
- $3(m - n) + 5m - 2(3m + 1)$

São exemplos de expressões algébricas.


A mathematical expression containing letters, numbers, and operations is an algebraic expression.
- $4a^3$
- $5a + 3b - 2c$
- $\frac{2}{5}xy + 7x^2$
- $3(m - n) + 5m - 2(3m + 1)$

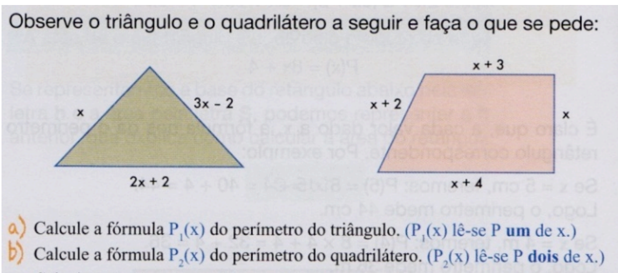
These are examples of algebraic expressions.

Figure 2: On the left, the first examples of algebraic expressions in a textbook (PNLD 2017), confirming the limitation of the universe of the coefficients to $\mathbb{Q}$ as well as the restriction to polynomial expressions in the first examples; on the right, the authors' translation.

It should also be pointed out that, once only polynomial expressions are presented as the first examples of algebraic expressions, the following doubts may arise later to the students, when the algebraic expressions called monomials, binomials, etc. are presented to them: To what extent do those new concepts (monomials, binomials, trinomials, etc.) differ from the concept of algebraic expression previously given?

The use of geometric context

None of the textbooks analysed in 2010 or in 2017 was careful to inform the units of measurement of the sides of the polygonal figures (Table 1 and Table 2), as shown in Figure 3. Although the units are sometimes mentioned in a few examples in one textbook in 2010 and in two textbooks in 2017, this information is not always present. It should be noted that the lack of any reference to the units of measurement causes ambiguity, as in the exercise presented in Figure 3. In fact, if x, 2, 3 and 4 represent the length in the same unit, then the algebraic expression corresponding to the perimeter of the quadrilateral is $4x+9$ units. However, if x is measured in meters and 2, 3 and 4 are measured in centimetres, then a possible expression for the perimeter of the quadrilateral is $4x+0{,}09$ meters.



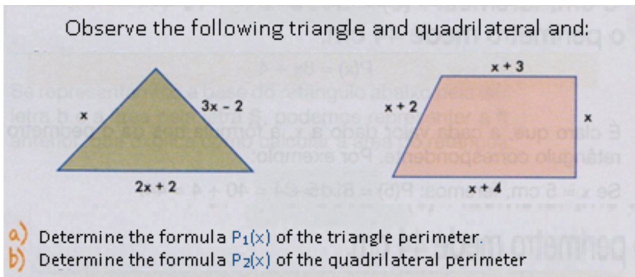


Figure 3: On the left, no information about the universe of the variables or about the units of measurement in a textbook (PNLD 2017); on the right, the authors' translation.

In none of the textbooks analysed in 2010, as well as in five of the ones analysed in 2017, can one observe any concern with any information about the universe of variables (Tables 1 and 2). For example, if x, 2, 3 and 4 represent the length in the same unit, then the triangle in Figure 3 exists only if $x > 2$ units.

The analyses carried out in 2010 and 2017 revealed five textbooks that present examples or exercises that do not offer any contribution to learning (Table 1). We could also observe that the excellent idea of using geometric context to support the learning of algebraic expressions is lost, for example, when the sides of squares are given by expressions like $6ab$, $\frac{2}{3}x^4y^2$ and $10t^3$ (Figure 4). Students may find those expressions representing the side of a square somewhat strange, since they usually make use of them to express volume or area.

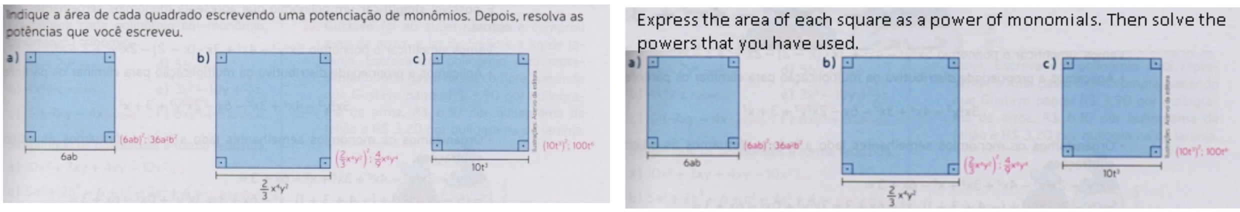


Figure 4: On the left, contexts that do not contribute to learning in a textbook (PNLD 2017), representing the measure of the side by 6ab; on the right, the authors' translation.

The reference to polynomials

In Table 1, there is no item named *reference to polynomials*. However, it is possible to extract some information related to this aspect from the text in Carvalho (2010).

Both in the 2010 and in 2017 analyses (Table 2), all the textbooks make inappropriate use of the term *polynomial*, which, in most cases, should be replaced by *polynomial algebraic expression*.

But, an important fact is that some of the textbooks are effectively operating with polynomials, using the Euclidean division of polynomials (Figure 5) or dealing with "the" factorization (suggesting uniqueness of factorization) or the "complete" factorization of a polynomial algebraic expression. Those concepts are not suitable for elementary school. It is worth remarking that those items and the word *polynomial* are not mentioned in either the PCN or the BNCC.

In the 2010 analysis, the use of the Euclidean division of polynomials could be detected in a textbook, where it is compared to the Euclidean division of integers. The same occurred with one textbook in the 2017 analysis (Figure 5), while all others deal only with the division of polynomial algebraic expressions by monomials, using the right-hand distributive property (without always mentioning it explicitly) and the properties of the powers (Figure 6).

The purpose of addressing factorization and/or division is clear in textbooks, namely to simplify algebraic expressions. However, regarding the simplification of algebraic fractions, the values of the variables that cancel out the denominator are not always considered (Figure 6).

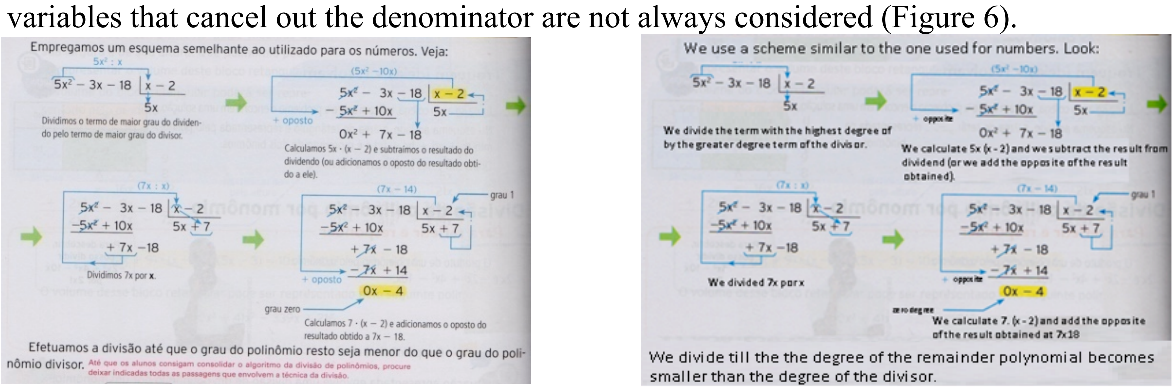


Figure 5: On the left, the comparison between the division of polynomials and the division of integer numbers in a textbook (PNLD 2017), and its translation on the right

$\frac{x^2 + 3x}{x + 3}$

A expressão que está no numerador pode ser fatorada:

$$x^2 + 3x = x(x + 3)$$

Então, essa fração algébrica fica:

$$\frac{x^2 + 3x}{x + 3} = \frac{x\,\cancel{(x + 3)}}{\cancel{x + 3}} = x$$

Lembre-se de que a simplificação é uma divisão! Por exemplo, quando simplificamos a fração $\frac{20}{28} = \frac{5}{7}$, estamos dividindo o numerador e o denominador por um mesmo número, no caso o fator comum 4.

$\frac{x^2 + 3x}{x + 3}$

The expression in the numerator can be factored:

$$x^2 + 3x = x(x + 3)$$

Then, the algebraic fraction can be rewritten as:

$$\frac{x^2 + 3x}{x + 3} = \frac{x\,\cancel{(x + 3)}}{\cancel{x + 3}} = x$$

Remember that the simplification is a division! For instance, when we simplify the fraction 20/28 = 5/7, we are dividing both numerator and denominator by the same number, in this case common factor 4.

Figure 6: On the left, simplification of algebraic expressions in a textbook (PNLD 2017), with no concern for the values of the variables that make the denominator equal to zero; on the right, the authors' translation.

The comparison between the Euclidean division of polynomials and the Euclidean division of integers is somehow dangerous for the student if it is not sufficiently detailed. For example, it may occur to the students that if they replace the variable by any number in the division of polynomial expressions, they will always find, as a remainder, the same remainder of the Euclidean division in $\mathbb{Z}$. However, this is not the case when $5x^2 - 3x - 18$ is divided by $x - 2$, for instance. In this case, by the Euclidean division of polynomials, we obtain the following polynomial identity $5x^2 - 3x - 18 = 5x(x - 2) - 4$. And, it is clear that, by replacing x with 6 in this identity, we will not find -4 as the remainder of the Euclidean division of the integer 144 (the result of $5.6^2 - 3.6 - 18$) by 4 (the result of $6 - 2$).

In the 2010 analysis, two textbooks give more than one answer to exercises that deal with the factorization of polynomial algebraic expressions. In all the others, the process of factorization always results in a unique answer. In the 2017 analysis (Table 2), one of the books does not mention factorization in the eighth year; five textbooks at some point mention the expression "a factored form of the polynomial", suggesting that there may be other factorizations for the same polynomial expression. Nevertheless, all the books that deal with factorization bring unique responses to the exercises (Figure 7). Regarding the issue of uniqueness of factorization/complete factorization of algebraic expressions, we remark that it is not correct to refer to those terms once the numerical universe of students is already at least $\mathbb{Q}$, and, hence, a field. Moreover, considering the later problem of solving the equation , the factorization may be more useful than , since it reveals the roots of the polynomial equation

$16a^2b^3x + 24ab^2x^2$

$4abx(4ab^2 + 6bx)$ é uma forma fatorada de $16a^2b^3x + 24ab^2x^2$, porém a fatoração ainda não está completa, pois o binômio $4ab^2 + 6bx$, que compõe o produto, ainda pode ser fatorado:

$2b(2ab + 3x)$

Então, $16a^2b^3x + 24ab^2x^2 = 4abx \cdot 2b(2ab + 3x)$.

Assim, a fatoração completa de $16a^2b^3x + 24ab^2x^2$ é $8ab^2x(2ab + 3x)$.

$16a^2b^3x + 24ab^2x^2$

$4abx(4ab^2 + 6bx)$ is a factored form of $16a^2b^3x + 24ab^2x^2$, but the factorization is not yet complete, because the binomial $4ab^2 + 6bx$ that makes up the product, can still be factored:

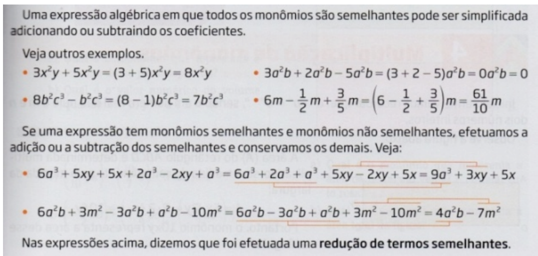
$2b(2ab + 3x)$

So, $16a^2b^3x + 24ab^2x^2 = 4abx \cdot 2b(2ab + 3x)$.

Thus, the complete factorization of $16a^2b^3x + 24ab^2x^2$ is $8ab^2x(2ab + 3x)$.

Figure 7: On the left, inadequate use of the nomenclatures "complete factorization" and "the factorization" in a textbook (PNLD 2017), since the domain of the variables is not mentioned; on the right, the authors' translation.

In the 2017 analysis, six textbooks present unnecessary rules and nomenclature in the presentation of operations with algebraic expressions, which may confuse students. For example, the so-called *reduction of similar terms* seems to suggest to the student the need for a definition of the addition of algebraic expressions, rather than merely suggesting the use of the properties of operations with numbers, since all the textbooks at the beginning suggest that the symbols in an algebraic expression represent numbers (Figure 8).



Uma expressão algébrica em que todos os monômios são semelhantes pode ser simplificada adicionando ou subtraindo os coeficientes.

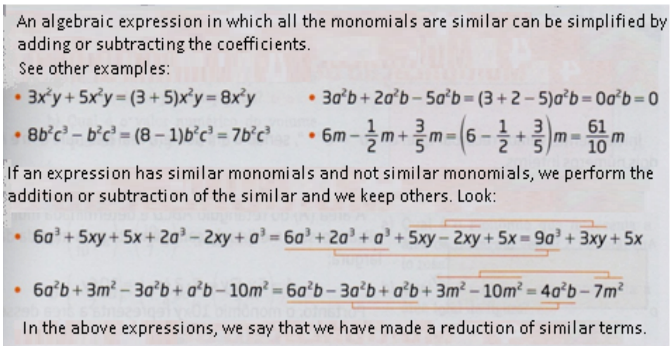
Veja outros exemplos.

- $3x^2y + 5x^2y = (3 + 5)x^2y = 8x^2y$
- $3a^2b + 2a^2b - 5a^2b = (3 + 2 - 5)a^2b = 0a^2b = 0$
- $8b^2c^3 - b^2c^3 = (8 - 1)b^2c^3 = 7b^2c^3$
- $6m - \frac{1}{2}m + \frac{3}{5}m = \left(6 - \frac{1}{2} + \frac{3}{5}\right)m = \frac{61}{10}m$

Se uma expressão tem monômios semelhantes e monômios não semelhantes, efetuamos a adição ou a subtração dos semelhantes e conservamos os demais. Veja:

- $6a^3 + 5xy + 5x + 2a^3 - 2xy + a^3 = 6a^3 + 2a^3 + a^3 + 5xy - 2xy + 5x = 9a^3 + 3xy + 5x$
- $6a^2b + 3m^2 - 3a^2b + a^2b - 10m^2 = 6a^2b - 3a^2b + a^2b + 3m^2 - 10m^2 = 4a^2b - 7m^2$

Nas expressões acima, dizemos que foi efetuada uma **redução de termos semelhantes**.



An algebraic expression in which all the monomials are similar can be simplified by adding or subtracting the coefficients.

See other examples:

- $3x^2y + 5x^2y = (3 + 5)x^2y = 8x^2y$
- $3a^2b + 2a^2b - 5a^2b = (3 + 2 - 5)a^2b = 0a^2b = 0$
- $8b^2c^3 - b^2c^3 = (8 - 1)b^2c^3 = 7b^2c^3$
- $6m - \frac{1}{2}m + \frac{3}{5}m = \left(6 - \frac{1}{2} + \frac{3}{5}\right)m = \frac{61}{10}m$

If an expression has similar monomials and not similar monomials, we perform the addition or subtraction of the similar and we keep others. Look:

- $6a^3 + 5xy + 5x + 2a^3 - 2xy + a^3 = 6a^3 + 2a^3 + a^3 + 5xy - 2xy + 5x = 9a^3 + 3xy + 5x$
- $6a^2b + 3m^2 - 3a^2b + a^2b - 10m^2 = 6a^2b - 3a^2b + a^2b + 3m^2 - 10m^2 = 4a^2b - 7m^2$

In the above expressions, we say that we have made a reduction of similar terms.

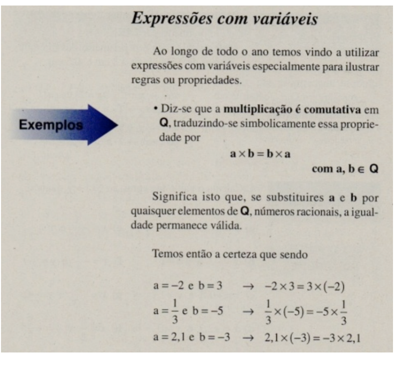
Figure 8: On the left, emphasis on a special rule for operating with algebraic expressions in a textbook (PNLD 2017); the right, the authors' translation.

The Introduction to algebra in textbooks for elementary and high school from other countries

An analysis on the theme *introduction to algebraic expressions* was also carried out on textbooks from other countries, namely Portugal, Chile and Cape Verde, as well as on two books written for primary schoolteachers, one from France and one from Chile. International comparison of textbooks is not new (see Fan, Zhu and Miao, 2013, for a survey on mathematics textbook research, including international textbook comparison). The purpose of analysing other countries' textbooks in this work was to check whether new approaches to this theme appear.

It was possible to observe some similarities with Brazilian textbooks regarding the criticisable items, but also some guidelines that confirm our position and the Brazilian documents' guidelines with respect to the numerical interpretation for the symbols in an algebraic expression in an initial and more systematic contact with algebra. The international texts also present the orientation of making use of the properties of arithmetic operations as support for operations with algebraic expressions (see, for example, Martinez and Vergnaud, mentioned in section 2).

The collection of Cape Verde deserves special comment, since it provoked a positive reflection for the authors of the present article concerning the question *What is really essential in the introduction to algebraic expressions*? Throughout the 7th grade textbook from Cape Verde, letters in algebraic expressions appear naturally as representative of numbers and operations with such expressions as are performed based on the properties of operations with numbers. There is no chapter or section specially dedicated to deal with those topics, and, hence, no mention of unnecessary rules or new nomenclature. Only on one of the final pages of this textbook, the term "Expressions with variables" appears as a title, followed by the sentence "Throughout the year we have been using expressions with variables specially to illustrate rules or properties", followed by an example of the commutativity of multiplication (Figure 9).



Expressões com variáveis

Ao longo de todo o ano temos vindo a utilizar expressões com variáveis especialmente para ilustrar regras ou propriedades.

Exemplos

- Diz-se que a **multiplicação é comutativa** em **Q**, traduzindo-se simbolicamente essa propriedade por

$$\mathbf{a \times b = b \times a}$$

com a, b ∈ Q

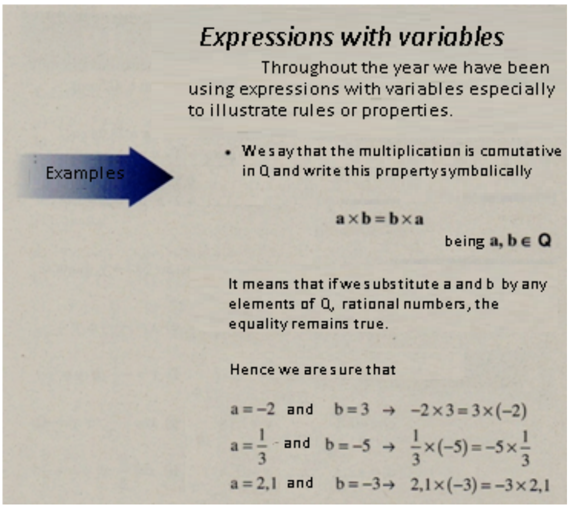
Significa isto que, se substituires **a** e **b** por quaisquer elementos de **Q**, números racionais, a igualdade permanece válida.

Temos então a certeza que sendo

$a = -2$ e $b = 3 \rightarrow -2 \times 3 = 3 \times (-2)$

$a = \frac{1}{3}$ e $b = -5 \rightarrow \frac{1}{3} \times (-5) = -5 \times \frac{1}{3}$

$a = 2{,}1$ e $b = -3 \rightarrow 2{,}1 \times (-3) = -3 \times 2{,}1$



Expressions with variables

Throughout the year we have been using expressions with variables especially to illustrate rules or properties.

Examples

- We say that the multiplication is comutative in Q and write this property symbolically

$$\mathbf{a \times b = b \times a}$$

being **a, b ∈ Q**

It means that if we substitute a and b by any elements of Q, rational numbers, the equality remains true.

Hence we are sure that

$a = -2$ and $b = 3 \rightarrow -2 \times 3 = 3 \times (-2)$

$a = \frac{1}{3}$ and $b = -5 \rightarrow \frac{1}{3} \times (-5) = -5 \times \frac{1}{3}$

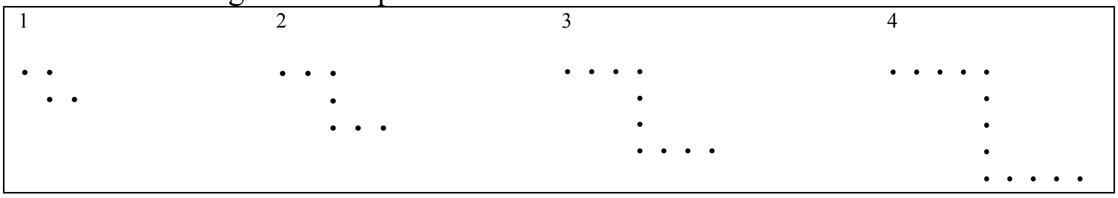
$a = 2{,}1$ and $b = -3 \rightarrow 2{,}1 \times (-3) = -3 \times 2{,}1$

Figure 9: On the left, an excerpt of a textbook from Cape Verde, showing how algebraic expressions are treated in the 7th grade; on the right, the authors' translation.

An example of an activity that uses generic thinking and develops the ability of constructing simple proofs

Differently from what we concluded from the textbook analyses that were carried out, we reinforce that, in the context of introduction to algebra, one can develop generic thinking and include results that meet the standard of proof with 8th grade students. In Figure 10, we show an activity that is reported in Carvalho (2010). It proved to be useful both for the development of generic thinking and the construction of simple proofs, naturally motivated by the desire to decide whether, after all, two algebraic expressions are equivalent.

Observe the four arrangements of points below



a) Find a pattern for the construction of these arrangements and draw arrangements 5 and 6, according to this pattern.

b) Explain the pattern that you have considered and a strategy for counting the number of points in any arrangement.

c) If the sequence of arrangements continues according to the pattern you have considered, how many points will arrangement 10 have? And arrangement 20?

d) Determine an algebraic expression for the number of points in arrangement *n*.

Figure 10: An activity in Carvalho (2010)

In fact, more than becoming familiar with the construction of a sequence - identifying and describing a pattern of construction and of counting the number of points involved in each term of the sequence and expressing the number of points involved in the arrangement *n* by an algebraic expression in terms of *n* - the activity also motivates the need to decide whether, after all, two different algebraic expressions given as answers to item *(d)* are equivalent or not. Since all the different patterns found by the students generated the same sequence of arrangements, all the expressions found should generate the same number of points, that is, they should be equivalent (see Figure 11 for some examples); hence, Carvalho´s students were motivated to construct simple proofs for those algebraic identities. In those proofs, the students made use of the properties of the operations with numbers, and no need for "rules for operating with algebraic expressions" was necessary or even expected by the students.

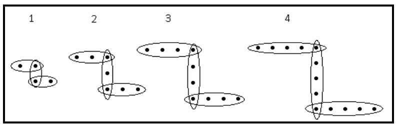
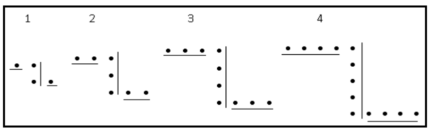
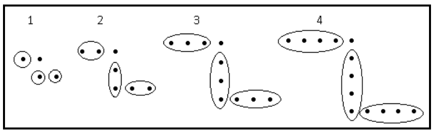
Strategy and answers of student A	Strategy and answers of student B	Strategy and answers of student C
		
b) "triple the successor minus 2" d) 3 (n+1) – 2	b) "double the position of the arrangement plus the successor of the position" d) 2n + (n+1)	b) "triple the number occupied by the arrangement plus one" d) n + n + n + 1 = 3n + 1

Figure 11: Some of the strategies used by the students

Finally, this activity also reinforced an aspect highlighted in the BNCC, that during the final grades of elementary school, students' capacity or abstract reasoning is greater (Brasil 2, 2017, p. 56). In fact, the search for an algebraic expression in item *(d)* serves to develop students' generic thinking. Therefore, the answer to the guiding question of the symposium Deductive Reasoning, Arguing and Proof in Textbooks, *Do textbooks include contexts, contents, results that meet the standard of proof?* should be affirmative with respect to the introduction of algebra.

FINAL COMMENTS

In this text, we report three different textbooks analyses with respect to the introduction to algebraic expressions: Brazilian textbooks in the years 2010 and 2017 and non-Brazilian textbooks in 2017. It was observed that, in both the analyses carried out that deal with Brazilian textbooks, there is no concern of their authors for helping student's development of mathematical thinking or abstraction abilities, in particular of the ability of proving. Thus, the answer is negative to the guiding question of the symposium *Deductive Reasoning, Arguing and Proof in Textbooks*, namely, *Do textbooks include contexts, contents, results that meet the standard of proof?*

Regarding the foreign textbooks, it was possible to find some similarities with the Brazilian textbooks with respect to the criticisable items. However, the analysis of a collection from Cape Verde provoked a positive reflection for the authors of the present article on the matter: *What is really essential in the introduction to algebraic expressions*?

Certainly, one restrains the students' development of abstract thinking and of the ability to perform simple proofs (such as verifying algebraic identities based on the properties of operations) by not considering the symbols in an algebraic expression as numerical variables.

The example of an activity mentioned in the last section stimulates the students' abilities of arguing in mathematics and constructing simple proofs. It also summarises and deepens the objectives of the first grades and aims to develop students' abstraction capacity in the final grades of elementary school; hence, it is also in accordance with the guidance of official Brazilian documents. Yet, most importantly, it shows that the answer to the guiding question of the symposium Deductive Reasoning, Arguing and Proof in Textbooks, namely, *Do textbooks include contexts, contents, results that meet the standard of proof?* should be affirmative with respect to the introduction of algebra.

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SYMPOSIUM C

TEACHER-RESOURCE USE AROUND THE WORLD

organised by

JANINE REMALLARD, HENDRIK VAN STEENBRUGGE AND LUC TROUCHE

TEACHER-RESOURCE USE AROUND THE WORLD

JANINE REMILLARD, HENDRIK VAN STEENBRUGGE, LUC TROUCHE

Purpose

This symposium presented research from different countries and school systems around the world, examining different aspects of teachers' interactions with and use of resources, factors that influence them, and their consequences for instruction. Representing studies of teachers' use of resources from Brazil, China, France, Mexico, South Africa, Sweden, and the United States, the symposium explored the relationship between individual and collective teacher capacity and the design of resources as factors that shape the enacted curriculum. The following questions guided the session:

- How might we understand the processes by which teachers engage with curriculum resources to design instruction?
- How does teacher capacity influence teachers' curriculum designs and how is this capacity developed and enhanced?
- How do resource features contribute to teachers' curriculum designs?
- How does collective use influence teachers' design decisions over time?
- How do teachers' curriculum designs contribute to the enacted curriculum?

Guiding Frameworks

The work represented in the session is supported by a participatory perspective on teachers' resource use, which views using instructional resources as a dynamic process involving reading, interpretation, appropriation, and design (Brown 2009; Gueudet & Trouche 2009; Remillard 2005). Resting on socio-cultural analyses of the agent-tool relationship (Vygotsky 1978), this perspective conceptualizes curriculum resources as cultural tools that mediate teachers' curriculum design work and are products of this work. Teachers' intellectual and cultural resources also mediate this process. Finally, teachers' curriculum design work occurs in a context, often with other teachers, and unfolds over time, leading to new designs, new capacities, and new curriculum enactments.

Significance and Theme

The symposium fits within the conference theme of "textbook use by teachers," however, we consider instructional and curricular resources that go beyond the textbook, including resources designed to guide, support, and enhance mathematics teaching and learning in schools and re-sources generated by teachers as they design instruction. In 2017, the types of resources used by teachers are diverse and include print, (more and more) digital, and online tools. The symposium considered teachers' use of resources across different cultural contexts and types of resources.

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Symposium Participants and Organization

The symposium was intended to highlight the work of early-career researchers from various parts of the world, having in mind that new researchers could be more sensitive to phenomena arising in the thread of digitalization. Nine distinct papers were included, representing research based in Brazil, China, France, Mexico, South Africa, Sweden, and the United States. The session began with a 10-minute introduction to the themes of the session, followed by 15-minute presentations of each paper, and concluding with sufficient time devoted to questions and discussion among the audience around the guiding questions, moderated by the session chairs. The nine contributions to this symposium are listed below and included in the ICMT-2 proceedings as short abstracts or extended papers.

Papers on Teachers' Interactions with Resources and Related Classroom Enactments

Design in use: from author intended to written to enacted lesson in Sweden, Hendrik Van Steenbrugge, Nina Jansson, Fredrik Blomqvist, and Andreas Ryve, Sweden

Analyzing teachers' collective engagement with resources through the lens of their documentational trajectories: the case of French teachers facing a new curriculum, Katiane de Moraes Rocha, Luc Trouche and Ghislaine Gueudet, France

An analysis of the engagement of pre-service teachers with curriculum resources in Brazil, Cibelle Assis and Verônica Gitirana, Brazil

From written to enacted lessons: A U.S. teacher's mobilization of a mathematical modelling-based algebra unit, Luke Reinke, USA

Papers on Teacher Capacity and Learning in Relation to Resource Use

Disaggregating teachers' pedagogical design capacity (PDC) in South Africa, Moneoang Leshota, South Africa

Knowledge of curriculum embedded mathematics: Exploring a critical domain of teaching in the U.S., Janine T. Remillard and Ok-Kyeong Kim, USA

An investigation of Chinese mathematics teachers' resources work in collectives and their professional development, Chongyang Wang, Luc Trouche and Birgit Pepin, China

Interdisciplinary program for professional development in mathematics teaching in Mexico, Daniela Reyes-Gasperini, Mexico

Resources for teaching, Jose Luis Cortina and Jana Visnovska, Mexico

Dissemination in addition to presentation in symposium

Four paper presentations (de Moraes Rocha et al., 2018; Wang et al., 2018; Assis & Gitirana, 2018; Cortina & Visnovska, 2018) have been extended as a full paper and are included as such in the conference proceedings.

Three paper presentations have been published meanwhile or are currently under review and are included in the conference proceedings as short abstracts:

Leshota, Moneoang & Adler, Jill, 2018. Disaggregating a Mathematics Teacher's Pedagogical Design Capacity. In L. Fan, et al., (Eds.), *Research on Mathematics Textbooks and Teachers' Resources: Advances and issues* (pp. 89-118). New York: Springer.

Remillard, Janine T. & Kim, Ok-Kyeong, 2017. Knowledge of curriculum embedded mathematics: exploring a critical domain of teaching. *Educational Studies in Mathematics*, *96*(1), 1-17.

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AN ANALYSIS OF THE ENGAGEMENT OF PRESERVICE TEACHERS WITH CURRICULUM RESOURCES IN BRAZIL

CIBELLE ASSIS AND VERÔNICA GITIRANA

Abstract

Within the theme "Teacher-resource use around the world: Understanding Critical Issues and Instructional Outcomes", of Symposium C we were particularly interested in: "How might we understand the processes by which teachers engage with curriculum resources to design instruction?" To answer it, we analyzed the trajectories of three pre-service teachers, focusing on the events in which they designed their resources for specific mathematics classes, during an initial teacher training course at a Federal University in Brazil. The research considered three final essays (TCC - Trabalho de Conclusão de Curso), which include a lesson plan and a worksheet, and other resources retrieved after the analysis of the TCC to compose their documentational trajectories. From them, we developed a categorization for their quotations considering the curricular resources (PCN and textbooks). The results indicated that the processes of engagement with these resources to design instruction was influenced by: specific characteristics of these resources; previous experiences with them at school and at university; and integration with other resources. These aspects evidenced a documentational perspective for prospective teacher in a training course context.

Introduction

The debate around different aspects of teachers' interactions with resources, proposed by the Symposium C, considers, essentially, the understanding of a teacher's work as a work with and on resources. For Gueudet and Trouche (2009), this work is the core of teachers' professional activity and professional change. For us, since we work and research within teacher training, this understanding is particularly important in order to design teachers training courses from the viewpoint of the future math teacher's engagement with resources.

In this paper, we will propose a reflection about the processes by which three pre-service teachers engage with a particular curriculum resource - the *PCN* (Brasil 1998) and the use of the textbooks as a curricular resource, in order to prepare a math lesson. Would they be similar processes, due to the use of the same resources within their particularities; or diverse processes, due to the diversity of resources used during their trajectory?

In order to answer it, we structured our contribution in five sections. In the first one, we present the Brazilian context considering the *PCN* and textbook. In the second section, we consider concepts related to Documentational Approach to Didactics (Gueudet & Trouche 2009), the Documentational Trajectory (Rocha 2016) and Forms and Modes of Engagement (Remillard 2012). Next, we describe the pre-service teachers and our associated methodological choices. In the fourth, we present the data and our analysis and discussion. Finally, we present some remarks and future perspectives.

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The Brazilian context: *PCN* and textbooks

Brazilian basic school system is structured in three school levels: *Educação Infantil* (2 - 5 years old), *Ensino Fundamental* (6 - 14 years old) and *Ensino Médio* (15 - 17 years old). In addition, the *Ensino Fundamental* level is divided into *Anos Iniciais* (1st - 5th year) and *Anos Finais* (6th - 9th) (Brasil 2013).
Regarding Brazilian national curriculum references, we have, as official programs, the following documents: *Diretrizes Curriculares Nacionais Gerais para a Educação Básica* (Brasil 2013, which encompass the whole basic school system, and *Base Nacional Curricular Comum – BNCC,* which is under development. In addition, *Parâmetros Curriculares Nacionais* - *PCN* (Brasil 1998) are curricular references for what is called *Ensino Fundamental*.
The *PCN* is a curricular reference used by teachers at school and preservice. In fact, they give teacher and their school community some support for discussion and development of educational projects, references for analyzing and selecting didactic materials and technological tools.
In Brazil, textbooks are used as a curricular resource. As in other countries, teachers are heavily influenced by textbooks and they have been fundamental to teacher's decision on which contents must be taught as well as the teaching approach to be developed in class (Lajolo 1996). Our States Schools receive textbooks from a Federal Government program called *PNLD* - *Programa Nacional do Livro Didático*. It is a national program responsible for the process of submission, evaluation, organization of the teacher's choice, acquisition and distribution of the textbooks. The collections are submitted by the editors, evaluated by a national commission appointed by the Government. The result of this evaluation is published in *PNLD* Guide of Textbooks (Brasil 2016), next the mathematics teachers' team of each school choose a textbook collection to be used from this guide (Brasil 2015).
In Brazil, both *PCN* (Brasil 1998) and textbooks are used as curriculum resources. This point of view will be analyzed later. In fact, we introduce some general *PCN* characteristics and then we make a parallel with the textbooks evidencing the differences between them considering modes and forms of address.

The Theoretical framework

Documentational Approach to Didactics and the Documentational Trajectory

Approximately ten years ago, the Documentational Approach to Didactics, introduced by Gueudet and Trouche (2009), considered teacher's work, in its specificity and continuity, as a work with and on resources. It brings a general reflective perspective focused on teachers' resources, their appropriation and transformation by a teacher or by a group of teachers working collectively. Similar issues have already been investigated by Adler (2000), whose notion about resources is a broad one: everything that sources teacher activity that appears particularly productive is widely recognized internationally. Adler (2000) suggests "think[ing] of a resource as the verb re-source, to source again or differently" (p. 207).
Following Adler (2000), Gueudet and Trouche (2008) propose a broad notion of resources but with some restrictions: teacher's knowledge is not considered as a resource (but what guides their work with the resource), and what is material or materializable entities are considered as resources. For example, teacher's classmates are not considered as a resource; but advices, messages, and proposals of colleagues are.
Like the instrumental approach (Rabardel 1995) that distinguishes what is available for the activity (the artifacts) and which is developed by the subjects (the instruments), Gueudet and Trouche (2009), in the documentational approach, distinguish what is available for the activity of the teachers: the resources; and what they develop to support their teaching activity: the documents. According to the Documentational Approach to Didactics, the evolution from resources to documents happens throughout a process they called a *documentational genesis*. A document is developed from the combined resources used in different contexts, and these usages, by a set of

knowledge and schemes. In fact, since documents incorporate teacher knowledge and schemes, the development of documents and teachers' professional growth are interrelated (Gueudet *et al.* 2012). The figure 1 is a representation of this documentational genesis process.

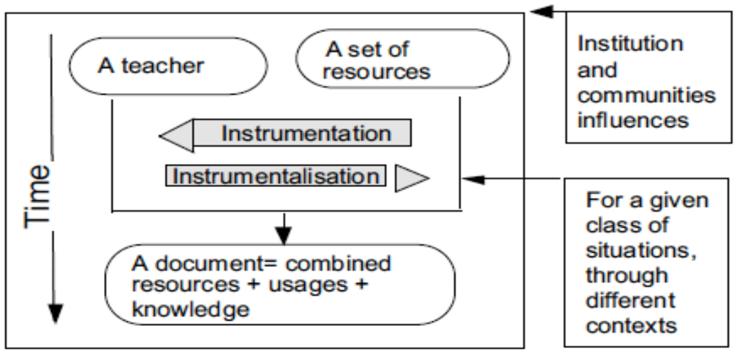

Fig. 2 Schematic representation of a documentational genesis

Figure 1 – Schematic representation of a documentational genesis
Gueudet *et al* (2012, p. 720)

For a given teacher, in order to perform a given type of task, there is a dialectic relationship between resources and documents and two dimensions intertwined: instrumentation - the resources act on the teachers (in helping them conceive their activity in a given way) and instrumentalisation - teachers also act upon these resources as they appropriate them (inverse process of instrumentation). We can also observe that there are external influences over the teacher whose origin is his institution or communities where he or she is integrated.

While studying the development of teachers' documental system, Rocha and Trouche (2017) propose a new concept for modeling the teachers' history of their resource usage: the teacher's documentational trajectory. Rocha (2016) defines the teacher's documentational trajectory as the interaction between events and resources that can be represented schematically following a timeline. In this perspective, a resource is something that supports the work of a teacher and an event is defined as something that has happened in the professional life of the teacher and that, at the time of the development of his trajectory representation, the teacher recalled as important in relation to his documentational work.

Rocha defined two types of representation of this trajectory: a reflective mapping (created by the professor himself) and an inferred mapping (made by the researcher) from the data and information collected. The term "mapping" of teacher resources integrating an idea of the progressive exploration of a new territory rather than the simple idea of representation for the teachers' resource.

Modes and Forms of Address and Engagement with curriculum resources

A curriculum resource is designed to provide an environment for helping to build the curriculum. Remillard (2012) conceptualized modes/forms of address and modes/forms of engagement to refer to what teachers develop through transactions with a curriculum resource. She developed her research considering NCTM Standards/USA and Standards-based curriculum materials, among which she mentioned Everyday Mathematics and Investigations in Number, Data, and Space (Remillard 2012).

The form of address of a curriculum resource "refers to the physical, visual, and substantive forms it takes up, the nature and presentation of its contents, the means through which it addresses teachers" (Remillard, 2012, p.108). The form of address is what teachers actually see, examine, and interact with when using a curriculum resource. She identified it in five important interrelated categories to be studied: structure, look, voice, medium, and genre.

- The *Structure* is the most widely studied feature of curriculum resources. The components of the structure are related to what a resource contains and what it offers for teachers and for students. The Structure is also related to how these offerings are organized;
- The *Look* refers to the simple visual appearance of a resource or what teachers see when they look at it. Some elements as bright pages, color photographs and pages that look like advertisements, fonts and type of paper result from a set of design choices.
- The *Voice* refers to the way the authors/designers' purpose is represented, and how they communicate with the teacher. In some curriculum resource, the authors are invisible and little information is given about themselves and their own experience. Anyway, the authors communicate their intentions through the actions they suggest for professors to undertake.
- The *Medium* refers to the form under which the resource is diffused. Unlike printed resources, digital resources allow and often support a non-linear path through what they offer, giving the user freedom of navigation. This element can influence interactions and resources' uses.
- The *Gender* refers to the nature of the curriculum resource within a broad classification of written materials for teachers. Gender is important because it has implications for the expectations of teachers, expectations that influence how they approach a resource.

In her terms, *modes of engagement* refer to "what a teacher does in her transactions with a particular curriculum resource, how she engages, infuses meanings, and makes sense of its offerings (p. 115)." Just as the mode of address of a resource can be seen in its forms, a teacher's mode of engagement can be understood through the forms that engagement takes up.

According to Remillard (2013), a teacher's mode of engaging with a curriculum resource as a reader includes four important features to describe their forms: what s/he reads for; which parts s/he reads; when s/he reads; and who s/he is as a reader. These questions related to "when" consider different moments of teaching (before, during, and after) and they are related to "why". In addition, the teacher, as a reader, is associated with an attitude or an orientation. According to Remillard (2012), the forms of engagement can evolve during the process of reading and the construction of experience. In addition, these forms of engagement can lead to different uses of the same resource.

The three pre-service teachers and methodological associated choices

We observed three pre-service teachers named here Ceci, Marcos and Carlos during the last year of their training course offered by a federal university in Brazil, which takes four years. The three pre-service teachers were classmates in many disciplines of the course between 2012 and 2017. They were chosen as subjects of this research because we supervised them during the semester where they produced their *TCC* reports (from April to June in 2016). In common, these pre-service teachers have already developed some projects with us using the software of dynamic geometry Geogebra. The *TCC* report - *Trabalho de Conclusão de Curso* - or final essay*,* is a mandatory report written by every pre-service teacher in the last year of the course. It comprises a report (around 100 pages in printed and digital version) about a teaching experience.

The pre-service teachers designed their resources for specific mathematics classes and produced a lesson plan (which contains goals, methodology, materials, references) and a worksheet (a script with activities) to be experienced in a class at school. Thus, the *TCC* report contains details about this designing which includes preparation of a worksheet until the implementation in class at a state school.

Ceci (case 1) aimed to design a lesson about triangles classification by sides and angles supported by Geogebra for a 8th grade class. Marcos (case 3) designed a mathematics lesson that aims at exploring square's and cube's properties supported by Geogebra with an 8th grade class. Carlos (case 2) designed a lesson to introduce feature aspects of linear and affine functions using Geogebra for at a 1st grade year of high school class.

We advised Ceci and Marcos to use the *PCN* (Brasil 1998) related to the 8ht grade, and *OCEM* (Brasil 2006) for Carlos, who decided, by himself, to use the *PCN* (Brasil 1998) aswell. Our analysis of their *TCC* reports started with the reading and identification of the quotations (direct and indirect ones) from the textbooks and the *PCN*, appearing in the theoretical framework section as well as in the lesson plans and worksheets, using Contents Analysis (Bardin 1977) for creating categories to infer the pre-service teachers as readers in the Remillard's sense (2005; 2013). We used the software MAXQDA in order to organise and categorise the quotations directly from the TTC reports, but we also had access to the three textbooks used.

After analyzing the *TCC* reports, we asked the pre-service teachers to build up their documentational trajectory. They presented and explained events and associated resources considering their importance for their professional development. They mentioned events from 2012 to 2016 which comprises the period between the beginning and the last year of their under graduate course. As a result, this trajectory gave us a general perspective on their experiences with resources, including curriculum resources and textbooks, over time. A first interview was done in order to get details about each documentational trajectory. It was recorded on audio and video and happened in November, 2016.

Finally, a last video-recorded interview was carried out to understand unclear aspects related to their engagement with textbooks and the *PCN,* which were previously analyzed by us from the *TCC* reports. It is important to say that we did not observe the pre-service teachers' resources while they were designing in class at school.

PCN and textbook: Modes and forms of address

The *PCN* (Brasil 1998) is organized in four cycles, separated into two volumes: the first volume comprises the cycle 1 (1st - 3th year) and the cycle 2 (4st - 5th year), and the second volume, the cycle 3 (6ht - 7th year) and the cycle 4 (8ht - 9th year). In our research, we considered the second volume of the *PCN* because it was used as a reference by the pre-service teachers.

The second volume is organized in three parts. The first one is related to general aspects about Mathematics which comprises: topics about curriculum and general aspects of teaching mathematics; characteristics of Mathematical knowledge; teaching-learning process; problem-solving as a methodology; mathematics teaching approaches within history, technology and games; general goals of teaching mathematics; some orientation of contents to be taught and evaluation principles. The second part considers almost the same topics being more specific according to the cycle and the mathematical subject (Numbers and Operations, Space and Form, Geometric Measurement and Dimension, Statistics). In addition, in this part, the *PCN* states that mathematics subjects must consider the concepts, proceedings and attitude dimensions. Finally, in the last part, for both cycles there are didactical orientations organized by subjects. The *PCN* approach points out didactical principles for some mathematical concepts related to specific goals that should be considered by teachers.

Considering the textbooks used by the pre-service teachers, in general, they were organized by chapters, sections and subsections. Their proposed activities are divided into: activities for exploration and investigation, for review and for contextualized problems. Their editorial design allows teachers and students easily identify them. For each textbook, there is a teacher's guide with didactic orientations to use students' textbook and also suggestions more accurate as sites, articles and others resources

The *PCN* and textbooks are distinct curriculum resources due to their different structure, look and voice. In fact, *PCN* proposals sound like advices for teachers, without details about which school grade contents can be exactly worked on, and also indicate very basic contents for each cycle without organizing them into grades and inside the grade. This organization of contents in general lead teachers and schools to follow the textbook collection as a curriculum resource. PCN does not present colorful photographs like the ones you can see in the textbooks, and the *PCN* language is more formal.

The *PCN* was written for principals, trainers and teachers, while textbooks were written having teachers and students as audience. The *PCN* is not commercial, while the textbooks are, and were the latter been approved by the PNLD program

PCN and textbooks: Modes and forms of engagement

Regarding *PCN* (Brasil 1998), in order to infer ways of preservice teachers engagement with it, we observed the sections quoted by them in their *TCC* reports. A quotation has a double sense: a real use of *PCN* (information mentioned in the *TCC* reports what includes the lesson plan or the worksheet) and a possible use of any *PCN* idea in their planning.

For example, Ceci wrote this paragraph in her *TCC* report (written in Portuguese and translated by us) where she states that she intends to construct an activity with Geogebra, which encompasses *PCN* assumptions (Brasil 1998, p. 26):

> In order to carry out the adaptations of the exercises for Geogebra, we were guided by some recurring keywords in all this work, such as constructing, testing, comparing, analyzing/ conjecture, observing, moving/dragging, among others, so that our activity attributed these possibilities. And that in a way, it could satisfy the following statement of PCN (1998):
>
> The exercise of induction and deduction in Mathematics is of importance in developing the capacity to solve problems, to formulate and test hypothesis, to induce, to generalize and to infer within a certain logic, which assures a relevant role for learning science at all levels of education (Brasil 1998, p.26).

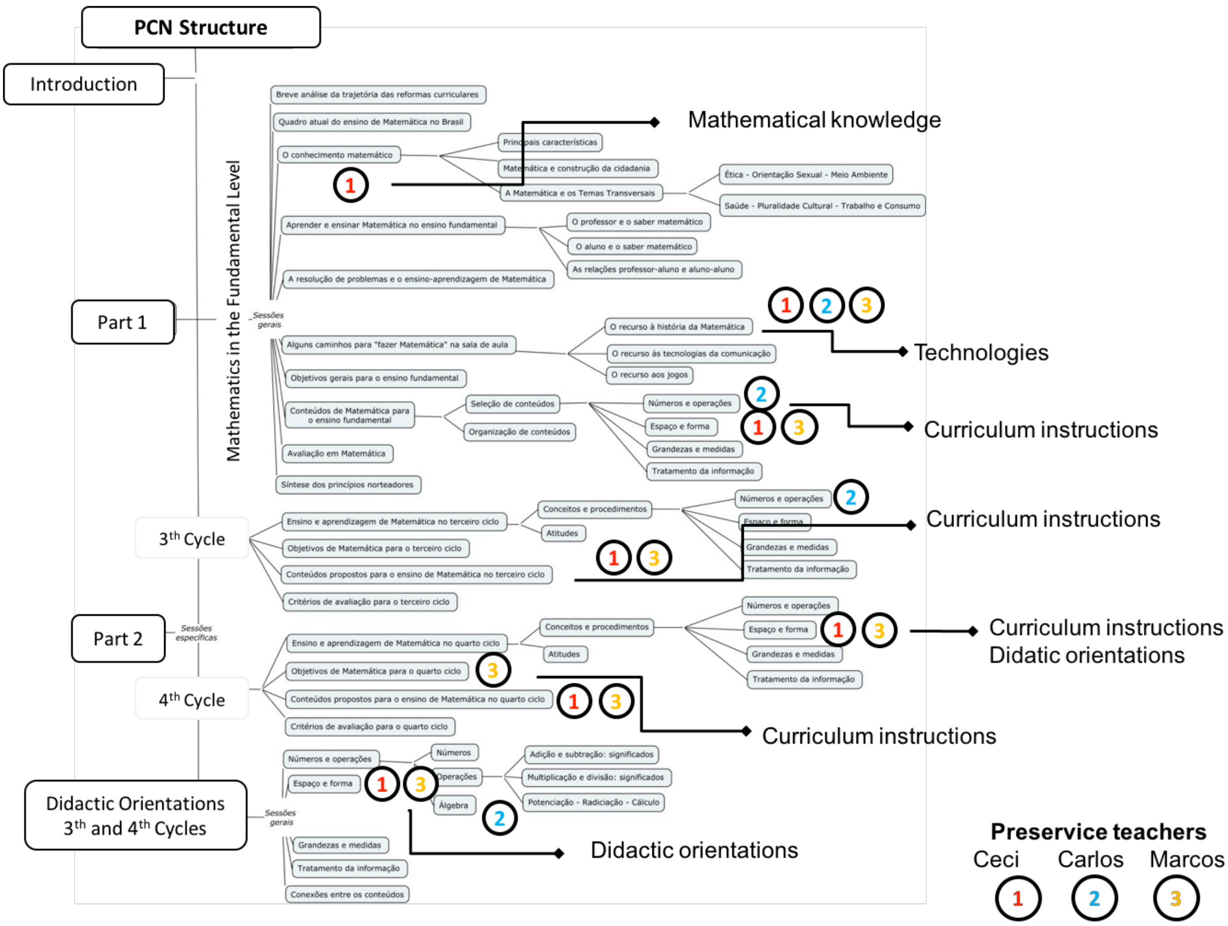

Figure 2 – *PCN* representation and mentioned sections in the *TCC* report

Beyond *PCN* quotation, we can also say that she applied the *PCN* orientations in her lesson plan. In fact, she proposed to construct examples of triangles to explore them by dragging and to observe invariant properties and to construct a classification of triangles. After the analysis of quotations, we organized them into three categories: *technologies in math classes, didactic orientations and curriculum information*. These categories were inspired by the PCN organization, despite a quotation can be found in a section with a different title from the category name. This categorization helped us to infer what the pre-service teachers had read and what for.
Figure 2 represents *PCN* structure (translated here) where we pointed out the visited *PCN* sections and also the categories for each identified quotation.
We can affirm that the three preservice teachers read the same subsection of the PCN - *Parte 1: O uso das tecnologias* - and used it in the TCC report. They also mentioned *Seleção de conteúdos* to find specific orientations about Numbers and operations (Carlos) and Geometry (Ceci and Marcos). The same happened in section *Orientações Didáticas* for cycles 3 and 4, where they mentioned aspects related to Geometry (Ceci and Marcos) and Algebra (Carlos). We can observe that Ceci and Marcos used common PCN information, we hypothesize that this is due to the fact that they were both planning a lesson on studying the characteristics of plane figures (triangles and squares).
However, regarding the lesson plan, including the worksheet, there were few evidences of *PCN* uses. Only Ceci and Carlos mentioned it in their lesson plan as a reference for planning without specifying exactly what. Because of this, we realized that the information read and used in their *TCC* reports were probably "implicit" in their plan lessons or worksheets. From this fact, we decided to investigate in detail considering the original source - *PCN* (Brasil 1998) and asking them, through an interview, how they used the *PCN* and which parts influenced their design.
From the first interview, which was conducted individually, we observed that Marcos and Carlos used the PCN as a resource to write their TCC reports, while Ceci used the PCN as a resource for designing, as we presented in the previous example. In fact, Marcos and Carlos revealed that they searched for information or orientation related to technologies, didactic suggestions and curricular aspects. In addition, they revealed that they seek for information through keywords (technologies, function, geometry) and after reading, they decide what is important to consider in the theoretical section of their *TCC* reports.
Concerning the mathematics textbook, their uses were easily identified in Ceci's and Carlos's lesson plans as a resource for proposing tasks for students, but not precisely in the worksheets. Ceci used the textbook in order to find some exercises about classification of triangles (sides and angles). In fact, she mentioned three exercises she adapted, and Carlos looked for problems related to linear function. In fact, he found a problem and then he added two items created by him.
Therefore, during the interview, new uses emerged. Ceci mentioned that she also used the textbook to learn more about triangles classification and to do exercises, while reading many textbook sections. Curiously, she mentioned that she did not use the teachers' guide of the textbook because she did not know about it. Carlos mentioned that he did not read any chapter apart from the specific section related to his subject. He did not observe the suggestions to use technologies or other resources, which are shown in the textbook next to the problem used by him, and he did not take advantage of the resources for teachers.
Different from the others, Marcos did not mention any exercise or example in his *TCC* report (lesson plan or worksheet). Therefore, during the interview, he mentioned how he used the textbook: he solved problems related to his subject, he looked for definitions and examples. He said that he used the textbook to study Mathematics and to be well prepared for teaching. He mentioned, for example, that he learned from the textbook that there are 11 planning forms for the cube. The textbook version used by him was not the teacher's guide of the textbook.
Considering a researcher's point of view, we realized that the teachers' use of resources comprised an observable dimension and a not observable one. In fact, as regards a lesson planning or a worksheet design for a specific lesson (examples of a resource-reference), it was possible to identify

some resources that were used and how they were used. It means that, depending on the level of details expressed on it, we can identify "sections or pieces" of resources through the quoted references (directly or indirectly). In addition, in this case, it is possible to identify "sections" in the original sources. For these aspects, we named an observable dimension of the use of the resources. Therefore, there are much more not observable elements related to resources uses. For example, a not explicit use of a resource, previous experiences with resources, strategies for searching for information, or else, elements of documental genesis are also not directly observable. This idea has conducted us to think about an iceberg model for uses of resources, considering a given resource as a reference (Figure 3).

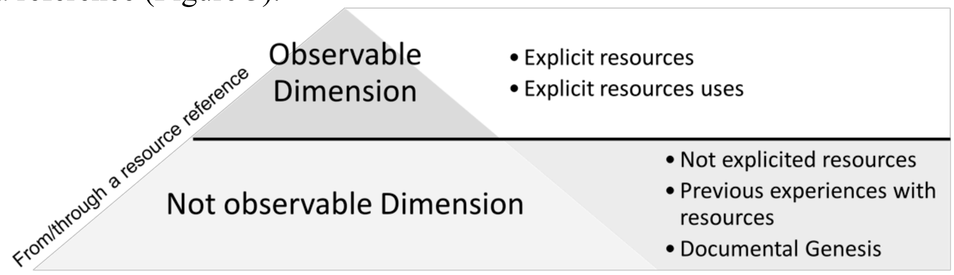

Figure 3 – Iceberg model for resources uses from and through a resource-reference

The small part of the iceberg represents the observable aspect of the resources and their uses; and the bigger one, what is below the water line, is related to what we cannot observe directly from or through the resource analyzed. A teacher's document itself is composed by both dimensions.

Previous experiences with PCN and Textbooks

In order to understand their *PCN* and textbook uses, we proposed to the pre-service teachers to construct a representation for their documentational trajectory. We requested them to present some important events for their professional development that happened during the training course.

In common, they mentioned more events related to experiences of teaching mathematics than events related to learning mathematics. We also identified, in each representation, events where the national references, including curriculum resources, and the textbooks were present. In each representation, naturally, we identified that Geogebra was a resource that was strongly present among the mentioned resources. As an example of this analysis, Figure 4 represents Ceci's trajectory with our categorization of the events.

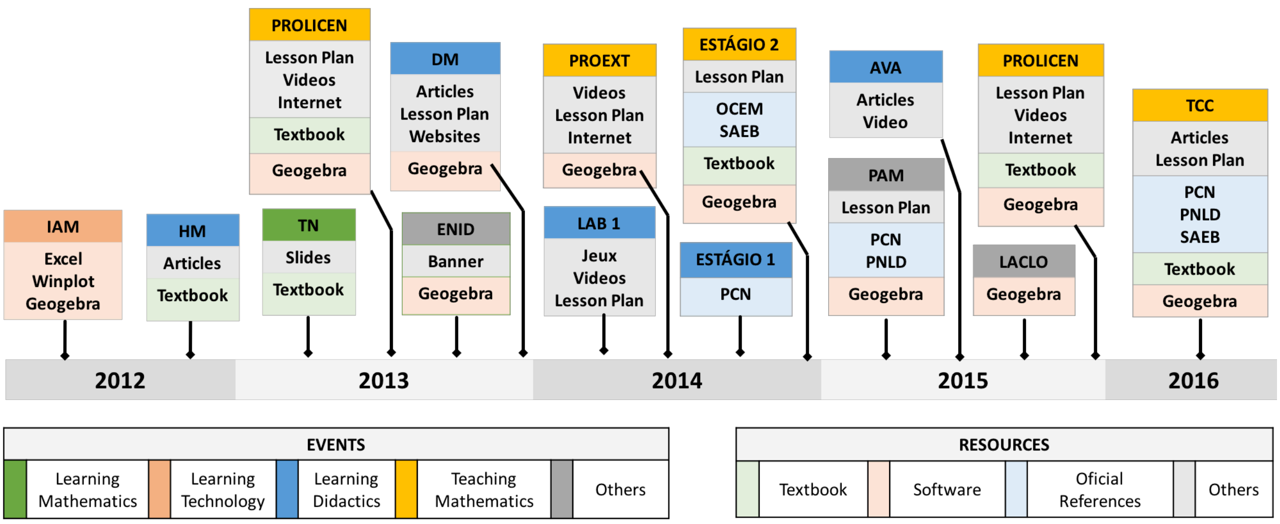

Figure 4 – Representation for Ceci's documentational inferred trajectory

Regarding the textbooks and considering their experiences at university, the pre-service teachers used them to study mathematics and to design activities; but they also had some lessons about textbooks as a subject of study. These experiences allowed them to have another idea about them.

In fact, during the interview, they mentioned that the textbook was a resource, but it was not the only one; the textbooks must be analyzed by the teachers considering possible mistakes; and the teachers can change the sequence of contents and also insert some themes.
From the interview, we observed that the pre-service teachers' conceptions about textbooks were related to their school's teachers when they were students. That is, a resource for guiding the teacher in class and also designing the lesson plan at home. They also mentioned their experiences with the textbooks during their school time. In general, the textbook was used to be the major resource for studying Mathematics and also an important resource for their teachers. Their teachers used to use it for planning the lesson at home. In class, the teacher used it to give a definition or some examples and also to propose exercises to be done.
Ceci said that she usually took notes from the blackboard and rarely did she study from the textbook. Using it as a reference, it was used to do exercises and to see check the answers. Marcos said that he appreciated the geometry classes because his teacher used drawing instruments and he was always engaged with solving problems. Carlos said that his teachers did not use any other kind of resource beyond the textbooks and, in class, they explored the problems and exercises in the textbook.
Regarding the curricular resources, they are introduced in the last two years of the course during four disciplines dedicated to curricular internship: two of them dedicated to internship at the 6^{th} - 9^{th} grades of elementary school (*Estágios* 1 and 2) and the others to High School (*Estágios 3 and 4*).
During these disciplines, the curriculum resources for the three pre-service teachers played an important role in the professional development, giving them the "feeling of" being a teacher. During the interview, they talked about responsibilities and the knowledge one needs to become a mathematics teacher, among them, knowing the *PCN* and understanding its conceptions.
Considering the inferred documentational trajectory, we conclude that Geogebra played an important role in the pre-service teachers' career. In fact, from the trajectory, we can observe that during the school time they had no math classes with the use of computers, particularly with dynamic geometry. On the other hand, at university, they developed a diversity of activities (events) involving Geogebra, being used to study mathematics or to develop teaching situations. As a consequence, we identified some influences of this new resource (and the dynamic geometry concepts) in their planning.
The first one is related to the contrast between the textbook (static objects) and the dynamic geometry (dynamic objects). This conception conducted them to engage differently with the textbook: Ceci adapted two textbook exercises to explore dynamic geometric constructions and, as a result, to allow students to understand the triangle classifications (by sides and by angles) as proposed in the textbook; Carlos considered a problem in a textbook which was about linear function but added two extra items to explore the coefficient influences over their graphic representation using the tool "selector".
The second influence is related to the idea that Geogebra might improve the learning in Mathematics through exploration of situations gained from the geometrical constructions. For example, Ceci proposed exploring the isosceles triangle characteristics considering just one geometrical construction which comprises a family of isosceles triangles; Carlos proposed identifying the influences of the coefficients through observation of a variety of graphical representations of linear function; Marcos proposed a comparison between a square as drawing and as a geometric construction to promote the students' perception about the square properties through the differences between them.

Final Remarks

Concerning our research question 'how are the processes by which three pre-service teachers engage with a particular curriculum resource (PCN) and textbooks, in order to prepare a math class: similar processes, due to a particular resource, or a diversity of processes, due to the diversity of resources? We identified three sources of influences over the engagement with resources.

This sources, altogether, indicate that the process of engagement is related to a diversity of resources but also to the particularities of each one and the integration between them, which is built over time through different experiences.

The first source is related to specific characteristics of the two resources - PCN and textbooks. Considering PCN, it was used as a resource for learning about general teaching aspects (technologies and education, didactic orientations and curriculum information) instead of having been used to plan with. This use is comprehensive because the PCN particularities (forms of address): it is a long text without objective suggestions for teachers, unlike textbooks.

The second one considers previous experiences with these resources, at school and at university. In fact, the experiences they had as students at school led them to conceive textbook as a resource for students to solve problems and, for a teacher, a resource to guide their class. For this reason, we affirm that their use of the textbook was limited to finding appropriate exercises for their lesson plans, despite changes as adaptation, addition of items related to the textbook exercises/problems.

The pre-service teachers were also influenced by the discussion about textbook at university. In fact, they created their own proposals adjusting the textbook exercise/problems, but these changes were also based on the desire of using the Geogebra. This aspect is related to the last source of influences under the engagement, the integration with other resources.

In fact, the contrast between the textbook (static objects) and the dynamic geometry (dynamic objects) but also previous experiences with the Geogebra were fundamental influences in their designing and in their engagement with the *PCN* and with the textbook guiding their design.

This research also allowed us to preset our methodological choices as a way to answer the question “How might we understand the processes by which teachers engage with curriculum resources to design instruction?” We started considering a resource-reference (TCC report), and from it, we identified the uses of the explicit resources. Since we did not follow the pre-service teachers’ activity itself while they were preparing their lessons or using resources in the classroom, we had access to only a part of their resources and their documents. Next, the representation of pre-service teachers’ documentational trajectory, followed by interviews, clarified the two first sources of influences over their engagement with the *PCN* and textbooks. The last interview about how they used the *PCN* and textbooks resources was fundamental to confirm the integration between them resources as a source of influence and to reinforce the others.

The use of not explicit resources in the resource-reference confirmed our perspective of the iceberg model for the uses of the resources.

This research opens two perspectives. The first, improving methodological choices to achieve the uses of resources, particularly the uses not explicit ones. Next, considering the documentational approach to didactics, our research indicates the importance of understanding the “production of the first document” by them, and probably of thinking about the process of formation of their first documentational systems.

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AN INVESTIGATION OF CHINESE MATHEMATICS TEACHERS' DOCUMENTATION EXPERTISE AND THEIR PROFESSIONAL DEVELOPMENT IN COLLECTIVES

CHONGYANG WANG, LUC TROUCHE AND BIRGIT PEPIN

Abstract

The boom of technology and a plethora of Internet resources are likely to enrich mathematics teachers' teaching resources and the forms for their collective work. This creates a new complexity for both the work of teachers and researchers who are interested in teachers' work and their professional development. Such situation has led to the emergence the Documentational Approach to Didactics (DAD) (Gueudet, Pepin & Trouche 2012), which provides a perspective for investigating both teachers' individual and collective work with resources. With a particular interest in the knowledge aspect of teachers' interaction with resources, we propose the notion of Documentation Expertise (DE) based on the notion of documentation work in DAD, and selected methodology tools, such as Documentation-working Mate (DWM), for studying it. To contribute to the symposium topic "teacher-resource use in the world", we situate our study in China, where teachers' collective work is part of teachers' daily work, which is evident in their various activities in the so-called Teaching Research Groups (TRG). We investigated two middle school mathematics teachers from the same school (one advanced teacher, and one novice teacher). We address (and compare) their individual work with their personal resources, and their collective work in the TRG. In terms of results, we develop insights into selected components of their DE, and how their DE is developed through collective work. This study is part of the first author's PhD project[1] (2014-2018).

Keywords: mathematics teachers' resources, teacher professional development, documentational approach to didactics, documentation expertise, pedagogical design capacity, documentation-working mate

Introduction

Aiming to contribute to the Symposium C "teacher-resource use around the word", we situate our research in the context of China. Moreover, we have a common interest in teacher expertise (Pepin, Xu, Trouche & Wang, 2016) and design capacity building in relation to resource use (Leshota & Adler, to be published; Pepin, Gueudet, & Trouche 2017; Remillard, to be published). The issues we address in our study are: (1) "how can we understand the process by which teachers engage with

[1] This PhD project entitled "An investigation of mathematics documentation expertise and its development in collective work: two contrasting contexts from China and France" started in September 2014, co-supervised by Luc Trouche at ENS de Lyon and Binyan Xu at East China Normal University, with Birgit Pepin at Eindhoven University of Technology as external supervisor.

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curriculum resources to design instruction?" and (2) "how does teachers' collective work influence teachers' design decisions over time?"

In 2014, we conducted a pilot study in China (Pepin et al. 2016), exploring the interactions with resources of three mathematics expert teachers from the same high school. Results show that (1) mathematics teacher (knowledge and) expertise has many facets, and one of them relates to their design activity (e.g. lesson preparation); (2) teachers' expertise of interacting with resources and their design activity seem to be enhanced by working in collectives (e.g. TRGs in this study); and (3) the 'institution' of TRG in China is a special form of mathematics teachers' collective work. In line with this study, we further investigated new Chinese (and French) cases with respect to their specific expertise in terms of teachers' resource work and its development in collectives.

We are interest to see: (1) From the individual perspective, what elements of the expertise in resource work could be found? (2) From the collective perspective, how does the collective activity promote such expertise development?

In this paper we start with our theoretical framework of DAD, where the notion of DE will be proposed and defined. In the second section, the 'institution' of TRG in China is described as the context of teachers' collective work in our case study. In the third part, we explain the methodology and tools inspired by the DAD. In the fourth section, we present our results from the individual perspective of the teachers' resource preference, and subsequently from the collective perspective of their cooperation in TRGs. In the fifth section, we discuss our results and develop insights resulting in our conclusions.

Theoretical Framework

In this section, first we situate our study in the frame of the *Documentational Approach to Didactics (DAD)*. Second, in order to investigate the expertise aspect of teachers' working with resources, we propose the concept of *Documentation Expertise (DE)* and develop a definition of DE.

The Documentational Approach to Didactics

Due to the difference in the language translation of 'resource' from English to Chinese, several researches (Xiang & Wei 2005; Jin 2013) discussed on teachers' conception of 'resource' in the field of technology education in China, aiming at reflecting and broadening teachers' understanding of available teaching resources that have emerged along with the fast development of technology. Moreover, there has been recent research investigating and defining digital curriculum resources (e.g. Pepin, Choppin, Ruthven & Sinclair, 2017), as compared to 'traditional' curriculum resources (e.g. texts) (see for example, Pepin & Gueudet 2014). Thus in this study, we keep the word "resource" rather than "curriculum resource" or "digital resource" which allows us to hold a broader definition of teachers' available resources. As Adler (2000) stated: a resource could be anything with the potential to "re-source" a teacher's activity.

Situating our work in DAD (Gueudet et al. 2012), we name resource as something encompassing materials and elements intervening "upstream" of teaching, such as emails, websites consulted, students' work etc., or even the products of interactions with their colleagues (Pepin et al. 2016). According to DAD, the interactions between teachers and resources, including retrieving, selecting, adapting, saving and sharing, were defined as *documentation work.* The work with resources results in *documents*, which consist of resources and the corresponding *schemes* of resource usage. A scheme is defined by Vergnaud (2009, p. 88) as "*the invariant organization of activity for a certain class of situations"*, which include the following four components: *goal(s)* and anticipations that guide the activity, *rules* that retain "*sequences of actions, information gathering, and controls*", *operational invariants* that is the "knowledge in action", and *inferences* that allow to take in account the singularities of the situation. Documents are considered to be developed through documentational genesis and articulated in a structured *documentation system*. Correspondingly, the *resource system* relates to the "resource" part of the documentation system (without the scheme part of the documents), and analyzing the documentation system and its evolution permits the study of the teacher's professional development (Gueudet & Trouche 2009). We propose two continuous dimensions for analyzing teachers' documentation work (see figure 1 below): (1) a time dimension, documentation work is a dynamic and continuous process along with teachers' professional development (however, the research analysis can be situated in a single moment (static), or part of thread in this dynamic process (long-term)); and (2) an individual-collective dimension, documentation work can be studied with individual teachers, and/or considering collectives where the individual participates as a unit.

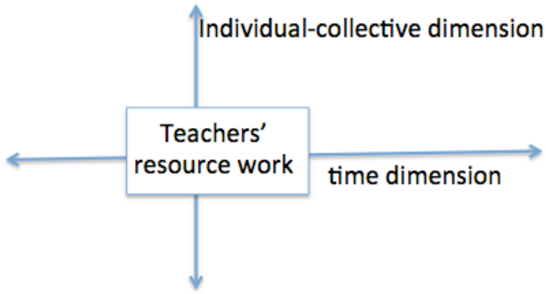

Figure 1 Two dimensions for analyzing teachers' documentation work

Documentation work refers to the complex and interactive ways that teachers work with resources, in and out of class, individually and collectively (Gueudet et al. 2012), because the work of teachers is neither isolated nor individual, but culturally and socially situated (Gueudet, Pepin & Trouche 2013): this is expressed in the "from individual to collective" continuum. Meanwhile, due to the mutual influences between teachers' documentation work and their professional growth, documentation work is a dynamic process consisting of continuous moments to capture and study: this is expressed in the "time" continuum, from static to dynamic. The methodology and tools will be developed along with these two continua.

Documentation expertise

As we have found in our pilot study (Pepin, et al. 2016), teacher expertise and its development can be evidenced by particular ways of resource sharing and reflecting on their usage. Linking to this study, we defined *documentation expertise (DE)* as the expertise in teachers' interactions with resources. As Berliner (1988) stated, expertise is "specific to a domain and developed over hundreds and thousand of hours", teacher expertise takes different forms in different cultures and teachers' working conditions exert a powerful influence on the development of their expertise (Berliner 2004). "Expertise is best thought of as a prototypical concept, bound together by the family resemblance that experts bear to one another" (p.16), and "there exists no well-defined standard that all experts meet and that no non-experts meet"(p.9) (Sternberg & Horvath, 1995). In the Chinese cultural context an expert mathematics teacher is expected to fulfill multiple roles: such as expertise in teaching, conducting research and publishing papers, and mentoring teachers, being a scholar of mathematics/theory/characteristics of learners/curriculum and exemplary models for students and colleagues (Yang 2013).

As our specific interest is on teacher expertise with respect to lesson design with resources, we refer to Brown's (2009) notion of Pedagogy Design Capacity (PDC) as "a teacher's skill in perceiving affordances, making decisions, and following through plans" (p.29) (in science education). Later studies develop PDC from different aspects: by adapting it in analyzing teachers' curriculum resources usage in classroom teaching (Remillard, to be published), by disaggregating PDC into different levels (Leshota & Adler, to be published). Pepin et al. (2017, b) defined, and refined, the notion of mathematics "teacher design capacity", distinguishing the following three components: (1) the goal/s of the design activity, (2) a set of principles, and (3) reflection-in-action. These components kept a clear resemblance with Vergnaud's conceptualization of operational knowledge and his notion of schemes.

The notion of DE we proposed in this study is similar to the PDC defined by Pepin et al. (2017, b), enlarging it by taking into account not only the lesson design phase and lesson implementation phase, but the whole process of documentation work: a life circle of producing a document through resource retrieving, selecting, organizing, modifying, adapting, implementing and sharing off, of an individual teacher, and also of her interactions in collectives. We assume DE as a type of teacher expertise towards resources, in both the resource conception (what could be a resource for the individual teacher), and the resources usage scheme (how the resource will be further developed and used). DE relates this specifically to ‘resources’, which makes it narrower than the expertise defined by Berliner (1988). It grows with teachers’ professional development over hundreds and thousands of hours, and there are no clear boundaries/levels of the process.

In this way, we define DE as the knowledge and skills in interacting (retrieving, selecting, organizing, modifying, adapting, implementing and sharing off) with resources. According to the four components of scheme (a goal, rules of action, operational invariants, and inferences), DE locates more in the part of operational invariants, namely the conceptions-in-action and theorems –in-action (knowledge in action) as stated by Vergnaud (2009). We assume DE as a developing process consisting of developing states, which means that also novice teachers could have DE. We hypothesize that the collective work is likely to offer the teachers a platform to contribute to/from expertise.

The Chinese Context

In this section, we introduce the context of collective work in China, both the origin of culture and support from institutions. Then we explain the history of TRG and its regular working modes, and finally, the principles and procedures of MOKE, as an emblematic way of working collaboratively in China.

The importance of collective work in Chinese culture

In China working collectively is considered as essential. Confucius says that “Whenever walking in a company of several persons, among them must be someone worth learning from (三人行，必有我师)”. From the cultural point-of-view the school-level working culture in China has been described as *collective* (Yang, 2013). Research on teacher education in China shows that Chinese teachers are benefiting from some school-based collective working means (An, Kulm & Wu 2004; Li & Huang 2008; Pepin et al. 2016; Wang 2013): they gain a deep understanding of basic mathematics and adequate pedagogical expertise through specific systemic structures.

The regular structures of teachers’ collective work in schools

The word “TRG” first appeared in Chinese Education Ministry regulation in 1952, aiming to “study and improve the way of teaching”. In 1957, the property and tasks of TRG were emphasized again and more clearly stated (Wang 2013). Since the 1990s, TRG undertook the work of carrying out post-1990 curriculum reform. From 2001, encouraged to participate in educational experiments, TRG slowly started some “real” research work, represented by “school-based research" (Gu & Wang 2006). Now the TRG has become the basic unit for teachers’ collective work in every school, the main platform where teaching resources are generated and shared through the regular collective activities. Generally, a TRG consists of teachers from the same discipline, such as mathematics TRG, or English TRG. In bigger schools, in addition to the TRG, there can be different Lesson Preparation Groups (LPG) based on grade, like a “mathematics LPG for grade 6”. In most of Chinese schools, teachers work full time, in a permanent office with their own office desk; teachers from the same LPG are usually organized to share the same office, which allows them to communicate and work collectively face-to-face often and conveniently.

MOKE: an emblematic activity of Chinese teachers' collective work

Among the Chinese studies of teacher expertise, several of these studies explored and defined teacher expertise in terms of stages (Berliner 1988, 2004), corresponding to a list of characteristics and roles, with common indictors towards collective work: e.g. being a leader in collectives with creative minds (Lian 2004); being an educator/mentor and moral/scholar example for other teachers (Yang 2013). This prototype way provides a "perfect" state, summing up almost each good quality. Moreover, they indicate that in the Chinese context teachers' work and the issue of expertise/professional development is never an individual one – this calls for special attention to collective activities that mostly happen in TRG.

The working modes of TRG could be classified into (1) "task-based activity", and (2) "operation mode of diagnose-based activity" (Hu & Wang 2014). The former (1) is represented by collective design of resources, such as the development of school-based exercise books (a series of exercises collection or lesson plans produced by the teachers collectively in the same school, some are published and some are printed and adapted only within their school); whilst the latter (2) is embodied by MOKE (an colloquial expression of "collective lesson preparation" used among teachers, see in Chen (2006). MOKE consists of:

(1) Individual lesson preparation by some teacher;

(2) Lesson implementation in front of the collective (as an open class);

(3) Collectively discussion seminar after the open class;

(4) Individual lesson refinement and modification;

(5) Repeat (2)-(4) till the lesson design reaches the expectation of most other teachers in the group;

(6) Resulting resources (e.g. lesson plans; courseware) submitted to TRG and shared by all teachers in TRG.

Several rounds of the process above constitute what is called a MOKE activity in China.

Generally, the leader of TRG will announce the topic or task of each TRG activity one or two weeks before the formal TRG activity time. Teachers can attend the lectures or teacher training sessions, and/or participate in MOKE activities. MOKE is considered the most important way for teachers' professional development (Hu &Wang 2013), especially for novice teachers (who are always asked to prepare open lessons) and experienced teachers (who are often expected to instruct the novices). In our study, an in-depth follow up of teachers' collective activity was situated in a MOKE activity; details of the choice will be presented in the methodology part.

Methodology

Drawing on the DAD, we first present the methodology of reflective investigation of DAD, and then propose the notion of *Documentation-working Mate* for a better understanding of the influences of collective work. Finally, we present our research design.

Reflective investigation

Reflective investigation involves the teachers as part of the study throughout the whole data collection. Teachers' reflections on their previous answers provide a link to previous data and hence a continuity aspect, and an opportunity to identify changes and developments. Four principles are emphasized: long-term follow-up; in- and out-of-class follow-up; broad collection of the material resources used produced throughout the follow-up; and reflective follow-up of the documentation work (Gueudet et al. 2013, p. 27).

To know the landscape of teachers' available resources, and how they organize and represent their resources, we drew on our pilot study (Pepin et al. 2016) and expanded the *Schematics representation of Resource System (SRRS)* to develop a specific tool, *"Inferred Mapping of Resource System"(IMRS)*. This is a structured mapping of the teacher's resource system drawn by the researcher first, based on the observation and the self-presentation of teachers' available resources. Then, on the basis of the IMRS, a Reflective IMRS (R-IMRS) is developed through a further interview where the teacher is asked to make modification/complementation and explanation on the IMRS. It should be noted that the IMRS and R-IMRS are not final representations, but can be improved, complemented, and reorganized continuously during the long-term follow up, along with different mathematics contents, and the development of teachers' reflections on their own resources.

Other tools were also developed for obtaining details of teachers' collective interaction. An online *"Reflective Investigation Box (RI Box)"* (Wang, to be published; Rocha, to be published) was set up and shared between the researcher and the teachers. Here the researcher could observe the messages/resources shared among the teachers, or propose online interviews with the teacher/s. The choice of technology for supporting RI Box depends on the applications used by the teachers: in our study, we adapted Wechat (an instant social communication application with functions of resource sharing and group chatting) for this purpose. In addition, field notes of teacher activity observations and school visits (by the researcher) were also added. The combination of field notes and RI Box provides the possibility of a long-term follow-up of the teachers' resources, for example, which resources are considered and incorporated, where these resources come from, and how they are integrated.

Documentation-working Mate

The notion of *Documentation-working Mate (DWM)* was proposed in order to better understand the influences of collectives on teachers' documentation work. In our study, collective is a group of teachers in TRG. We define DWM as a person who works closely with the targeted teacher, with mutual influences on their documentation work and DE development. *Mate* (in the Oxford Dictionary) refers to "a fellow member of joint occupant of a specific thing, like table-mate" (with the "underlying concept being that of eating together)". This indicates four characteristics of DWM: (1) s/he is chosen by the targeted teacher, not the researcher; (2) each teacher could have several DWMs, such as mentor/apprentice, trainer/trainee; (3) the most important indicator for choosing DWM is "interacting most frequently"; (4) unlike the notion of "peer" in the field of "peer education"(Turner & Shepherd, 1999), there is no boundary or constrains of age or education background or expertise levels for becoming DWM, they could be both advanced or novice teachers, or colleagues with different working experiences.

Research Design

In the school where we selected our targeted teachers, there were 12 mathematics teachers in the mathematics TRG, with three mathematics teachers in each grade (from grade 6 to grade 9). Each Tuesday afternoon was the regular time for collective TRG activities. We chose two teachers who worked closely and frequently as a mentor-apprentice relationship, Gao and Yao. According to the five-stage model of pedagogical expertise of Berliner (1988), Gao belonged to the group of expert teachers (with 24 years teaching experiences and ability to deal with teaching problems effectively and effortlessly), while Yao was a novice teacher (with less than two years' working experience).

Gao had been a mathematics teacher in middle school since 1993, and she was one of the most experienced teachers in her school. Compared with other mathematics teachers, she was not a very "traditional" teacher with professional training, because she never majored in mathematics, neither in her college education, nor in her undergraduate education (in 2003, she continued her bachelor study in education management after a "top-up exam"[2]). After graduation she started to work as a middle school mathematics teacher. She got her "first-class title" in 2000. She was a mathematics teacher of two classes in grade 8 when we conducted our data collection. She was the ex-leader of Mathematics TRG in her school, and she worked as the leader of Lesson Preparation Group in grade 8 then.

[2] An exam allows the students from vocational colleges to upgrade to university.

As for her DWM, Gao chose Yao as one of her DWMs, who worked as the apprentices of Gao since September 2015 after she graduated with a master degree of mathematics education. She taught two classes in grade 6. Although working in different grades with her master Gao, she always turned for Gao's help and instructions when she met problems. In the interview for choosing a DWM, Yao also chose Gao as the first choice. The two-way choice between Gao and Yao helped us to select a smaller collective in their TRG, bringing the opportunities to see Gao's expertise from Yao's perspective, with the following tools:

- 1) At the beginning of the follow-up, we spent a three-week full-day observation of Gao's work in both her office and classroom teaching, with a particular interest in the resources Gao prepared and used for her classroom teaching, and also those she shared with her colleagues.
- 2) For both Gao and Yao, we adapted the tools of IMRS and R-IMRS. We conducted an interview with each of them for one hour: about their working experiences especially with resources, their resources for classroom teaching, lesson preparation, and the collectives or persons who inspired their resources work. Then, the researcher drew the first version of IMRS based their self-presentation on resources. With the IMRS, the teachers were asked to modify and complement this IMRS through a further reflective interview. In this way, we got two versions of mapping of resource system for each teacher: IMRS and the R-IMRS.
- 3) Between Gao and Yao, we filmed a series of videos on the collective MOKE activities. Generally each year (from March to April) in this school was the period when new teachers prepare and give open lessons. When Yao got this task, she was required to prepare it with the instructions of Gao. They conducted a total of three rounds of MOKE activities, and we filmed their discussions.

In the following section, we analyze the data with respect to our research questions: (1) the elements of DE found in their documentation work and (2) the way that collective MOKE activity promotes the development of it.

Findings

Our analysis is based on the two dimensions of documentation work: (1) the elements of DE shown in the R-IMRSs at a given moment (static) by the individual teachers; and (2) the elements of DE and influences of the intense MOKE activities as collective work over a period of time.

Analysis of R-IMRS

In this section, we propose an analysis of teachers' resource systems based on their R-IMRSs, and then a comparison of the results from a DE point of view.

(1) Gao's case

We analyze the main features of Gao's resource system, justifying our drawing with the follow-up interview. Then we focus on the changes and complements made by Gao herself in the R-IMRS. Finally, we infer some consequences of the analysis for the individual-collective dimension.

- There exists a structure of resource "input" and resource "output" (see figure 2), which could be evidenced in Gao's self-presentation of her resources: her resources for lesson preparation and classroom teaching are mainly in form of material, including official curriculum resources (e.g. textbook, teaching guidance book and exercise book along with textbook), various learning aide books[3], and most importantly, the self-owned resources that she purchased and accumulated herself, which is the only part linking to her resource "output", self-developed resources. She selected and

[3] Learning aid book is one type of learning aid materials, which is edited by some educators or teachers, mainly along with the teaching contents or exams, and could be bought in the bookstores.

accumulated the resources (i.e. the exercise items) from the learning-aide exercise books that she bought for the bookstore (she considered these as her own resources), and she developed the accumulated resources into school-based exercise books (printed and sent only to the students in her school) and published exam/exercise book (by national publishers and available in bookstores). TRG activities and participation in research projects with research institutions and superiors (Teaching Research Office in Pudong District) was also a way to reflect and show off her resources.

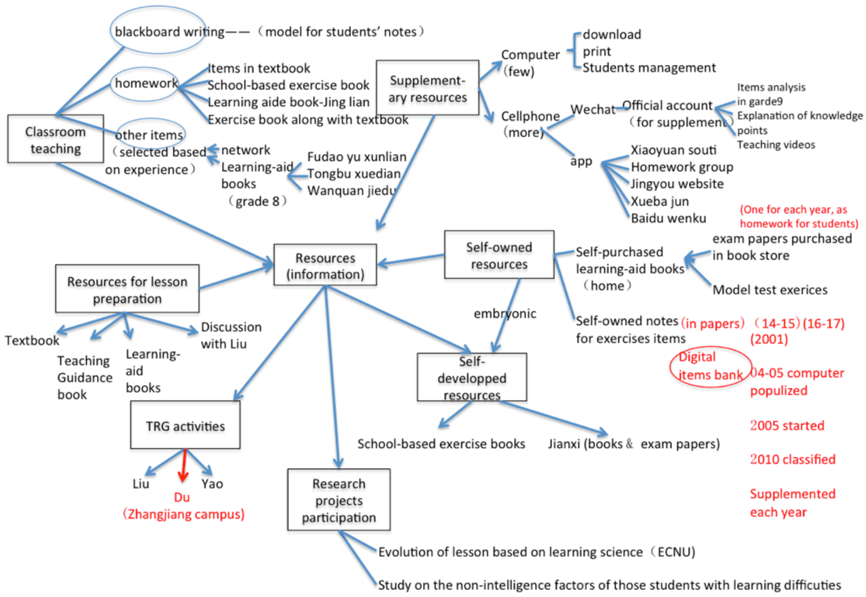

Figure 2 The R-IMRS of Gao

She used traditional material resources more than online digital ones; online resources (both from computer and cell phone) worked as supplementary resources. She said that although for her "resource" was "a kind of information" (cited in interview 1 with Gao), she was positively inclined towards traditional resources:

> "I seldom use my computer, neither the software like GeoGebra, I prefer to draw graph on the blackboard with chalk and teaching instruments, so that the students can observe the drawing process with deeper impression" (cited in interview 1 with Gao, personal translation).

She explained her scheme of using the material resources and those from the Internet:

> "With the development of cell phone, a lot of websites and forums have their own applications or Wechat official account, so I use cell phone more than computer, it is more convenient, look, I can take pictures with it and send it out at once" (cited in interview 1 with Gao, personal translation).

She explained why she considered the online resources only as a supplement to her teaching resources, "because the contents is not specially for some specific lesson, for what I will teach each day." (Cited in interview 1 with Gao, personal translation)

She had a habit of systematically and efficiently accumulating and referencing her resources, which allowed her to be a resource-developer for others (both the students and the teachers). It was clear that she knew the learning aid materials market well because she bought a lot, and she insisted on visiting the bookstore at least once each semester, "to see whether there is any changes, because they modify it almost each year after the exams, if there is, I will buy them" (Cited in interview 2 with Gao, personal translation). At the same time she also acknowledged and knew the feedbacks from the users – her students: With two classes of about 70 students to teach, and 10 lessons each week, Gao spent a large amount of hours (of her working day, from 7 am till 5 pm) on marking her students' homework. She carefully selected the exercises and assigned to her students, "because there are too many choices for the students in the book markets." (Cited in interview 2 with Gao, personal translation). Subsequently, she collected all her students' homework, checked one by one, marked each mistake, and asked the students to correct the mistakes in her office (vis-à-vis her classroom. In this way she identified and selected "truly valuable" (in terms of identification of misconceptions) exercises for students, and the experiences how to teach the students to learn from mistakes. She wrote down all the items in a paper notebook, and screened them to store in computers or share with others.

(2) Yao's case

The R-IMRS of Yao was less complex than Gao's (see Figure 3): she considered resources as "the things available to be used for teaching" (cited in interview 1 with Yao).

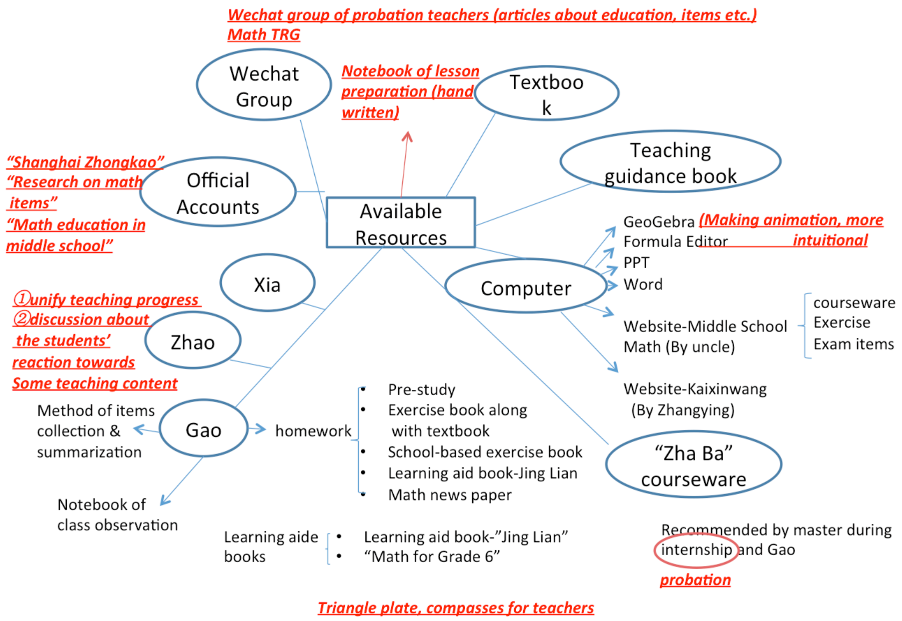

Figure 3 The R-IMRS of Yao

- In her self-presentation there was a category of "available resources": official curriculum resources (textbook and teaching guidance book that each teacher has); resources from computer and cell phone (mainly Wechat Group and Official Accounts); Resources from other persons (like Xia, Zhao, Gao), including "Zha Ba[4]" courseware which was shared by Gao.

- Resources from collective work with other teachers occupied a large part in her resource system, which could be evidenced by the list of names in the left-down part, and also besides other resources, such as "Zha Ba" courseware, websites in the "computer" parts, she mentioned in detail who recommended these resources.

- The usage of the resources she obtained from other teachers was mainly a kind of "imitation", which could be evidenced in her explanation of the resources from Gao: she used the same series of learning aide books as homework for students in a same way; she prepared her lessons mainly reference on her observation on Gao's lesson (where she kept Gao's teaching procedures, examples, exercises etc.); she was also trying to follow the suggestions from Gao about how to collect and accumulate the exercises items from the learning aide books.

(3) A comparison of the two R-IMRSs

In this section we will compare the two cases from an aspect of collective mutual benefit.

[4] "Zha Ba" is the name of a middle school in Zhabei district of Shanghai, "Zha Ba" courseware was developed collectively by the mathematics teachers in this middle school.

Gao and Yao were resources for each other. When we proposed the question of "in your teaching experience, who is the one who influence you most", both Gao and Yao chose their masters, an experienced teacher who instructed them a lot at the beginning of their career. This was evidenced in Yao's resource system: they appeared in the R-IMRS of each other, but in a different position, Gao for Yao was a main "human resource" who provided various resources continuously, while Yao for Gao was an "output" for providing suggestions and varying the effects: "When I instruct her, I also experience a reflection" (cited in interview 1 with Gao, our translation).

Interactions with experienced teachers and participation in collective work were a main resource for the novice teacher, Yao. In her self-description of her available resources (see in Yao's R-IMRS), she categorized her available resources according to the providers, with the specific suggestions of usage from these providers, which for her, is more like to obtain operational knowledge more directly.

Technology hold a clear influences on teachers' resource work and their resource system, and this could be evidenced by a social communicate software, Wechat, which appeared in both of their R-IMRSs: Gao introduced several favorite official accounts that often publish useful articles about mathematics teaching; while for Yao, besides the official accounts, she also introduced the Wechat groups, which is a group chatting function of Wechat, allowing teachers discussed online instantly in a same group. It also evidenced in the "resources output" of Gao's R-IMRS, she often shared resources (like articles, exam items, or messages, pictures of her notes or exam papers etc.) in the group chatting, which for Yao was an important way to receive resources from other teachers.

Reflecting on the knowledge-in-action (i.e. operational invariants), we could find some DE components from Gao: DE in systematically and efficiently resource accumulation by knowing well the learning aide material markets well and regular collection based on her homework marking experiences and feedbacks from the students; DE in sharing off resources as well as the usage by donating instructions face to face and sharing related articles online instantly; DE in using digital software such as GeoGra critically by considering the needs of the students' better learning in process.

In the following section, we pay particular attention to a moment of intense collective work between Gao and Yao, and analyze how the two teachers could feed each other.

An analysis of the MOKE activity

The lesson to be prepared collectively in the MOKE activity we followed was assigned to Yao, entitled "the properties of inequality" (mathematics content in grade 6). It was Yao who decided the topic, with the recommendations/instructions of Gao (they could choose one topic that would possibly be taught during the MOKE period according to their teaching plan). The three rounds of MOKE activity resulted in three versions of lesson plan. Evidenced by the three lesson plans, the biggest changes appeared in the parts of "introduction or warming up activity" and "the chosen/difficulties of example items and exercises". In this section, we choose some moments that promoted the changes in Yao's lesson plan, and evidenced the elements of DE and the influences from collective interactions.

- The consideration on combining the "difficulty" and students' performance level ran through all the lesson design of Yao and Gao. According to the arrangement in the textbook, there should be three properties of inequality to be taught, but considering the performance and basics of the students (they were not top students), Gao suggested Yao to teach only the first one: "for a given inequality, when adding or subtracting a same number, the direction of the inequality sign will not change". Her scheme of grouping the difficulties and the quantity of exercises for this lesson appeared to be: "(She suggested me to) try the most typical and difficult items and adjust them according to the reactions of the students" (interview 3 with Yao, our translation). Gao added several exercise items in Yao's first lesson plan, and then she suggested deleting almost 1/3 of them after she felt the students were tried and less interested after her observation.

- Yao showed her advantage, as a young teacher with sensitive towards news, in the choice of the introduction activity for the students: according to the lesson plan, Yao planed to introduce the concept of "in-equality" through an activity, she designed and tried two but Gao was not satisfied with the effects. In the last version Gao adapted the "air pollution index" that she saw in the metro screen everyday, combining the lesson topic with students' daily life. This arrangement was highly appraised by the experienced teachers including Gao. Before this, Gao used to suggest to Yao to operate an experiment with a balance, and let the students observe. But she herself denied this arrangement, after Yao conducted this activity in the second round of MOKE, based on her observation on the students' reactions and classroom atmosphere: "it is too far when you operate it, the students who sit behind were not listening to you, and there is also the problem of the weights, what if putting on different weights, like one side with 10g, and another side with 20g? The direction of the inequality did not change, when we put on different weighs… too complex…" (cited in Gao's words in the 2nd MOKE, our translation). This process evidenced the mutual beneficiation for both of the novice and experienced teachers.

As a DWM of Yao, in the MOKE activity, Gao was not only a master in instructing her, but also a co-defender with Yao, in front of other teachers, especially in the 3rd MOKE discussion. Among the exercises in the students' work sheet, there was an exercise about "the properties of equality", so a teacher doubted that this was off-topic. Gao insisted to maintain this exercise: "I would never prepare a lesson only for a lesson, knowledge should never be isolated, it should be linked to the previous, and we should remind the students do not forget the previous knowledge when they learn new things." (cited in Gao's words in the 3rd MOKE discussion, our translation). This evidenced Gao's principles or conceptions towards open lesson, and also her roles of both the instructor in lesson design and resource co-producer.

Some details of how Gao instructed the other teachers were also found. For instance, she observed Yao's lesson in the first round of MOKE, and took pictures of Yao's blackboard writing, then she showed this picture to Yao in the following discussion: there was a mistake in her writing and she emphasized that "teachers' blackboard writing must be precise, because it is the model for students' note, if you write wrongly, they will misunderstand." (cited in Gao's words in the 1st MOKE discussion, evidenced also in Gao's R-IMRS).

Reflecting on the knowledge-in-action (i.e. operational invariants), some elements of DE could be evidenced from Gao: (1) When designing a lesson, the principle of 'stick to the topic' is important, but the needs of students is deserved more attention, any open lesson should be designed and linked with what the students had learned before; (2) when instructing novice teachers' teaching practices, knowing the characters of teachers is an important precondition for giving specific instructions, Gao stated in her 1st interview that "Novice teachers are easier to make mistakes in details like wrong blackboard writing or inaccuracy in oral expression." To arise Yao's self-reflection, she took pictures and notes then showed the 'evidences' to Yao in the following discussion.

Discussion of results and conclusions

By analyzing the static R-IMRS and dynamic MOKE activity from the aspects of individual and collective activity, we draw inspiration from the topics in Symposium C:

About the issue of "process teachers' engage with resources in lesson design": The lesson design for teachers was not an isolated task, but organized into their "lived" and developing documentation system. For both Yao and Gao, they "perceive[d] and interpret[ed] existing resources, evaluate[d] the constraints of the classroom setting, balance[d] trade offs and devise[d] strategies (Brown 2009, p. 18)", and this process was an exchange between their practice and their documentation systems: they gained new resources, or new usage schemes.

About the issue of "influences from collective to teachers' design decisions": The process of MOKE, for all actors, was a crucial way to influence others, and each other. For the novice teacher, Yao, she experienced the complete process of carefully preparing a good lesson, with many ideas/suggestions/reminders from other members of her TRG. For the experienced teacher, Gao, she gave suggestions, observed the effects of implantation, and provided new suggestions for modification in her reflective comments, thus she contributed at a level of instructor and spectator. This evidenced the way of Chinese mathematics teachers obtaining "deep understanding of basic mathematics and adequate pedagogical expertise through some school-based means" (Wang 2013).

In this study, we have explored DE through the lens of Vergnaud's knowledge-in-action (i.e. operational invariants), by studying how the teachers expressed and organized their resource systems individually, and how they interacted with their DWMs collectively. Hence, for us two issues emerge: (1) the (notion of) DWM was useful for better understanding teachers' specific resource usage scheme, by comparing their resource system and their usage of specific resources in similar situations; and (2) in terms of DE, there were selected resource usages/ working habits with resources that supported their resource work in an efficient way. However, in order to obtain deeper understandings of the knowledge underpinning these actions (that were easy to observe), we claim new tools needed to be developed that could be useful, such as interviews that allowed for deeper reflection (based on the R-IMRS, for example), or logbooks for resource usage reflection (e.g. a designed table with the information of resources and its usage to be filled by the teachers daily).

A more developed and refined definition of DE with specific components and structure, as well as the elements for developing DE, is still in progress. Further follow up of mathematics teachers (in France and China) and related studies will be conducted to investigate DE further.

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RESOURCES FOR TEACHING: SUPPORTING A MEXICAN TEACHER'S LEARNING

JANA VISNOVSKA and JOSÉ LUIS CORTINA

Abstract

We analyze the role played by a teaching resource in effectively supporting a professional development collaboration with Irene, a Mexican public-school teacher. The resource is an instructional sequence on fractions that was developed through a series of design experiments in Mexican classrooms. As a result of this collaboration, Irene modified significantly her instructional practices. We discuss how the instructional sequence contributed to Irene's renewed view of a mathematics classroom by providing her with guidance that was explicit, specific, and achievable.

Introduction

Mathematics teaching is not the same around the world. National educational systems have their unique histories and are organized differently. Mathematics teachers work under different institutional conditions, receive dissimilar opportunities for their professional development, and engage with students whose cultural, social, and educational backgrounds often vary substantially.

We explore the case of a Mexican public-school teacher, Irene, who agreed to collaborate with us in a dual design experiment (Gravemeijer and van Eerde 2009), aimed at supporting both her professional development as a mathematics teacher, and her fifth-grade students' understanding of measurement and fractions. As a result of this collaboration, Irene modified significantly her instructional practices. In addition, she successfully supported her fifth-grade students in making sense of important mathematical ideas. This was especially significant given that very few Mexican children seem to get a fair opportunity to understand the targeted ideas as they go through compulsory education.

In this paper, we analyze the role played by an instructional resource in effectively supporting the professional development collaboration with Irene (cf. Pepin 2018). The resource in question is not a printed textbook, but an instructional sequence on fractions as measures. We elucidate the difference and discuss how the instructional sequence contributed to Irene's renewed view of a mathematics classroom by providing her with guidance that was *explicit*, *specific,* and *achievable*.

After we situate our study, we introduce the instructional sequence, the key principles that guided its design, and anticipations for its use. We then overview the dual design experiment, focusing on its professional development component and outcomes resulting from Irene's work in her classroom. Finally, we introduce our analysis of supports that facilitated Irene's transition to more ambitious and equitable instructional practice (Jackson, Gibbons, and Sharpe 2017).

Background

To situate our contribution, we refer to the state of mathematical learning in Mexico, particularly with regard to pupils living in harsh social and economic circumstance. For more than fifteen years, national and international assessments have shown a rather disturbing image. For instance, in a recent assessment conducted by the National Institute of Educational Evaluation (Instituto Nacional para la Evaluación de la Educación 2015b), 60.5% of sixth graders perform in mathematics below

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the “basic” level. The data also shows that the great majority of those low performing pupils are the children of families living in poverty.

Studies in which instructional resources are developed are not often situated in severely underprivileged classrooms, but teachers in these classrooms require support. Moreover, such classrooms are not unique to Mexico. Data form PISA (Organisation for Economic Co-operation and Development 2013) suggests that just within Latin America, where the ICMT took place, Argentina, Brazil, Chile, Colombia, Costa Rica, Peru, and Uruguay face similar challenges.

In Mexico, more than a decade of efforts at the public policy level included changing the curriculum, developing new official textbooks, implementing high-stakes tests, and establishing programs of special rewards for good teachers. Yet, these efforts did not result in a clear indication of improvement of students’ learning or performance. In this context, we hope to systematically explore the role that instructional sequences that are a product of careful design and experimentation in classrooms, can play in supporting teachers who work with low performing students. We situate our contribution within the space of teachers’ resources (Trouche and Fan 2018) and discuss implications of our analysis for conceptualization of educative curriculum materials (Davis and Krajcik 2005) designed for these teachers.

THE INSTRUCTIONAL SEQUENCE

We developed the instructional sequence on fractions as measures (Cortina, Visnovska, and Zúñiga 2014, 2015) to respond to concerns about limitations in the mathematical competence that most Mexican students achieve in their formal education. The results from both national and international assessments indicate that very few Mexican students develop the necessary mathematical understandings to mathematize and solve problems that involve continuous magnitudes, or require the use of rational numbers or multiplicative reasoning. For instance, in PISA 2012 (Organisation for Economic Co-operation and Development 2013) only 17% of Mexican sixteen-year-olds achieved proficiency levels (Levels 3 and above) that involve ability to handle percentages, fractions, and decimal numbers, and to work with proportional relationships. In terms of quantitative literacy (Steen 2001), this suggests that the majority of Mexican children leave schooling able to soundly deal with only a few types of quantitative situations – those limited to natural numbers, additive relations, and discreet quantities.

The instructional sequence was developed through a series of classroom design experiments in Mexican classrooms, following Cobb and colleagues’ methodological guidelines (Gravemeijer & Cobb 2006). Framed by the design theory of Realistic Mathematics Education (RME; Gravemeijer 1994), the sequence entails classroom activities that are experientially real for students, can guide them to reinvent mathematics by bringing in their everyday experiences, and provide the students with opportunities to create their own mathematical models (Cobb 2003).

In addition, the sequence was developed under the assumption that teachers necessarily adjust the instructional resources they use to the actual circumstances that they encounter in their classrooms. Hence, the sequence was not conceived as an instructional asset that could influence students’ learning directly. Instead, it was developed as a resource for supporting teachers in pursuing a fruitful instructional agenda (Cobb, Zhao, and Visnovska 2008).

As an instructional resource, the instructional sequence is not a printed collection of lessons, problems, and exercises that a teacher and her students can follow. Instead, it outlines a progression of students’ learning goals, along with the rationale for this progression, which includes the means of supporting the students’ learning at each step. Following RME, the purpose of this rationale is to support the teacher in making informed instructional decisions as she adjusts the instructional activities to the contingencies she encounters in her classroom. This would include decisions such as when to start pursuing a new learning goal and how to do it.

The instructional sequence on fractions as measures is intended to support a classroom community in reinventing length measurement, including the complexities of measuring the reminders of units.

The students are first expected to develop the need to *standardize the unit* that serves as a reference when measuring lengths. They then confront the problem of measuring the lengths that the reference unit does not cover exactly (i.e., remainders). Unit fractions are then introduced as a means of producing measurement subunits—smaller than the reference unit—in a systematic way. Within the sequence, it is expected that common fractions will come to be construed by students as quantities that express the number of times that a specific subunit was iterated when measuring a certain length. For instance, it is expected that the fraction 7/5 will come to be construed as seven iterations of the length of a subunit *1/5 as long as the reference unit*. Finally, this way of construing fractions is expected to allow students to gauge the lengths that fractions account for as either being shorter than (e.g., 5/7), as long as (e.g., 7/7), or longer than (e.g., 7/5) the reference unit. The rationale for this sequence includes notions that typically fall within teacher's pedagogical repertoire, such as how classroom activities within the sequence can be organized to be most productive and why, and what types of classroom discourse made it possible for students to progress in past trials.

Working in four different Mexican schools, we have documented how the instructional sequence can be a powerful resource in supporting low-achieving and disenfranchised children, in developing relatively sophisticated understandings of fractions as measures (Cortina, Visnovska & Zúñiga 2014). These understandings include the inverse order relation amongst unit fractions (Tzur 2007), and fractions as numbers that may account for the size of quantities bigger than one unit (Norton and Hackenberg 2010). We have also recognized that the effective use of the instructional sequence in Mexican classrooms entails great teaching challenges. It requires making instructional decisions based on students' reasoning, as well as supporting pupils to participate in ways that are new to them and often uncommon in their regular classrooms. They are expected to actively listen to what others say, ask questions, express non-understanding, and articulate and communicate their own thinking.

Collaborating with Irene

In several ways, Irene can be regarded as a typical Mexican elementary-school teacher. Almost all of her students were the children of low-income families, and most of them were low achievers in mathematics. Irene graduated from Mexico City's Normal School, and became the first member of her family to be a teacher and to obtain a college-level degree. She was amongst the 68% of Mexican elementary school-teachers who are graduates of public normal schools, with no postgraduate education (Instituto Nacional para la Evaluación de la Educación 2015a).

Throughout her sixteen-year teaching career, Irene had worked in public elementary schools located in the hilly western suburbs of Mexico City. As many other Mexican teachers, she rarely collaborated with her colleagues on issues directly related to instruction. As it is also typical, school authorities held her accountable mostly on matters concerning administrative issues, such as completing paperwork correctly and on time. She received no support for her teaching, but also a very little oversight of how she taught on a daily basis. Even though official regulations are rather restrictive in the Mexican educational system, Irene had relatively high autonomy in deciding what to teach, when to teach it, and how to teach it in her classrooms.

In other ways, Irene can be regarded as an atypical Mexican teacher. She was amongst the few who held two teaching positions, one in a school with a morning shift (8am to 1pm) and the other in an afternoon shift (2 to 7pm). She was also a teacher who was unsatisfied with her students' mathematical achievement, felt responsible for it, and believed that by furthering her education she could improve on what her students can learn.

In 2014, Irene obtained a paid leave of absence in her morning-shift school position, to enroll in a master's program at the National Pedagogical University (NPU). She continued with her teaching in the afternoons. Our collaboration with her commenced at the beginning of her master's studies, with the second author as her academic adviser.

The NPU master's program has a strong professional development orientation. It is expected that enrolled students' research projects will further their capacity as mathematics teachers. The dual design experiment thus presented an ideal framework for supporting and researching Irene's learning. It allowed us to collect data on Irene's adoption of a new resource, while she conducted the classroom design experiment on fractions, for her master's project, in her fifth-grade afternoon classroom. Irene obtained ethical clearance from the university and consent from the students, parents, and school administrators, to collect students' work and videotape all the classroom sessions.

Methodology

The data collection for the dual design experiment primarily consisted of two design research logs. Irene's *planning and teaching log* included elaborated lesson plans, in which she specified the learning goals for each upcoming classroom session, and the activities she planned to use. After each classroom session, Irene annotated her lesson plan, reflecting on classroom events. Irene also video-recorded all the teaching sessions, and collected copies of her students' work.

The second author produced a *research log*, which included design conjectures and notes related to both students' and Irene's learning. First, the log documented the second author and Irene's conversations during weekly debriefing and planning meetings aimed at understanding students' learning progress. Second, this log documented weekly to bi-weekly debriefing sessions between the two authors, which focused on Irene's teaching and planning, and on the ways in which her work was supported.

In the retrospective analysis of the data, we relied on an adaptation of the constant comparative method described by Cobb and Whitenack (1996) that involves testing and revising tentative conjectures while working through the data chronologically. As we analyzed new teaching episodes, we compared these with conjectured themes and categories. This process resulted in a set of the theoretical assertions that remained grounded in the data. For present purposes, we focused on the key episodes, which highlighted features of the instructional sequence on which Irene relied as she supported the learning of diverse learners in her classroom.

Irene's Classroom Design Experiment Overview

In six one-hour weekly sessions with the second author, Irene first became acquainted with the instructional sequence, including how it was developed and used in prior classroom design experiments. She worked through all the instructional activities as a student, deepened her understanding of measurement and fractions, and got acquainted with the different aspects of instructional practices that place student mathematical reasoning at the center of decision-making in the classroom. She also became familiar with the classroom design experiment methodology and the importance of documenting her rationales for instructional decisions made in the process.

Irene started to trial the instructional sequence in her fifth-grade classroom about four months into the school year. The results of her students' initial assessment suggested that the great majority of them had made little progress in making sense of fractions as numbers that express quantities. For instance, all but four of her 20 students would not correctly and consistently recognize which of two fractions (e.g., 1/4 and 1/2) represented the bigger amount. Moreover, only two of her fifth-grade students seemed to recognize that improper fractions account for quantities that are bigger than one.

Irene worked with the instructional sequence during 18 weekly sessions, each lasting about 35 minutes. After each classroom session, she met with the second author, for one hour, to analyze the learning that took place, and plan for the upcoming session. Irene described the different ways in which students participated and how they reasoned during the instructional activities. When planning for the upcoming session, particular attention was placed on supporting the participation of the students that struggled the most.

The results of a final written assessment suggested that, similar to other classrooms in which the instructional sequence had been used, Irene's students developed relatively sophisticated understandings of fractions as measures. They could all easily and correctly compare unit fractions.

In addition, all but two students could readily gauge the size of a common fraction as being smaller than, as big as, or bigger than one, and could use this knowledge to make accurate estimates of where to place a fraction on a number line (Visnovska & Cortina 2017).

Trialing the instructional sequence had a profound impact on Irene's teaching. We became aware of this by noticing that in the final three instructional sessions, the students in Irene's class made explanations that were clearer and more articulate than those made by students in any of the other groups in which we had worked with the instructional sequence. When we asked Irene about her thoughts on why this had happened, she responded that for several months now, she had been asking students to express non-understanding and to communicate their thoughts, regardless of what she was teaching. Whole-class conversations had become a common aspect of her everyday teaching, in all subject areas. She mentioned that she now felt uncomfortable teaching without knowing what her students were understanding, particularly those students who struggled the most.

Many factors can indeed be recognized as being critical in influencing Irene's teaching, including her dissatisfaction with the results she was obtaining, her willingness to learn, and the intense professional development support she received during her classroom design experiment. Nonetheless, as professional development facilitators, we recognize that the instructional sequence became a particularly powerful means of supporting Irene's learning. In the following section, we identify and analyze some of the key roles that the sequence played.

The instructional sequence as a resource for teaching

To understand the supports the sequence provided, we need to discuss ways in which Irene's instruction changed. At the beginning of the collaboration, Irene was mostly concerned about delivering instruction appropriately. She would decide what to do and how based on the contents specified in the program of studies, and on the lessons included in the official textbook and in other teaching resources that she regarded as valuable. She believed that if she taught "well," her students would learn. Unfortunately, many of her students made little progress in mastering the content she taught. At the time of her entry to the masters' program, the pending issue for her was to learn what kind of instruction would be more effective, and how can it be implemented properly.

During the classroom design experiment, Irene's focus on proper enactment was replaced by her emerging need to understand how her students were thinking. We illustrate this through an episode from the very beginning of her classroom design experiment, where Irene set out activities for her students to measure with body parts. She followed the instructional sequence where the first learning goal for students was to become aware of the shortcomings of using body parts as units of measure. Noticing these shortcomings was an important step for students to come to see the introduction of a standard (informal) measurement unit, *the stick*, as a reasonable innovation.

The instructional activities Irene used in her classroom were introduced in a conversation about how people measured before there were rulers and tape measures. Based on the prior experiences in using the instructional sequence, it was expected that students would come to value practicality of measuring with body parts as they measured objects in the classroom with their hands. In addition, it was expected that some pupils would become aware of disadvantages in measuring with body parts. Specifically, Irene expected that some students would find it problematic that the class sometimes recorded different measures for the same measured length.

For the most part, the session unfolded as Irene expected. Students had considered using body parts, and had eagerly engaged in measuring and recording the measures they made with their hands. However, in the whole-class conversation, students had only said positive things about measuring with body parts. Irene first interpreted this classroom session in a way consistent with the orientation she typically followed in her teaching. She considered that she did not properly implement the activities, since the students had not come up with the expected contributions. Even though the learning goal had not been achieved, she planned to proceed to activities in which students would now start using a standard unit of measure that she would introduce, a wooden stick.

In the debriefing session, the second author shared with Irene how the instructional sequence aims to structure students' experience of instructional activities as being *coherent*. He explained how this aim guides us to only introduce a new tool such as the stick to students once they can see it as a solution to a problem that they already identified. Otherwise, from students' perspective, an introduction of a new tool would break the storyline, and the tool would be seen as an object important to the teacher but with no particular value for students. Irene considered the classroom situation from the students' perspective, and agreed to co-design and trial in her classroom additional instructional activities with the goal of helping her students recognize the shortcomings of measuring with body parts.

When Irene and the second author met again, she reported that the instructional activities had worked fairly well. In one of them, she had told the students how a window had once broken in their school. The principal in the morning shift had measured its width, obtaining five hands. Then the principal of the afternoon shift had measured its width and obtained six hands. They were puzzled about which of them had made a mistake when measuring.

Irene related that most of the students had recognized that both principals could have measured correctly, and that measuring with hands might thus not always be a good idea. Surprisingly to the second author, Irene did not yet want to proceed to the following mathematical goal (this would involve Irene asking how we could improve on measuring with hands, and later introducing a new measurement tool, the stick). Instead, she suspected that for several of her pupils the shortcomings of using non-standardized units of measure were unclear and she wanted to create additional scenarios in which the problem could be discussed. She now wanted to make sure that *all* her students had reached the first learning goal before pursuing a new one.

From this point on, a significant shift in Irene's orientation as a mathematics teacher was noticeable. She was now making instructional decisions based on what she considered her students were understanding and what they needed to learn next. She no longer based these on a plan about what had to be taught. For this shift to happen, it was critical that the instructional sequence she was trialing had an explicit rationale that specified a progression of clearly formulated and specific learning goals, and that these goals were achievable in her classroom. We now unpack each of these supports in turn.

Learning goals: Explicit rationale

For more than twenty years, the Ministry of Education has been providing Mexican teachers with "sequences of problem situations" that are expected "to arouse students' interest, and invite children to reason mathematically, find ways of solving problems, and formulate arguments that validate their discoveries" (Secretaría de Educación Pública 2011, 67). Although teachers have been provided with guides that include many recommendations about how to best enact the sequences of problem situations in their classroom, and even to complement them, little information is usually offered to the teachers about the rationale that guided the design. For the most part, this knowledge is implicit in the provided sequences, as it seems to be the case with many carefully crafted instructional resources. According to Remillard (2018), accessibility to design rationales remains problematic even in resources that are accompanied by teaching guides, such as ones produced as part of NSF-funded textbook series in the US.

The rationale for the instructional sequence that Irene trialed with her students was explicitly formulated during the process of initial design and subsequent modifications in a number of classrooms, and consists of conjectures about the collective mathematical development of the classroom community, and how this development can be supported. Availability of this rationale made it possible for the second author to discuss with Irene her role in supporting each of the learning goals that students were expected to sequentially achieve, and the resources on which she may draw in doing so, including the ideas that students generate. Had the instructional sequence been limited to a set of instructional activities, manipulatives, worksheets, and other tools, about which we knew merely that they worked in other classrooms, it would have been very difficult to guide Irene in responding to the unexpected student ideas that emerged in her classroom.

Learning goals: Specific and understandable

In addition to availability of the rationale for the instructional sequence, it was also important that the student learning goals were specific. Efforts for reforming mathematics education, at least in Mexico, have often included comprehensive recommendations for students' learning, such as to help pupils build their own knowledge and skills with sense and meaning (Secretaría de Educación Pública 2011). Although valuable at some level, goals that are this broad might not be of much use in making specific instructional decisions – including how to know whether to move to the next learning activity.

During the debriefing after the initial instructional session, it was apparent that Irene understood the first learning goal, and that it was specific enough for her, so much so that she clearly recognized that it had not been reached. Had the goals been formulated in a way that was not easy for her to assess – because she found them to be vague or confusing – it would have been much harder for her to focus her instructional agenda on her students' learning.

The specificity of the learning goals positioned Irene as a teacher fully capable of reflecting accurately on her progress. Her advisor, in turn, became a collaborator in a shared problem-solving situation, rather than a person exerting judgment or control over her work. Irene identified that her students did not accomplish the learning goal. Discussing pedagogical issues related to the sequence rationale helped her re-evaluate why this was problematic, and realize that the problem would not be addressed by moving to the next type of measurement activity. Similarly to students realizing that different-sized hands were problematic, Irene's refined awareness of the problem enabled her to look for a more adequate solution. The importance of sequentially achieving each of the learning goals then became more meaningful to her.

Learning goals: Achievable in the classroom

Once Irene recognized the importance of reaching the first learning goal, it was crucial that she also regarded it as achievable by all the students in her classroom. This seemed to have happened during the second instructional session, when she noticed how many of her students, including ones she did not expect to do so, started to realize that measuring with non-standardized units might not always be a good idea. At that point, she seems to have realized that, with more support, everyone in the classroom could come to the same conclusion. This realization allowed her to put aside her concern for what needed to be taught, and start making instructional decisions based on her observations and conjectures about what were her students understanding. Importantly, observing how they accomplished the first learning goal seems to have provided Irene with confidence in the instructional sequence, and a resolve that the ensuing goals could also be sequentially reached.

It is important to point out that the achievement of Irene's students on the first learning goal was not a fortuitous occurrence, by any means. As we mentioned before, the sequence she was trialing was a product of careful design and experimentation in classrooms with students very similar to hers. As designers, we were not only fairly confident that the expected goal was achievable for Irene's students within several classroom sessions, we could also draw on the sequence rationale to advise Irene about various means of support that the sequence provided for her use (e.g., ways of creating additional problem scenarios).

Concluding remarks

The instructional sequence used in our professional development collaboration with Irene is a rather sophisticated resource for teaching, in many ways different from traditional textbooks. As noted throughout this paper, its effective use entails great challenges. It requires a kind of teaching that is complex, not typical, and involves substantial learning (Maaß & Artigue 2013).

It might thus seem unsound to consider the instructional sequence on fractions as measures as a worthy resource in an educational system like the Mexican one, where ambitious and equitable instructional practices are not typical. However, this kind of instruction can also be regarded as necessary for significant improvement in Mexican students' mathematical understanding to take place, particularly for those that are living in harsh social and economic circumstance. From this perspective, our professional development collaboration with Irene can be seen as an illustrative case of a specific kind of resource that can successfully support Mexican teachers' transition towards instructional practices that better support their students' understanding of worthy mathematical ideas. We discussed several features of the instructional sequence on fractions as measures and how these facilitated teacher learning in relation to the instructional design rationale for the sequence.

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ORAL COMMUNICATIONS

SECTION TEXTBOOK RESEARCH AND TEXTBOOK ANALYSIS

HOW COMBINATORIAL SITUATIONS ARE REPRESENTED IN BRAZILIAN PRIMARY AND MIDDLE SCHOOL TEXTBOOKS[1]

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Abstract

In this article, results of a research concerning combinatorial situations in Brazilian primary and middle School textbooks are presented. The main concern are symbolic representations – such as natural language, drawings, tree diagrams and numerical expressions – and how they are dealt with in the study of Combinatorics. Three textbook collections, directed to students aged 6 to 14, were analysed, focusing on the combinatorial situations presented and the symbolic representations conversions required. In lower primary school textbooks, all the problems were Cartesian products. Combinatorial situations in upper primary school textbooks were better distributed and in middle school textbooks an even better distribution was presented with Cartesian products; arrangements, combinations and permutations. The distinctive properties of different combinatorial situations were not discussed neither with teachers nor with students. Positive aspects were observed, such as conversions of varied symbolic representations. However, a better distribution of problem types and more symbolic representation conversions still need to be present.

Reasons for constant textbook analysis

Textbooks in Brazil, as in many other countries, play a central role in mathematics teaching and learning and, in this sense, need to be constantly analysed in order to evaluate the quality of this resource used in classrooms. Constant textbook analysis is also justified by the way that textbooks reflect, as pointed out by Harries and Sutherland (1999),

a) prospects of what mathematics is;

b) what one needs to know about the use of mathematics; and

c) how mathematics should be taught and learned. In this sense, when books are analysed, which and how mathematics is proposed can be analysed.

Another reason for constant analysis is that textbooks assist in planning classroom activities throughout the school year and book collections suggest the development of content throughout schooling. They are also a source of continuing education for teachers (Gérard and Roegiers 1998).

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Thus, the importance of textbooks is reflected on what is effectively taught and what teachers also learn by using this resource.

Carvalho and Lima (2010) highlight four poles, related to textbooks, that must be linked to produce effective teaching and learning processes: 1) the textbook and the author, 2) the teacher who uses the book, 3) the student user of the book and 4) the area of knowledge and concepts considered in the textbook. The present paper addresses the fourth pole, specifically knowledge concerning Combinatorics and, in particular, how combinatorial concepts are symbolically represented in textbooks.

Textbook evaluation policy in Brazil

The Brazilian National Textbook Programme (PNLD – Plano Nacional do Livro Didático), since the late 1990s, has been concerned with distinct aspects to be considered in textbook evaluations and Carvalho and Lima (2010) observed advances in textbook quality, due to this policy. The Brazilian Ministry of Education has provided textbook analysis by specialists in different areas (portuguese, mathematics, natural sciences and social sciences, amongst others) and information about textbooks is published in guides (Guia do Livro Didático). The guides help teachers in choosing the textbooks to be used in their classrooms, as the specialists describe what is contained in each textbook collection and also point out how the content is presented and the orientation provided.

The guide for primary school provides information on textbook collections: one set of collections consisting of the volumes for 1st, 2nd and 3rd grades and another set of collections containing the volumes for 4th and 5th grades. There is also a guide for the four years of middle school and still another guide for the three grades of High School.

In the initial evaluations, in the late nineties, many conceptual errors and methodological incoherencies were observed, such as theoretical approaches defended by the authors that were not coherent with the activities proposed to the pupils in the textbooks. Initially, most textbooks did not consider recent tendencies in mathematics education. In 1995, the main characteristics of the mathematical textbooks were: conceptual approach was based on memorization, learning was based on repetitive exercises, situations were not socially contextualised and students were led to mechanically use procedures – mainly formal algorithms with no valuing of personal strategies. Advances in textbook quality were observed in the editions that followed of the PNLD (Programa Nacional do Livro Didático – National Textbook Programme) and this constant evaluation can provide still more advances in the quality of mathematics teaching and learning.

Symbolic representations as means to think about mathematical concepts

An important aspect of analysis concerns how symbolic representations are dealt with in textbooks as means to think about mathematical concepts. As defended by Vergnaud (1996), symbolic representations have a very important role in mathematical thinking and aid problem solving, in particular when there is a large amount of data and/or the answer to the question requires several steps.

We only have access to mathematical objects through its representations, which leads to the question: How not to confuse an object with its representation? Duval (1995) posed and answered this question by emphasising the importance of semiotic representation transformations: internal transformations in which the system is the same (treatment) and transformations in which there are processing changes to the system (conversion). By these different types of semiotic transformation, the same mathematical concept can be recognised and dealt with through different symbols and the students can learn that the same concept is present in different forms of representations.

Combinatorial situations in primary, middle and high school

Concerning Combinatorics, Brazilian curriculum orientations, Brasil (1997), recommend the teaching and learning of different situations in all school levels (primary, middle and high school), i.e., *Cartesian products (CP), arrangements (A), combinations (C)* and *permutations* (P). The differences in these distinct combinatorial situations is related to *choice* and *ordering* of elements (Borba, 2010). In Cartesian products elements are chosen amongst distinct sets and in arrangements, combinations and permutations, elements are chosen from a single set of elements. The order of elements determines different possibilities in arrangements and permutations, the distinction being that permutations involve all the elements of the set. In Cartesian products and in combinations the order of elements does not determine distinct possibilities.

Examples of each problem type are as follow (presented by Borba, Azevedo & Barreto 2015):

Cartesian product: At the square dance three boys and four girls want to dance. If all the boys dance with all the girls, how many pairs will be formed?

Simple permutation (without repetition): Calculate the number of anagrams that can be formed with the letters of the word LOVE.

Simple arrangement: The semi-finals of the World Cup will be played by: Brazil, France, Germany and Argentina. In how many distinct ways can the three first places be formed?

Simple combination: A school has nine teachers and five of them will represent the school in a congress. How many groups of five teachers can be formed?

Combinatorial reasoning enables hypothetical deductive thinking and this, as other types of reasoning, needs a long period of time for its development. In this sense, Combinatorics teaching can start at the first years of primary school and continue until the final years of high school. Research (Borba, Pessoa, Barreto & Lima 2011 and Azevedo 2013) has shown that children in initial schooling are able to deal with different problem types: *Cartesian products*, *arrangements*, *combinations* and *permutations,* not using formulas but other efficient strategies – such as drawings and lists – and these investigations also show that older students using tree diagrams and other symbolic representations can build a solid understanding of formal Combinatorics procedures. Other investigations, presented in Borba, Pessoa & Rocha (2012), also show that some teachers present difficulties in solving combinatorial situations. In this sense, textbooks may provide continuing education to enable, conceptually and pedagogically, teachers of all school levels to teach Combinatorics.

Investigating how combinatorial situations are represented in primary and middle school textbooks

This study analyses three textbook collections: directed to lower primary school (1^{st}, 2^{nd} and 3^{rd} grades), to upper primary school (4^{th} and 5^{th} grades) and to middle school (6^{th} to 9^{th} grades).[2] These collections are directed, respectively, to children aged 6, 7 and 8; to 9 and 10 year-olds; and to students aged 11 to 14. The analyses concern which combinatorial situations are presented and what symbolic representations conversions are required.

How combinatorial situations are proposed in textbooks

Concerning the occurrences of combinatorial situations, 25 situations were observed in the lower primary school textbooks, 75 in the upper primary school textbooks and 58 in the middle school textbooks. This distribution is, in some way, surprising because it is expected that more situations be presented in middle school textbooks. However, more situations were presented in upper primary school, that is when multiplicative problems are formally introduced. It would be better that more

[2] The research also involved the analysis of the high school collection (directed to students aged 15 to 17), but this analysis was not yet concluded at the time of ICMT 2.

combinatorial situations be presented in middle school, preparing students for the future use of more formal procedures in high school.
Table 1 shows how combinatorial problems are distributed along the three textbook collections.

School grades/ Problem type /	Cartesian Products	Combinations	Permutations	Arrangements
Lower primary school	100%	-	-	-
Upper primary school	54%	36%	10%	-
Middle school	32%	22%	24%	22%

Table 1 – Combinatorial problems distribution in the collections analysed

Although it is a positive finding that combinatorial situations are dealt with in lower primary school textbooks, as recommended in the national curriculum orientations (Brasil 1997) and reinforced by Borba (2010), a negative aspect was observed: 100% of the problems were of one only kind: *Cartesian products*. Combinatorial situations in upper primary school textbooks were better distributed amongst *Cartesian products*; *combinations* and *permutations*. However, no *arrangements* were presented. In middle school textbooks, an even better distribution was presented with problems of all four types. This is what research results ((Borba, Pessoa, Barreto & Lima 2011 and Azevedo 2013) have pointed out: that in the teaching and learning of Combinatorics distinct combinatorial situations can and must be worked with the students of all school levels.
Despite the variation of problem types, neither teachers nor students are alerted that there are different combinatorial situations that have distinctive properties. This is necessary, especially in teacher manuals, so that the teachers are aware of the distinct situations posed to students and how they can deal with different combinatorial relations. In this way, textbooks can aid teachers to overcome their difficulties in solving problems in Combinatorics, as pointed out by Borba, Pessoa & Rocha (2012), and also help them in the teaching of this area in mathematics. The textbooks, thus, can be a source continuing education – as pointed out by Gérard and Roegiers (1998).
Most of the combinatorial situations are proposed in chapters that deal with *Numbers and Operations*. If presented in other chapters (*Measurements*, *Geometry* or *Information Processing*) combinatorial problems are presented in sections aimed at revising subjects already studied. If the problem posed is a *Cartesian product*, the situation is presented in a section that deals with multiplication. In other parts of the textbooks there are some *combinations, permutations* and *arrangements*. In this sense, one recommendation for authors would be to present combinatorial situations in other parts of the textbook, in order to defend the idea of the presence of Combinatorics in a broad sense.

Conversions of symbolic representations in textbooks' combinatorial situations

In the textbook collections analysed, all problems involve at least one conversion being the most common converting natural language and drawing into numerical expression, usually a multiplication that determines the total number of possibilities in a combinatorial situation. It is also common to convert natural language and drawing to list, especially in lower and upper primary school.
Presenting problems in only natural language and asking to convert data to lists or to numerical expressions was not as frequent (only 27% of the activities) in primary school textbooks but was very frequent (88%) in middle school textbooks. Early schooling textbooks tend to use natural language with drawings, tree diagrams or tables to aid the understanding of combinatorial situations. These representations are very appropriate for primary school children. Tree diagrams

and tables are proposed mainly with *Cartesian products*, but not commonly for other types of combinatorial situations. In particular, Azevedo (2013) and Borba, Azevedo and Barreto (2015) showed that tree diagrams are appropriate for solving simple combinatorial problems of the four types: *Cartesian Products*, *arrangements*, *combinations* and *permutations*. Commonly in primary school textbooks some of the situations proposed involve multiple conversions, such as natural language and drawing to list and numerical expression. Figure 1 shows two examples from the 6th grade textbook. The first problem is presented in natural language and the conversion requested is to a list (Problem 45: How many 2 digit numbers, without repetition in the same number are there using the digits 1,3,5 &7?) The second problem is presented in natural language and drawing and a list is also requested (Problem 46: How many different ways, in relation to order, can three people sit in a 3 seat sofa?)

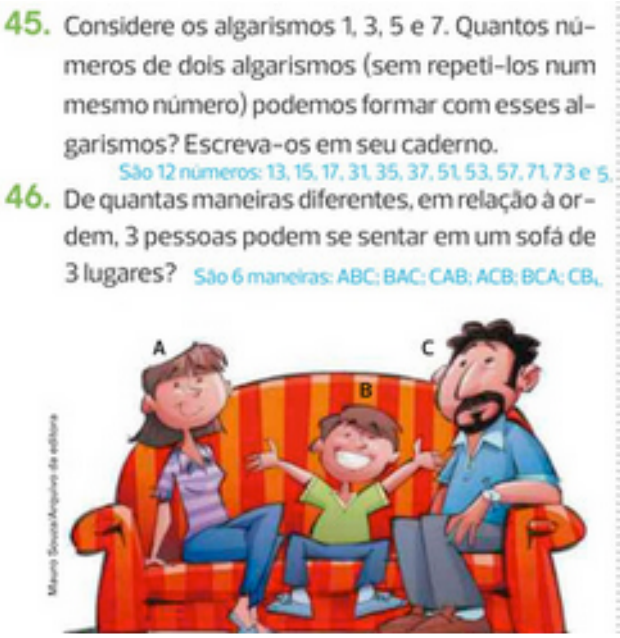
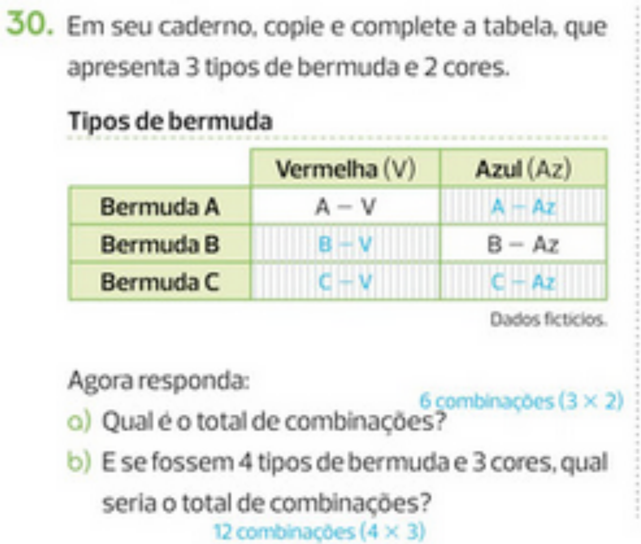


Figure 1 – Conversions for natural language and natural language and drawing to lists.

Other conversions are presented or requested in the problems shown in Figures 2 and 3, also from the 6th grade textbook. In the first of these examples the conversion is from natural language to tree diagram and to numerical expression. From four different fruits and three glass sizes, the tree diagram and the numerical expression (4 x 3) present the answer: 12 distinct possibilities. In the second example the conversion requested is from natural language to table and to numerical expression. With three types of shorts (A, B and C) and two colours (red and blue) the table and the numerical expression (3 x 2) represent the total (6) of possibilities. The following request is a generalization: How many are the possibilities of combinations from four types and three colours of shorts?

Figure 2 – Conversion for natural language to tree diagram and to numerical expression.


30. Em seu caderno, copie e complete a tabela, que apresenta 3 tipos de bermuda e 2 cores.

Tipos de bermuda

	Vermelha (V)	Azul (Az)
Bermuda A	A – V	A – Az
Bermuda B	B – V	B – Az
Bermuda C	C – V	C – Az

Dados fictícios.

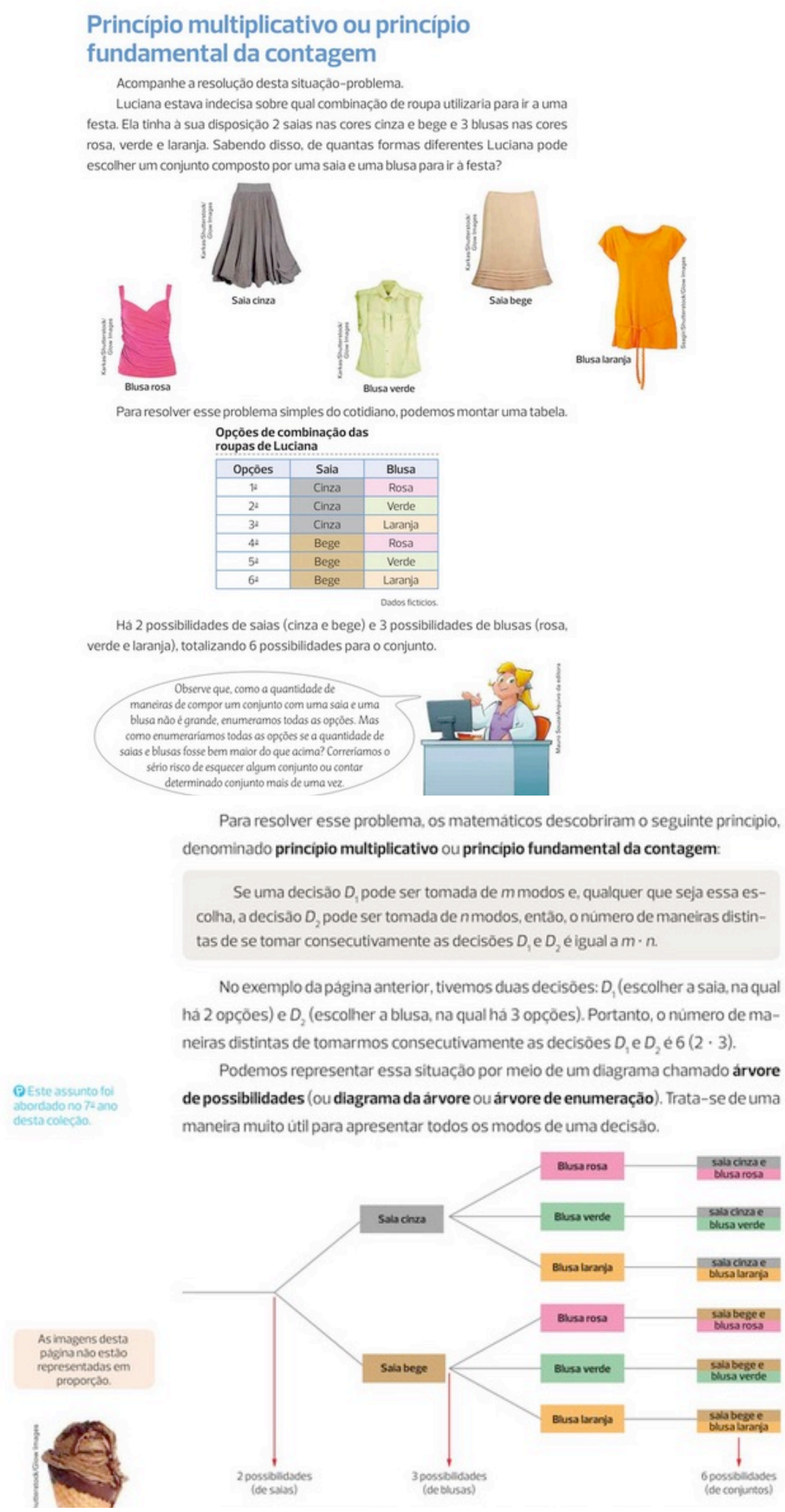
Agora responda:

a) Qual é o total de combinações? 6 combinações (3 × 2)

b) E se fossem 4 tipos de bermuda e 3 cores, qual seria o total de combinações? 12 combinações (4 × 3)

Figure 3 – Conversion for natural language to table and to numerical expression.

A final example, from the 9th grade textbook, is presented in Figure 4 in which natural language and drawing is converted to a table, to numerical expression and a tree diagram.



Princípio multiplicativo ou princípio fundamental da contagem

Acompanhe a resolução desta situação-problema.

Luciana estava indecisa sobre qual combinação de roupa utilizaria para ir a uma festa. Ela tinha à sua disposição 2 saias nas cores cinza e bege e 3 blusas nas cores rosa, verde e laranja. Sabendo disso, de quantas formas diferentes Luciana pode escolher um conjunto composto por uma saia e uma blusa para ir à festa?

Para resolver esse problema simples do cotidiano, podemos montar uma tabela.

Opções de combinação das roupas de Luciana

Opções	Saia	Blusa
1ª	Cinza	Rosa
2ª	Cinza	Verde
3ª	Cinza	Laranja
4ª	Bege	Rosa
5ª	Bege	Verde
6ª	Bege	Laranja

Dados fictícios.

Há 2 possibilidades de saias (cinza e bege) e 3 possibilidades de blusas (rosa, verde e laranja), totalizando 6 possibilidades para o conjunto.

Observe que, como a quantidade de maneiras de compor um conjunto com uma saia e uma blusa não é grande, enumeramos todas as opções. Mas como enumeraríamos todas as opções se a quantidade de saias e blusas fosse bem maior do que acima? Correríamos o sério risco de esquecer algum conjunto ou contar determinado conjunto mais de uma vez.

Para resolver esse problema, os matemáticos descobriram o seguinte princípio, denominado **princípio multiplicativo** ou **princípio fundamental da contagem**:

> Se uma decisão D_1 pode ser tomada de m modos e, qualquer que seja essa escolha, a decisão D_2 pode ser tomada de n modos, então, o número de maneiras distintas de se tomar consecutivamente as decisões D_1 e D_2 é igual a $m \cdot n$.

No exemplo da página anterior, tivemos duas decisões: D_1 (escolher a saia, na qual há 2 opções) e D_2 (escolher a blusa, na qual há 3 opções). Portanto, o número de maneiras distintas de tomarmos consecutivamente as decisões D_1 e D_2 é 6 (2 · 3).

Podemos representar essa situação por meio de um diagrama chamado **árvore de possibilidades** (ou **diagrama da árvore** ou **árvore de enumeração**). Trata-se de uma maneira muito útil para apresentar todos os modos de uma decisão.

Este assunto foi abordado no 7º ano desta coleção.

As imagens desta página não estão representadas em proporção.

Figure 4 – Conversion for natural language and drawing to table, numerical expression and tree diagram.

A summary of the analysis is that positive aspects were observed in combinatorial situations proposed in the textbooks, such as conversions of varied symbolic representations, but a better distribution of problem types and more symbolic representation conversions still need to be present in the books. This better distribution and more conversions can allow opportunities to discuss different types of combinatorial situations and encourage the use of varied symbolic representations to deal with Combinatorial problems.

Conclusions

As main conclusions, we reinforce statements previously made:

- Transformations of semiotic representations are necessary in pupil's conceptual development (Duval 1995) and need to be addressed in textbooks.
- In combinatorial situations, conversions of one form of representation to another allows students to recognize common combinatorial relations present in different problem types: *arrangements*, *combinations*, *permutations* and *Cartesian products*.

Positive aspects were observed in the textbooks' Combinatoric activities proposed – such as multiple representations of situations – but a wider range of problem types and symbolic representation conversions still need to be present in textbooks for a broader development of combinatorial reasoning. The study is in progress, analysing the combinatorial situations proposed in high school and this last stage of the research will allow a still better understanding of how symbolic representations dealt with in textbooks can allow a wider development of combinatorial reasoning in basic schooling.

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ON PREVALENCE OF IMAGES IN HIGH SCHOOL GEOMETRY TEXTBOOKS

MEGAN CANNON and MILE KRAJCEVSKI

Abstract

Despite ample evidence that images aid in guiding our intuition and designing a logical chain of arguments for proving process, there is a lack of research on how to explicitly formulate the characteristics of an image that fosters the cognitive process. We examine the visual representations of triangles, parallel lines and transversals, parallelograms, and trapezoids in 14 geometry textbooks, focusing on the corresponding sections in each of these textbooks. After defining a typical image in each category of the above listed mathematical notions, we determine the percentage of the typical images among all images in the lesson and exercise portion of each textbook. Results show the percentage of typical images among all images in each section varies from 52% in the category of parallel lines and transversals to 75% in the category of trapezoids. We indicate some confusing characteristics stemming from typical images, and provide suggestions for overcoming these obstacles.

On Visualization

As Phillips, Norris, and Macnab (2010, p.22) noted in their review of the notion of *visualization*, many terms –including visual aid, image, and visual literacy- have been used frequently and interchangeably throughout the literature. They also indicate Bishop's (1989) distinction between the noun and the verb forms of visualization. For example, Zimmerman and Cunningham's 1991 Editor's Introduction provides a definition of visualization as a verb, meant to "describe the *process* of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated" (p. 1). Similarly Clements (2014) describes visualization, as "something which someone does in one's mind-it is a personal *process* that assumes that the person involved is developing or using a mental image" (p.181). Presmeg (2006) states that the method of "visualization is taken to include *processes* of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics" (p. 2). On the other hand, the noun directs our attention to the product, the object, the 'what' of visualization, the visual image. After examining various definitions in the research literature from 1974 to 2009, Phillips at al. (2010) found 23 explicit definitions pertaining to different uses of the term visualization. They divided these into three categories: the category in which visualization was used as a noun and referred to as an object (visualization object) and two other categories in which visualization was used as a verb and referred to as a process, in so called introspective visualization and interpretive visualization. We adopt their definition of visualization objects as "physical objects that are viewed and interpreted by a person for the purpose of understanding something other than the object itself. These objects can be pictures, 3D representations, schematic representations, animations, etc. Other sensory data such as sound can be integral part of these objects and the objects may appear on many media such as paper, computer screens, and slides" (Phillips et al. 2010, p. 26).

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We also found Tall and Vinner's term of *concept image* to be particularly useful in our analysis of the overuse of some images in geometry textbooks. They defined the term concept image as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is build up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall & Vinner 1981, p.152) They contrasted this notion with the notion of *concept definition,* which is a "form of words used to specify that concept" (ibid., p. 152). One of the challenges associated with creating valuable concept image is the struggle of trying to sort whether images are or are not relevant to be included in the set comprising the concept image (Vinner & Dreyfus 1989). This difficulty in determining relevance is problematic because a concept image is considered functional only when it allows a student to classify an example or a non-example (Gutierrez & Jaime 1999).

A specific type of concept image is a prototype, also referred to as a prototypical image (we show some prototypes in Figure 1). A prototype is the most generic representation of an object from a category of objects. Prototypes are particularly useful in the cognitive process because they allow students to attribute properties to all objects in a given category of objects using one single representative of the group.

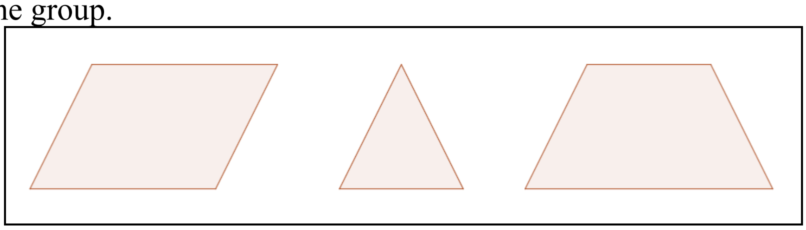

Figure 1: Example prototypes of a parallelogram, triangle, and trapezoid

Prototypical images are useful for classifying groups of mathematical objects, but they may also present challenges for students who inappropriately utilize them and assume their properties are shared by every element of the class. Prototypes can interfere with identification of somewhat atypical examples (Presmeg 1986). For example, if students are used to seeing most trapezoids drawn to look isosceles, they may not recognize other quadrilaterals that still belong to the category of trapezoids.

Geometry textbooks often present only the most prototypical shapes and lack a sufficient number of non-standard examples (Cunningham & Roberts 2010). If a visual representation of a certain object is always presented in the same way, students will have difficulty recognizing the object or identifying some of its properties when this object arises in a problem or application. This may lead students to form inadequate concept images of an object by attributing characteristics like relative position of a figure to the actual properties of the object. As an example from geometry textbooks, altitude of a triangle is often presented as in the left-most triangle of Figure 2. So, when students encounter a triangle like either of the two triangles on the right in Figure 2, they may have difficulty indicating or drawing the altitudes of these triangles.

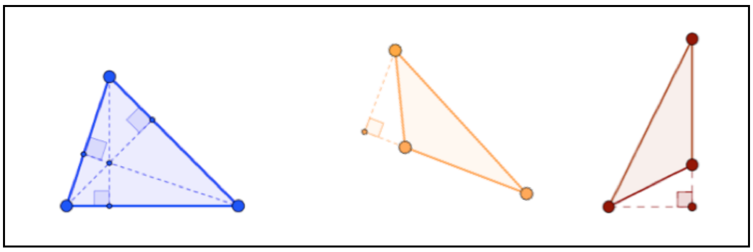

Figure 2: Representations of altitudes of triangles

The Role of Visualization in Problem Solving

Research in mathematics education demonstrates that visualization can aid students and mathematicians in problem solving (Lakin & Simon 1987), reasoning (Presmeg 1992), showing relationships between concepts (Phillips, Norris & Macnab 2010), discovering mathematical ideas

(Zimmerman & Cunningham 1991), offering intuitive visual evidence (Duval 2006), and possibly even providing basic proofs (Arcavi 2003).

According to Presmeg (1986), "A visual method of solution is one that involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are also employed" (p. 42). This definition is contrary to what one might expect of as a "visual method" in that a visual representation does not need to be drawn, only imagined as assistance to the solution. In other words, visual processing could be useful for solving some problems that are seemingly nonvisual (Presmeg 1992).

Visualization is an important component of the critical thinking process. When working on a mathematical puzzle or a problem, many students innately tend to use visual representations as a reasoning tool (Lakin & Simon 1987). These representations may take the form of sketches on paper, drawings on a whiteboard or chalkboard, or visualizations using dynamic geometry software. Students may also use internal or introspective visualization, without producing a material object by transforming images in their mind.

Visualization promotes a deeper conceptual understanding of certain mathematical ideas. For example, in Figure 3, by taking a parallelogram and reconfiguring it to produce a rectangle, we give visual explanation of the area formula for parallelograms. For many students, this method of thinking comes naturally, but for others it takes time and practice. The process of visualization is a powerful tool for comprehension (Zimmerman & Cunningham 1991).

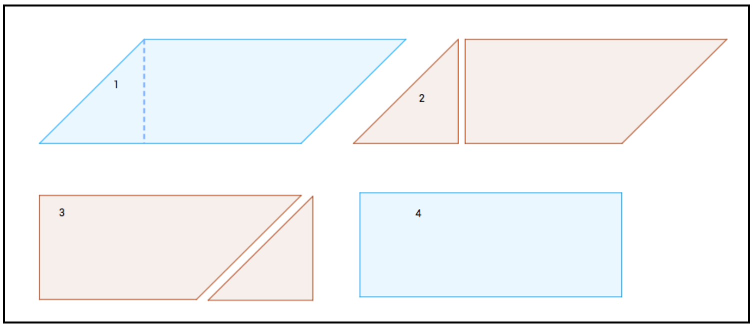

Figure 3: Visual method for demonstrating how area of a trapezoid relates to area of a rectangle

Challenges Using Visualization

One problem facing students is the phenomenon of *compartmentalization*, occurring when students have conflicting schemes or concept images and therefore may provide inconsistent answers or solutions. For example, if a student thinks of a parabola as a curve that is "opening upward" and then inappropriately applies that mental image to a problem where a "downward opening" parabola would be applicable. Both mental images are images of parabolas, but only one variation is currently appropriate. Vinner and Dreyfus (1989) completed a quantitative study on concept images of definitions of functions involving 271 college students and 36 mathematics teachers. In one particular example, their study found 56% of students struggled with the problem of compartmentalization, causing them to provide somewhat conflicting responses to questions.

Presmeg (1992) asserted that many of the difficulties experienced by visualizers are related to the one-case concreteness of an image prototype that students use (p.603). This one-case correctness occurs when a student observes that a particular property holds for a particular object and then incorrectly generalizes this property to the whole class of objects.

Nardi (2014) expanded on the issue of compartmentalization and added a list of challenges most commonly encountered by students: "The one-case concreteness of an image may be tied to irrelevant details or introduce false information, a prototypical image may induce inflexible thinking, an uncontrollable image may persist, thus preventing more fruitful avenues of thought, and imagery needs to link with rigorous analytical thought processes to be effective." (p. 213).

Using visualization to reinforce the cognitive process, supports students' comprehension of the mathematical formalism, guides a successful proof strategy, helps students understand when their intuition is misleading, and helps students appreciate concepts without the use of a mathematical formalism (Arcavi, 2003). Visualization needs to be combined with critical thinking. If imagery is not linked to analytical thinking the result can be harmful to students (Presmeg 1986). The use of imagery will reach its full potential when it is used to benefit the abstraction of mathematics (Presmeg 1992).

Technology provides an opportunity to benefit from dynamic imagery by using visualization with incorporated movement. Dynamic Geometry software such as The Geometer's Sketchpad (www.dynamicgeometry.com) or GeoGebra (www.geogebra.org) can use movement to demonstrate how properties in images are preserved or changed (Guvan & Kosa 2008). Figure 4 shows how GeoGebra can be used to illustrate the Inscribed Angle Theorem. GeoGebra allows students to construct an angle inscribed in an arc, and drag the vertex of the angle to different points along the circle's arc. It can also display the angle measure as students move the vertex along the arc of the circle to show that the measure is maintained.

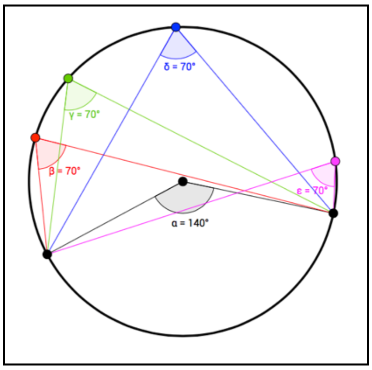

Figure 4: GeoGebra illustration of the Inscribed Angle Theorem

These software packages can be used as a tool for students to transform figures and to examine how certain properties remain even avoiding the consequences of one-case correctness of an image. Dynamic Geometry Software can also link visualization to critical thinking by allowing students to see how changing certain portions of the image affects the overall properties of the image.

Typical Images in High School Geometry Textbooks

Our basic research question is: What are the 'typical' images in High School Geometry textbooks? We began by defining what we mean by the term typical image.

A typical image of a particular mathematical object is a visual representation of that object that is drawn a certain way in the majority of the instances with no content-based reason.

For example, if in a section about equilateral triangles we notice that over 50% of the triangles are positioned such that one of their sides is aligned with the text, we say that an equilateral triangle with this "horizontal" alignment is a typical image of an equilateral triangle. However the fact that every triangle in this section is equilateral would not imply that equilateral triangles are typical images of triangles because there is a content-based reason for this representation in that particular section.

We decided to look at geometry textbooks because of the wealth of images they provide. To determine the typical images, we looked at a relatively small sample of geometry textbooks and examined what images were presented in these textbooks. For example, we found that most of the images of polygons were drawn in such a way that at least one side of the polygon was aligned with the text, with no content-based reason for this alignment. This may lead students to assume that polygons must always be drawn with this particular alignment and will affect the concept image a student has of this mathematical notion.

In mathematics classrooms across the nation, there are three major publishing houses: Houghton Mifflin Harcourt, McGraw-Hill, and Pearson. They provide the majority of secondary mathematics

textbooks accounting for approximately 75% of the market (Banilower et al. 2013). Two other publishers, University of Chicago School Mathematics Project (UCSMP) and CK-12 Foundation, were included in the sample in order to provide books that are not as commonly adopted. UCSMP has created a student-centered and research driven curriculum that is currently being used by 4.5 million students nationwide. CK-12 Foundation is a non-profit organization developing free and customizable materials designed to align with the state standards. They develop so called *FlexBooks*, which are interactive online textbooks that can include multimedia activities.
We display the above information in a tabular format, and gave each textbook a short code as shown below in Table 1.

Textbook and Publisher	Short Code
Pearson: Geometry Common Core Charles, R. I., Kennedy, D., Hall, B., Bellman, A. E., Bragg, S. C., Handlin W. G., . . . Wiggins, G. (2015). *Geometry Common core.* Upper Saddle River, NJ: Pearson Education.	P1
Pearson: CME Project Geometry Cuoco, A., Baccaglini-Frank, A., Benson, J., Antonellis D'Amato, N., Erman, D., Harvey, B., . . .Waterman, K. (2009). *CME Project Geometry.* Upper Saddle River, NJ: Pearson Education.	P2
Pearson: Informal Geometry-Classics Edition Cox, P. L. (2006). *Informal geometry Classics edition.* Upper Saddle River, NJ: Pearson Prentice Hall.	P3
Pearson: Blitzer, Thinking Mathematically Blitzer, R. (2015). *Thinking mathematically* (6th ed.). Boston, MA: Pearson Education.	P4
McGraw Hill: Glencoe Geometry Carter, Cuevas, Day, Malloy, and Cummins (2010). *Glencoe McGraw Hill Geometry.* Columbus, OH: The McGraw Hill Companies.	M1
McGraw Hill: Geometry Concepts and Applications Cummins, J., Kanold, T., Kenney, M., Malloy, C., and Mojica, Y. (2006). *Geometry Concepts and application.* Columbus, OH: Glencoe/McGraw Hill.	M2
Houghton Mifflin Harcourt: Holt McDougal Geometry Common Core Edition Burger, E. Chard, D. J., Kennedy, P., Leinwand, S., Roby, T., Seymour, and D., Waits, B. (2012). *Holt McDougal Geometry.* Lewiston, NY: Houghton Mifflin Harcourt Holt McDougal.	H1
University of Chicago School Mathematics Project (UCSMP): *Geometry* Benson, J., Klien, R., Miller, M. J., Capuzzi-Feurstein, C., Fletcher, M., Marino, G., Usiskin, Z. (2009). *University of Chicago School Mathematics Project: Geometry* (3rd ed.). Chicago, Il: Wright Group/McGraw Hill.	U1
CK-12 Foundation: Geometry 2nd Edition Jordan, L., and Dirga, K. (2015). *CK-12 Geometry - Second edition.* Palo Alto,	C1

CA: CK-12 Foundation.	
CK-12 Foundation: Basic Geometry Concepts Greenberg, D., Jordan, L., Gloag, A., Cifarelli, V., Sconyers, J., and Zahner, B. (2015). *CK-12 Basic geometry concepts.* Palo Alto, CA: CK-12 Foundation.	C2
CK-12 Foundation: Geometry Concepts-Honors Spong, K. (2016). *CK-12 Geometry concepts- Honors.* Palo Alto, CA: CK-12 Foundation.	C3
CK-12 Foundation: *Geometry-Basic* Jordan L., Zahner B., Cifarelli, V., Gloag, A., Greenberg, D., and Sconyers, J. (2014). *CK-12 Geometry - Basic.* Palo Alto, CA: CK-12 Foundation.	C4
CK-12 Foundation: Geometry-Concepts Dirga, K., and Jordan, L. (2015). *CK-12 Geometry concepts.* . Palo Alto, CA: CK-12 Foundation.	C5
CK-12 Foundation: Foundation and Leadership Public Schools, College Access Reader, Geometry Fauteux, M., and Zapata, R. (2015). CK-12 Foundation and leadership public schools, College access reader: Geometry. Palo Alto, CA: CK-12 Foundation.	C6

Table 1: Short codes used for textbooks

Topics Examined

We focused on the visual representations of the following four topics: parallel lines and transversals, classification of triangles, parallelograms, and trapezoids. For each textbook we found the sections that most thoroughly covered these four topics, and we looked briefly at each of the topics to get an idea of what would be considered *typical*. The majority of images were consistently drawn with at least one line or line segment of the image being horizontal. For each of the sections we defined the typical images (shown in Figure 5) as follows:

- *Parallel lines and transversals*- We defined a typical image in this section to be an image in which the parallel lines are aligned with the text.
- *Classifying triangles*- We defined a typical image of a triangle in this section to be one drawn with one side aligned with the text.
- *Parallelograms*- We defined a typical image in this section to be a parallelogram that is drawn with one pair of parallel sides aligned with the text.
- *Trapezoids*- We define a typical image in this section to be a trapezoid that is drawn with a pair of parallel sides aligned with the text.

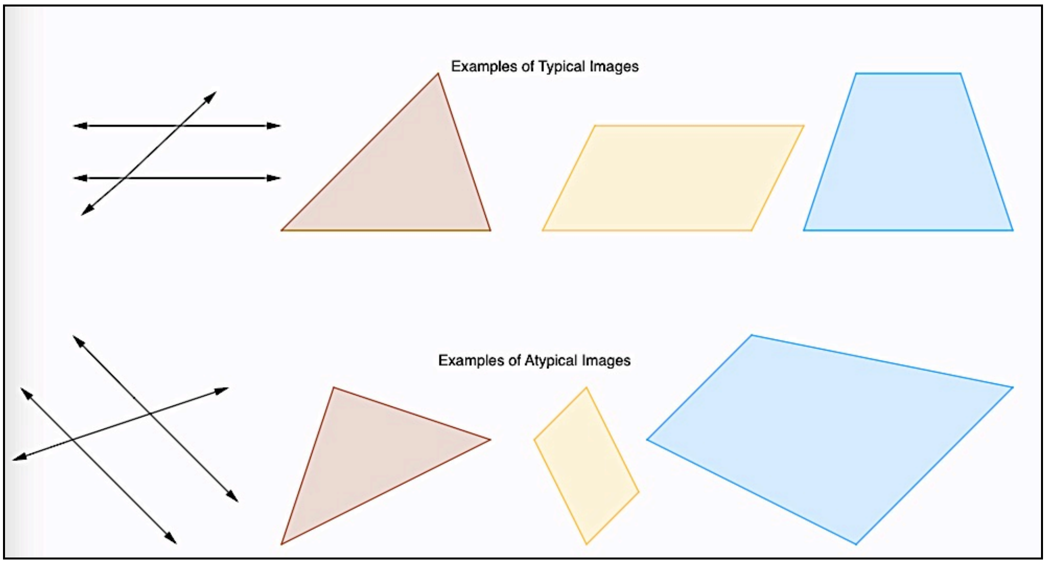

Figure 5: Examples of Typical and Atypical images in each section

Methodology

For each of the textbooks we examined both the lesson and exercise portions of the four previously listed sections. In total, we examined 1444 images in 14 geometry textbooks. We decided to look at these two portions of each section separately to see if there was any difference in the number of typical images presented. Each image in the textbook section was coded as either typical or atypical according to the previously discussed specifications and a tally was kept.

When an image presented ambiguity in coding because it was not a single figure but a composition of more than one figure put together, we examined the questions asked about the represented notions on the figure. We illustrate this with the example on Figure 6. If the question asked is about the triangle ABD then we would code triangle ABD as a typical image of a triangle. If the question asked is about parallelogram AEFB then this image would not be typical because the parallelogram is not aligned horizontally. This was done for each of the questions asked in the composition images, in order to avoid inconsistencies.

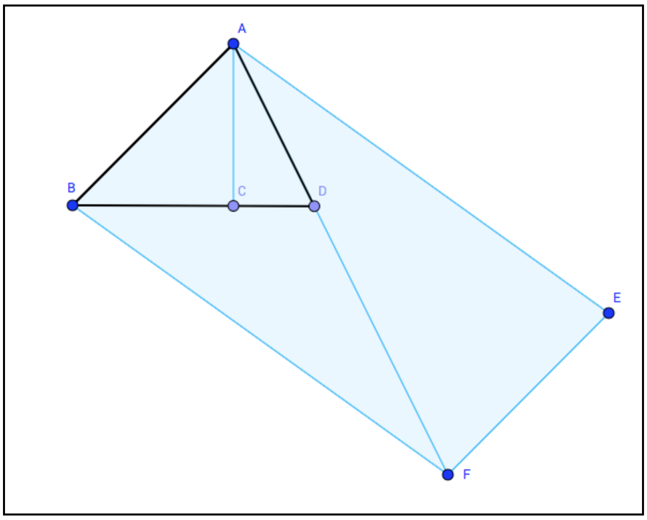

Figure 6: Example of a compilation image needing interpretation for coding

Typical Images by Topic

Table 2 shows the percent of images that are typical by topic for each of the four topics examined: parallel lines and transversals, classifying triangles, parallelograms, and trapezoids.

Topic	Number of Images	Number of Images that	Percent of Images that

	Analyzed	are Typical	are Typical
Parallel Lines and Transversals	270	140	51.9%
Classifying Triangles	576	333	57.8%
Parallelograms	316	236	74.7%
Trapezoids	282	212	75.2%
Total	1444	921	63.8%

Table 2: Percent of images analyzed that are typical by section.

Percent of Typical Images in Lessons versus Exercises

We examined each section's lesson and exercise portions separately in order to see if there were more typical images in one or the other. Table 3 shows the results of this inquiry.

Textbooks	Percent of Typical Images in Lessons	Percent of Typical Images in Exercises
P1	89.5%	67.9%
P2	60.0%	22.9%
P3	90.9%	69.9%
P4	83.3%	98.8%
M1	50.0%	54.4%
M2	80.0%	61.2%
H1	67.9%	65.9%
U1	66.7%	53.6%
C1	51.4%	57.4%
C2	67.2%	68.3%
C3	66.7%	45.5%
C4	63.2%	62.1%
C5	62.5%	64.7%
C6	61.5%	72.7%

Table 3: Percent of images that are typical in lessons versus exercises by textbook

Typical Images in Physical Textbooks versus FlexBooks

Of the 14 high school geometry textbooks examined, 8 were physical printed textbooks and 6 were FlexBooks, which are open sourced, web-based, customizable, and interactive textbooks. One of our suggestions for overcoming limitations in visualization coming from the static nature of images is to use interactive figures that the FlexBooks have ability to provide. However, the CK-12 FlexBooks contained image content that was similar to that of the physical textbooks. FlexBooks

could have easily differentiated themselves from physical textbooks, but they did not fully utilize that capability.
Table 4 compares the percent of typical images found in the physical textbooks versus the CK-12 FlexBooks.

	Number of Images Analyzed	Number of Images that are Typical	Percent of Images that are Typical
Physical Textbooks	859	561	65.3%
Flexbooks	585	360	61.5%

Table 4: Percent of typical images in physical textbooks versus FlexBooks

Conclusion

Discussion

Textbooks play an important role in developing the visual literacy of students so it is appropriate to give more attention to this area of research. If students are not given the opportunity to learn visual strategies for analyzing images then they are less likely to understand mathematical notions that are highly visual, or use the power of visualization in the cognitive process.
The textbook image analysis we have provided used 14 high school geometry textbooks, 8 physical printed textbooks from publishers who dominate the textbook market and 6 FlexBooks from a relatively new and innovative publisher CK-12 Foundation. We examined the use of images in four major geometry topics: parallel lines and transversals, classifying triangles, parallelograms, and trapezoids. For each of these four topics we looked at the section in each book that was most relevant to the concept and counted the number of typical images out of the total images presented for the lesson portion and exercise portion separately. We established what percentage of images seen by students are typical images.
Examination of the textbooks revealed that approximately 63.8% of the images used in corresponding sections of these textbooks are typical images. This average can be specified further to show 65.3% of all images are typical images in the physical textbooks and 61.5% are typical images in the relatively new CK-12 FlexBooks. With such a large portion of the images being typical, students' visual literacy is negatively affected. So in a case where a shown figure is presented with an unusual alignment or representation, students may not recognize the figure (Krajcevski, Sears & Kardes 2017).
Most of the textbooks examined did not have much difference in the number of typical images shown in the lesson versus the number of typical images in the exercises. However, P1, P2, P3, and M2, (Pearson Geometry Common Core, Pearson CME Project Geometry, Pearson Informal Geometry-Classics Edition, and McGraw Hill Geometry Concepts and Applications respectively) had significant differences (more than 15%) between the percentages of typical images found in lessons versus exercises. All four of these textbooks with marked differences had more typical images in the lessons portions than in the exercises portions of the section. This may present major challenges to students. If a lesson shows only typical images, a student facing a problem with an image presented in atypical representation, student may have difficulty recognizing the image as having the same properties as the typical form of the image.
Of the four sections we analyzed, at least 50% of the images were typical. The sections on parallel lines and transversals and classifying triangles were in the lower end of this having 51.9% and 57.8% typical images respectively. Of the parallelograms examined 74.7% were all drawn as typical images and 75.2% of trapezoids were typical.

Limitations

Our sample for the study used a range of textbooks but the sample is still not fully representative of all the textbooks currently in use in the country. While these choices provide a wide range of textbook types, the sample of books could be expanded. Our study also examined four sections of these textbooks, parallel lines and transversals, classifying triangles, parallelograms, and trapezoids.

The typical images in these sections may not represent the percentage of typical images in other topic areas. We included only geometry textbooks because they provide the largest number of images for analysis, but the results could be different if we looked instead at Algebra or Calculus textbooks.

The examination of the CK-12 FlexBooks occurred until March 1st 2016. The FlexBooks open-sourced web-based format allow for quick changes and revisions to the content. It is conceivable that these books may have changed from the time they were originally analyzed.

Implications and Conclusions

This study shows that visual material presented to students needs more variety. This lack in diversity of images could lead to the creation of misconceptions if students are presented with images that are not aligned in the typical fashion or have an unusual orientation (Nardi 2014). Teachers should be active in incorporating more varied imagery into their classroom lessons and show connections between different ways of representing mathematical notions. Our study also indicates that more varied imagery should be included in the geometry textbooks. It is our belief that there is potential for FlexBooks to be used successfully in the classroom. While they do not yet differ much from physical textbooks, FlexBooks have the ability to adapt and change rapidly at the demands of the learners. In addition, teachers can customize them to the needs of their classroom and students. More interactive material could be added to the FlexBooks to provide diverse opportunities for visualization to enrich the mathematical learning process.

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WHEN IS AN EXPLORATION EXPLORATORY? A COMPARATIVE ANALYSIS OF GEOMETRY LESSONS

LESLIE DIETIKER and ANDREW RICHMAN

Abstract

This paper presents a comparative analysis of two textbook lessons on the same topic from U.S. textbooks to learn how differently-designed "exploratory" lessons may structure content to enable or constrain student inquiry. One lesson, representative of a "reform-based" textbook, contains investigations of conditions of triangle congruence. The second is a "technology lab" on triangle congruence from a "traditional" textbook, the design of which is atypical for the that textbook. Framing a lesson as a mathematical story, this analysis exposes three distinct ways that these lessons are different: (a) the proportion of the lesson in which mathematical questions remain unanswered, (b) the manner in which content unfolds to address each question, and (c) the way in which open mathematical questions overlap to increase the dynamically-changing number of questions that are pursued. This contrast of the two lessons illuminates how a lesson structure can prevent an "exploration" from being exploratory.

Introduction

In the United States, calls for written curriculum that is designed to support inquiry and problem solving have been increasing in recent decades (e.g., NCTM 2000, NCTM 2014, CCSS 2012). Curriculum designers and publishers have responded to this call by including "explorations" in their lessons, which we define as activities without proscribed procedures that are focused on answering at least one large mathematical question as the focus. However, given inevitable variations in the designs of explorations, we are concerned that some lessons that appear exploratory may not fully support student inquiry. As the call for shifting the mathematical experiences of students toward inquiry and problem solving becomes increasingly international, understanding how the designs of written curricula enable or constrain these experiences is needed.

To begin to address this concern, this study compares two mathematical lessons on the same topic from U.S. textbooks to explain how the sequence of tasks in an activity may impact the experience of a learner. One lesson, representative of a "reform-based" textbook ("A"), contains investigations of conditions of triangle congruence. The second is a "technology lab" on triangle congruence from a "traditional" textbook ("B"), the design of which is atypical for the that textbook. Comparing these lessons offers the opportunity to distinguish between differently-designed "explorations" and allows us to recognize how the structure of content may enable or constrain student exploration.

Theoretical Framework

In order to compare the unfolding mathematical structure of these lessons, we interpret the mathematical content of written curriculum as a form of narrative. Specifically, we interpret how mathematical content unfolds sequentially for a reader as a *mathematical story* (Dietiker 2013, 2014 & 2015). This conceptualization focuses on the mathematical content, interpreting mathematical objects such as numbers or triangles as characters of the mathematical story. This framing also considers the manipulation of mathematical objects as mathematical action. The setting of a

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mathematical story can be thought of as its representational space, such as a number line on paper (for adding integers) or on a plane represented with dynamic geometry software.

Drawing from the literary theory of Barthes (1974), we propose that a reader asks questions of a story (based on what limited information has been introduced so far) and continues reading in the hopes that these questions will be answered. As mathematical questions are raised and tackled, the changing moment-to-moment tension felt by a reader between what is known and what is desired to be known can be interpreted as the *mathematical plot*. Thus, a mathematical plot attends to two characteristics of the mathematical story: the sequence in which events occur within the story (i.e., the way the story controls what the reader learns and when they learn it) and what is known and not known by the reader throughout the story as the sequence unfolds (i.e., realizations made in the moment). In terms of a written mathematics lesson found in a textbook, the mathematical plot represents the coordination of the *acts* of the mathematical story (the sequential parts of the lesson) and a reader's *questions* that are raised and addressed throughout the story (what is known and not known by the reader).

Methods

In order to begin to address the research question, "How do mathematics written materials potentially enable or constrain explorations?" we compared two textbook lessons with explorations, both of which are the first lesson in their respective textbooks to introduce the side-side-angle triangle congruence ambiguity principle (sometimes referred to as "SSA"). We chose lessons focusing on SSA anticipating that differing treatments of the ambiguity might provide interesting variations in the lessons. One lesson was selected from Textbook A, a book that regularly features investigations. This textbook also is designed to focus on inductive reasoning whereby students are expected to conjecture based on the results of an experiment that involved geometric construction. The lesson in this textbook that was analyzed in this study ("Lesson A") is the first of two that explore triangle congruence shortcuts. This lesson first introduces the idea of a congruence shortcut by describing a potential use of congruence shortcuts in building trusses for houses. It then lists the six potential triangle congruence shortcuts and structures paper and pencil explorations that test three of them – Side-Side-Side (SSS), Side-Angle-Side (SAS), and SSA.

The other textbook lesson was selected from Textbook B, which is generally designed so that most lessons provide definitions, examples, and theorems directly before presenting tasks or questions for students. However, the lesson analyzed in this textbook ("Lesson B") deviated from the majority of lessons in Textbook B by consisting solely of two explorations. These explorations, which require the use of dynamic geometry software, ask students to manipulate specific dimensions of triangles while holding others constant. After the figures are manipulated, the text asks students to observe that Angle-Angle-Side (AAS) and SAS combinations are congruence shortcuts while SSA is not.

For each textbook lesson, only the recommended instructional components of the lessons were analyzed as part of the mathematical story of the lesson, as opposed to optional parts or homework problems. The mathematical plot of each lesson was analyzed by identifying three aspects of each mathematical story which are described below: the acts, mathematical questions raised by the story, and all forms of progress on each question. The mathematical plots were then compared to learn how the lessons were similar or different.

Acts. First, each mathematical lesson was interpreted as a mathematical story that connects start to finish. In order to determine the sequence of the textual parts, the lessons were assumed to flow from top to bottom of each page. For callouts, such as teacher notes on the side of the page, the text was linked to corresponding portions of the student text in order to determine its placement within the sequence of the mathematical story.

With the sequence determined, each mathematical story was then subdivided into multiple, sequential acts. A new act was identified each time new story elements (i.e. mathematical characters, actions, settings, or relationships) were introduced or became the focus of the lesson.

Mathematical Questions. Once the acts of the mathematical story were defined, the mathematical questions that emerge throughout the story were identified. Both explicit questions raised by the text and implicit questions suggested by goal statements or other content were coded. For example, the goal statement “In this lesson you will learn about congruence shortcuts for triangles” can raise the implicit question, “What is a congruence shortcut?” These questions were identified by individual coders, who then met to resolve differences.

Coding Forms of Progress on Mathematical Questions. Then, for each question, the research team tracked throughout the lesson any change in what is known about that question, whether it is some part of its answer (which we referred to as *partial answer*) or some form of misdirection that subverts progress (e.g., when a textbook implicitly sets up an incorrect assumption, which we refer to as *equivocation*). Table 1 lists the mathematical plot codes, adapted from Barthes (1974), that describe how progress can be made (e.g. *partial answer*) or thwarted (e.g., *equivocation*) toward the answer to each question. Collectively, the transition from the formulation of each question to its answer forms a *story arc*. Story arcs can last for just one act if the question is immediately answered, or they can extend for the entire lesson. By juxtaposing the acts as columns with the mathematical questions as rows, the mathematical plots are represented as diagrams in Figures 1 and 2. The shading in each row illustrates in which acts the question is open and thus represents a story arc for a mathematical question.

	Code	Description
0	Proposal	A hint or undefined mystery that sets up anticipation.
1	Formulate Question	A question that is raised explicitly or implicitly in the text.
2	Promise	An explicit indication that a question will be answered later.
3	Partial Answer	Progress is made toward an answer without endorsement.
4	Equivocation	Misdirection through ambiguity that leads to an incorrect assumption.
5	Delayed Disclosure	A question is formulated and answered by the reader in the same act but is disclosed more than one act later.
6	Disclosure	An explicit revelation of the answer in the text (or a teacher is directed to disclose the answer).

Table 1. Mathematical Plot Codes adapted from Barthes (1974)

	Lesson A	**ACT 1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**beyond**
1	What is a congruence shortcut?	1			3		3	3							
2	Are there congruence shortcuts?	1			3		6								
3	Must all six parts of both triangles be compared?		1	3			6								
4	What are professional applications of triangle congruence?		13							3					
5	Does AAA guarantee triangle congruence?			13					3					2	
6	What more about the sides do we need for AAA?			1							3			2	
7	What is the least amount of information needed?		0		13		6								
8	How can we show 1 or 2 congruent pairs are not sufficient?				136										
9	Are three pieces of information sufficient?					1	6								
10	What are all the combinations of S and A?					13	6								
11	Is SSS a shortcut?						12	13	6						
12	Is SAS a shortcut?						12				13			6	
13	Is ASA a shortcut?						12							132	
14	Is SAA a shortcut?						12							132	
15	Is SSA a shortcut?						124					13	3	6	
16	Is AAA a shortcut?						124		3					132	
17	How do I construct a triangle with these given lengths?							16							
18	Are reflected triangles congruent?							136							
19	How does determining relate to congruence?								13						
20	How do I construct a triangle with SAS?										13				
21	How do I construct a triangle with SSA?											13			
22	What is a counterexample for SSA?											13			
23	When is SSA sufficient to prove congruency?												13		
24	What are the other arrangements of three parts?													136	

Figure 1. Mathematical plot of Lesson A, where each digit in the story arcs refers to a mathematical plot code in Table 1.

	Lesson B	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	What are other triangle congruence relationships to learn?	1					3	3	3		3			3			
2	How do I construct these two angles?		136														
3	How do I measure angle DGA?		136														
4	What happens to ∠ DGA as moves ∠ CAB from left to right?		135			6											
5	What is the measure of the distance from A to D?			136													
6	Can you change the shape without changing AD, ∠ A, or ∠D?			136													
7	Are your results the same with a different angle measure?				135				6								
8	Do the results change if one of the given angles is 90º?				135				6								
9	What theorem proves that m∠DGA will always be the same?					13											
10	Does AAA, like SSS, make only one triangle?	0					13		6								
11	Why does AAA make only one triangle or not?						13										
12	Are the results the same if you measure length AG instead?							136									
13	Does the result in Q12 lead to a new congruence postulate?	0						13									
14	Given AA, what additional piece of information is needed?							13	3							6	
15	What conjecture can you make based on your findings?							13									
16	How do I construct YZ with a length of 6.5cm?									13							
17	How do I construct XYZ measuring 43º?									13							
18	How do I construct circles with radiii 5cm and ZW?									13							
19	What are the possible shapes of ΔYZW as W moves?									13							
20	How many different triangles were possible (given SSA)?										136						
21	Does SSA make only one triangle?	0									14	3	3	3	3		
22	How many different triangles are possible if A= 90º and r=7?											135			6		
23	What kind of triangle do you have if A=90º and r=8.25 cm?												13				
24	What must be true about the angle for SSA congruence?													14	6		
25	How are the ASA, AAA, and SSA the same or different?																13

Figure 2. Mathematical plot of Lesson B, where each digit in the story arcs refers to a mathematical plot code in Table 1.

Findings

By analysing the mathematical plots of these selected lessons as represented in Figures 1 and 2, we have identified three mathematical plot characteristics that describe structural ways in which these explorations differ that can potentially impact the experiences of a learner.

Story arc length. In general, Lesson A has longer story arcs, keeping mathematical questions open for longer and thus allowing and fostering sustained curiosity. On average, story arcs in Lesson A remain open for 4.5 acts whereas the story arcs in Lesson B remain open for 2.4 acts. A majority (63%) of its story arcs are open for more than one act. In fact, Lesson A is particularly interesting as it raises mathematical questions that are not addressed until the next lesson, which creates the opportunity for students to wonder about these congruence shortcuts independently after the end of this lesson.

In contrast, despite having two explorations, Lesson B is designed for questions to be answered relatively quickly. In its mathematical plot, a majority of its story arcs (60%) are only one act long. Even some story arcs that are longer than one act are shortened by delayed disclosures, which occur when the textbook shifts the focus after students address a question, but then discloses the answer one or more acts later.

The unfolding of content within story arcs. Within the story arcs of both lessons, the nature of the unfolding content further demonstrates how these lessons enable or constrain exploration. For example, the story arcs in Lesson B are much more active than those in Lesson A. On average, 76% of each Lesson B story arc, vs. 47% of each Lesson A story arc, contains at least one code describing some changing aspect of the content. This difference is reflected in Figures 1 and 2 as there are more shaded cells that are empty (i.e., without codes) in Lesson A than Lesson B. Thus, Lesson A provides more opportunities for readers to wonder about questions even when they are not being actively addressed. This structure helps build in students a habit of inquiry whereby they are looking for answers rather than expecting to be guided to them.

One prominent way that Lesson A keeps some questions open for long portions of the mathematical story without changing what is known is through promises. For many of the key mathematical questions of Lesson A, such as Questions #12 – 16, the mathematical story promises answers (code "2") that will not become realized for a reader for as many as seven acts. These promises assure students that the questions will be answered, thus encouraging students to wonder about them even when the questions are not the explicit focus of the current activity.

Although both lessons have two *equivocations*, that is, instances in the lessons when students are encouraged to make an incorrect assumption, the equivocations in Lesson A potentially support sustained curiosity while those in Lesson B do not. The equivocations in Lesson A occur in Act 6 when the book displays the six potential triangle congruence shortcuts. All six are displayed by marking the congruent parts on triangles that appear congruent. This suggests that all six potential shortcuts guarantee the congruence of triangles. This is coded this as an *equivocation* for questions 15 and 16 (*Is SSA a shortcut?* And *Is AAA a shortcut?* respectively) since these pairs of known information do not guarantee congruence. This equivocation supports student inquiry because it contradicts an accompanying statement in the text that indicates that some of the combinations of information (AAA, AAS, ASA, SSA, SAS, and SSS) will guarantee congruence and others will not. Thus, students are directly challenged to resolve this contradiction and carry these questions into the next act.

In Lesson B, no such contradiction is created by the equivocations. The first equivocation occurs in Act 10 when students construct an SSA triangle where the included angle has two possible measures. This enables students to answer "no" when asked whether constructing a triangle given Side-Side-Angle can result in only one triangle. This is an equivocation because there are, in fact, conditions under which SSA does form only one triangle. However, unlike in Textbook A,

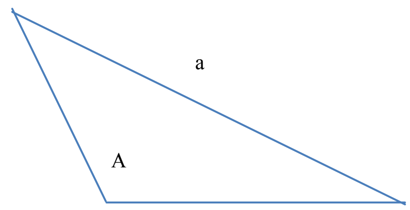

Figure 3. A case where two side lengths *a* and *b* and non-included obtuse angle *A* are given and only one triangle is able to be constructed.

Textbook B offers no indication that there is any other possibility, so the equivocation does not support student curiosity. A second equivocation in Lesson B, in Act 11, occurs when students learn that SSA right triangles form only one triangle and use this fact to answer a problem that asks students to identify what type of non-included angles result in only one triangle. Since the expected answer to this problem is a right angle and since the tasks never propose the investigation of other types of angles, the lesson enables a student to assume that right triangles are the only case of SSA that only form one triangle. This precludes a student from recognizing that when the given non-included angle is obtuse and only one triangle is possible (see Figure 3). Again, since there is no indication that another answer is possible, the equivocation does not spur student curiosity.

Density of open mathematical questions. An analysis of the density of each lesson, which is a measure of the number of open story arcs in each act, also reveals how these mathematical stories are different. The average density in Lesson A, that is, the average number of open questions per act, is 8.3 as compared to 3.7 for Lesson A. In other words, on average, each act in Lesson A has twice as many open questions as Lesson B. This means that Lesson A has a "thicker plot," making it more likely to offer a general sense of a multifaceted lesson. A reader is offered much to consider at any given moment and the potential for reader anticipation is likely to be higher.

The overlapping story arcs in Lesson A also tend to cluster more than in Lesson B. In the plot diagram of Lesson A, large blocks of adjacent multi-act story arcs are evident, while in Lesson B, the overlapping story arcs are more scattered. This suggests that the story arcs in Lesson A are more related to each other, and build on each other progressively, providing an enhanced feeling of coherence to the lesson.

The changing levels of density as the lessons progress suggest that Lesson A might provide a more engaging structure of inquiry than Lesson B. As can be seen in Figure 4, the number of open mathematical questions in Lesson A increases for nearly the first half of the lesson. It reaches a peak of 15 open questions in Act 6, after which it then begins to resolve more questions than it opens. This decrease in density continues for approximately a quarter of the lesson, after which it levels off to have about 9 open questions through the end of the lesson. In comparison, Lesson B instead maintains a relatively constant amount of open questions (either 3 or 4) for a majority of the lesson. It does increase to 6 open questions midway through the lesson (in Acts 7 through 9) before returning to a level state in Acts 10 through 14. Finally, the density of Lesson B drops to 1 question as it ends.

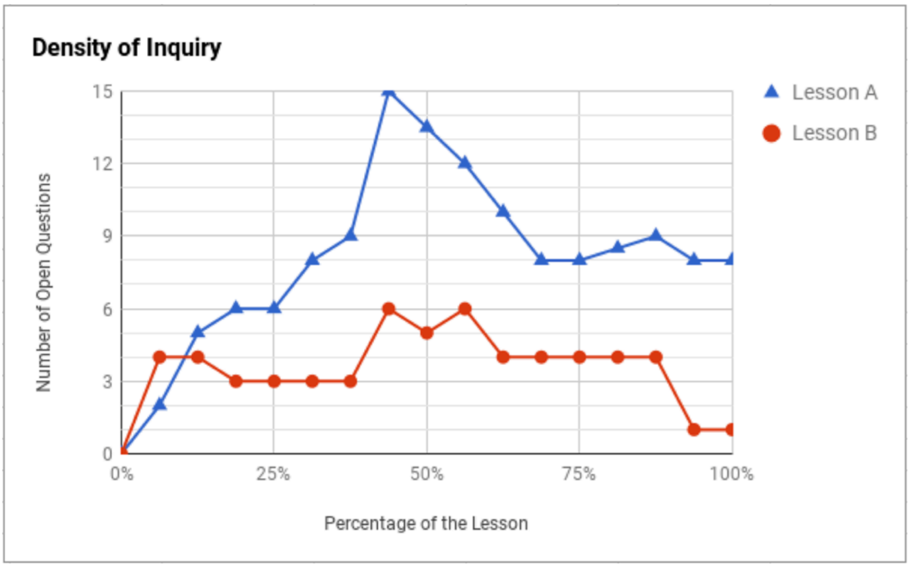

Figure 4. The density of inquiry of Lessons A and B.

The difference in the shape of density between these two lessons highlights a potential felt difference in the way in which the lesson could be experienced by a student. When mounting questions increase in a literary story, a deepening sense of mystery can result in a feeling of suspense. In a similar way, when a mathematical story enables a reader to recognize an increasing number of questions that are unanswered and under consideration over a prolonged period of time, there is a similar potential for heightened anticipation for the questions to become answered. For this reason, we conclude that the design of Lesson A, with its numerous overlapping story arcs, enables a reader to develop a sense of growing mystery throughout the first part of the lesson. These layers of related, unanswered questions that remain open for long periods of time can allow students to anticipate a “climax” during which the plot “turns” to resolve tension. In contrast, Lesson B feels “flat,” and therefore may be experienced as predictable and uninteresting.

Discussion

Our study illuminates how a lesson structure can prevent an “exploration” from being exploratory. The mathematical plot of Lesson B is designed in a way that does not facilitate sustained curiosity. Readers are prevented from developing an interest in where the lesson is headed due to short story arcs that are only open when a question is being addressed and a lack of promises and contradictions. A consistently thin plot means that Lesson B provides little to wonder about at any given time and does not build to any dramatic moment. In contrast, the mathematical plot of Lesson A is sequenced to slowly reveal and withhold information in a way that cues the learner of what lays ahead, which can potentially increase student curiosity and sustain the student’s interest in completing the story. Lesson A raises and keeps open multiple questions through significant portions of the story. It also uses ambiguity, promises and a progressively thickening plot to potentially increase anticipation and curiosity for what is to come.

We propose that the design of Lesson B may be constrained by the broader mathematical story within which it is placed (i.e., the sequenced lessons that come before and after), which likely creates an expectation from readers that mathematical questions will be asked and answered quickly and in a straightforward manner. The regularity with which explorations are used in Textbook A, however, may prepare readers to persevere though story arcs that not only last longer but stay open in the background and even take an indirect path to disclosure. Further study is needed to learn whether, in general, the broader design principles of textbooks impact the extent to which explorations support inquiry in the way that Lesson A does in this study.

Thus, we argue that the mathematical story framework offers new understanding of how designed textbook content incrementally emerges and changes as a lesson unfolds. This understanding is important as it can distinguish between textbooks that add only superficial features that appear to be consistent with reform from textbooks that are structured throughout to provide students with the experiences that reforms are designed to promote. This distinction is not only important for researchers as they examine curricular materials and decision makers as they choose materials, but can also help curriculum writers and teachers make the design and pedagogical decisions that shape the daily instruction of students in the classroom.

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HOW BOOKS FROM THE 6TH TO THE 9TH GRADE PROPOSE HORIZONTAL TREATMENT OF COMBINATORICS

ANA PAULA LIMA and RUTE BORBA

Abstract

In the present study, part of a doctoral research in progress[1], it is proposed an investigation on how combinatorics is treated in textbooks used by mathematics teachers of the Colégio de Aplicação (CAp), a middle and high school maintained by the Universidade Federal de Pernambuco – UFPE. In this sense, the aim of the present study is to analyse how *Horizon Content Knowledge*, one of the six dimensions of teacher knowledge proposed by Ball, Thames and Phelps (2008), is encouraged in textbooks, so that it may help teachers in a gradual teaching of combinatorics along basic education. Thus, it is expected that textbooks~~s~~, being instruments widely used by teachers for the teaching of mathematics, should propose a gradual deepening of concepts, in particular those present in combinatorial situations, from one school grade to another and from one level of schooling to another.

Keywords: Textbooks; Combinatorics; Teacher's Knowledge.

Introduction

In Brazil one of the public policies focused on education, is the free distribution of textbooks for students of basic education. These books are evaluated and distributed by the Brazilian National Textbook Programme (PNLD), established by the decree nº 91.542, of 19/8/85, that periodically analyses textbooks for the different segments of Brazilian basic education. For middle school, the evaluations of new books occurred in the years 1999, 2002, 2005, 2008, 2011, 2014 and 2017.

For Gérard e Roegiers (1998), a textbook can have different functions that vary according to its user (government, students or teachers). For teachers, Gérard e Roegiers (1998, p. 89) believe that "the textbooks contribute to the development of pedagogical innovations. [...] the evolution of pedagogical knowledge, the sensitivity of each teacher and the specificity of the contexts". Another function attributed to the textbook is to take to the classroom the didactic and pedagogical changes that are indicated by official curricular documents and by the research developed in the universities (Brasil 2016). Oliveira & Bittar (2015, p. 133) also affirm that "the textbook exerts a strong influence on the didactic and pedagogical activity of the teacher and the acquisition of knowledge by the student".

The study of combinatorics is usually treated less formally in the early years of primary school as guided by the Parâmetros Curriculares Nacionais - PCN (Brasil 1997; 2017), and in high school this content comes to full formalization, being expected in middle school (6th to 9th grade) an intermediate stage of formalization. For middle school, the PCN (Brasil 1998) recommend that the study of combinatorial situations should not be done based on the definition of the different types of problems (*cartesian product, permutation, arrangements* and *combination*) And neither with the use

[1] The general aim of the research is to investigate collaborative activities among mathematics teachers of the CAp - UFPE and how, from these activities, the teachers mobilize different forms of knowledge for the teaching of combinatorics.

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of formulas, thus enabling the development of combinatorial reasoning and the perception that the *fundamental principle of counting* is applicable to the different problems studied. These guidelines are also reinforced by the National Curricular Common Base – BNCC (Brasil 2017) and pointed out in the studies of Pessoa & Borba (2009) and Borba (2010; 2013; 2016). A possible horizontal presentation can help the teacher in the development of a gradual teaching of combinatorics along basic education.

Content Horizon Knowledge is the teacher's knowledge and ability to perceive the relationships between mathematical fields and how connections are present throughout the curriculum. For Ball, Thames & Phelps (2008) the teacher also needs to have a prediction about future mathematical content to be able to conduct a deepening that is being worked on in the classroom.

In combinatorics, for example, it is the knowledge that the teacher has about the teaching of this content of mathematics throughout all basic education, how this should be approached in the different stages of schooling, which recommendations regarding the contents are placed in the official documents and how it is presented in the collections of textbooks adopted by schools in which they work. According to Phelps, Weren, Croft & Gitomer (2014), this type of knowledge also includes the different ideas that are connected in a mathematical field, starting from the most basic to the most complex or advanced ideas.

Thus, it is expected that textbooks, being instruments widely used by teachers for the teaching of mathematics, should propose a gradual deepening of concepts, in particular those present in combinatorial situations, from one school grade to another and from one level of schooling to another.

Method

The main aim of the paper is to analyse how *Horizon Content Knowledge* is encouraged in textbooks, so that it may help teachers in a gradual teaching of combinatorics along basic education. Another aspect observed will be about how this horizontal treatment can stimulate collaborative actions among teachers (from middle school and high school).

Analysis concerning the collection used at middle school, from the Colégio de Aplicação – CAp of Universidade Federal de Pernambuco - UFPE, in the chapters and topics about *data processing*, are presented. The collection was evaluated and approved by the PNLD/2014 and adopted by CAp for the triennium 2015/2016/2017.

Results and Discussion

The analyzed collection presents in the Teacher's guide one table (Table 1) indicating the work with combinatorics in each one of the years of the second stage of middle school.

Content	6th grade	7th grade	8th grade	9th grade
Count of possibilities and probability	Double input table	Trees of possibilities; multiplicative reasoning; notion of probability	Chance calculation	

Table 1- Proposal of work with Combinatorics in the teacher's guide of the collection analysed.

In the Teacher's guide of the 6th to 9th grade textbooks of the analysed collection, the authors point out that the contents presented in Table 1, with the proposal of development of activities, are only the central topics that will be worked on each year. The cells of the table are organized to show only the advances in the study of each of the topics. Thus, the absence of content in the table does not mean that the theme will not be worked on, but rather, that no different idea has been added to the respective year.

In this collection, were identified some activities involving combinatorial situations, distributed throughout the different years, as indicated in Table 2.

Grade	6th grade	7th grade	8th grade	9th grade	Total
Number of problems identified	13	20	4	19	56

Table 2- Activities of Combinatorics present in the collection.

From the proposal presented in Table 1 and the quantitative of combinatorial situations presented in Table 2, it can be affirmed that combinatorics is present in all volumes of the collection, thus ensuring, that the teacher should work in final middle school, with combinatorial situations. It is this constant presence along the collection that can indicate a *horizontal* work with combinatorics. To confirm this hypothesis (of the intention of the authors of the collection to point out a constant work with combinatorics), the guidance given to teachers and the approach to the situations identified will be analysed.

The following examples are presented in order to identify the horizontal treatment of the 6th to 9th grade combinatorial situations and if the guidelines given to the teacher can, in any way, encourage collaborative work among mathematics teachers who work in the classes of middle school.

For the 6th grade, the authors claim to start the work on counting possibilities from the study of simple situations in which the double entry table, (Figure 1) is an initial resource to calculate all the required possibilities. In that same year, it is already advanced to the teacher that in the 7th grade

Figure 1: Example of a combinatorial situation in which a double entry table is used as a resolution strategy (Source: 6th grade, p. 28).

the students will come in contact with possibilities trees and the multiplicative reasoning to solve counting situations. According to the collection, the possibilities tree will provide students with "the perception that multiplication relates to certain counting problems" (6th grade, Teacher's guide, p. 37) and thus students can solve situations a little more complex involving a greater number of possibilities.

This type of orientation presents indications of a horizontal work of combinatorics. It starts the study from simple situations and with the table as resource to solve these problems, but already anticipates to the teacher that the students the following year (7th grade) will come across another type of resource, which is the possibilities tree. It is suggested, therefore, that the situation will be more complex than those presented in the book of the 6th grade and thus it is recommended to use a representation that enables more systematization.

In this way, indirectly, there is encouragement for collaborative work among teachers, since a teacher should take into consideration the work done previously by another teacher and seek, from there, to expand the concepts to be studied by students.

The textbook of the 7th grade, according to Table 2, has more combinatorial situations (20 in total) and points out that the count of possibilities, tables, possibilities three and multiplicative reasoning will be used as resources to solve the different situations presented. According to what is presented, the idea is that students perceive patterns in simple situations and that can generalize to more complex cases. This proposal can stimulate, in teachers, the development of the *Horizon Content Knowledge*, which is when teachers must understand how from the double entry table, e.g., can stimulate their students at other times to build trees, perceive patterns and use multiplicative procedure.

The suggestion for the teacher to discuss with students different resolutions, can be seen in the orientation given to the problem of Figure 2, “Some students will solve the problem without using a table or possibilities tree. Always try to discuss with the class the different solutions to a problem” (7th grade, Teacher’s guide, p. 73). The idea is for the teacher to stimulate his students in the perception of patterns from simple cases to the generalization of more complex cases, It is believed, therefore, that this type of orientation can take the teacher to develop *Horizon Content Knowledge*.


40. Um membro do serviço secreto de certo país, chamado LIN, manda mensagens em código. As palavras do código têm 3 letras e são formadas pelas letras L, I e N.

a) Escreva as três palavras começadas por LL. LLL, LLI, LLN

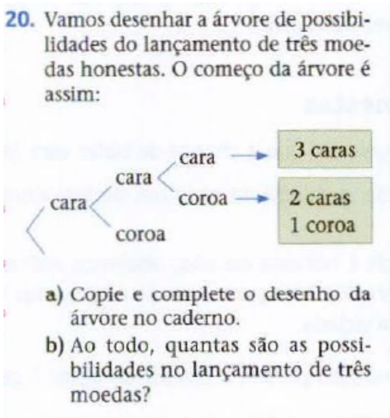
b) Escreva todas as palavras começadas por LI. LIL, LII, LIN

c) Quantas são as palavras começadas por L?

d) No total, quantas palavras podem ser formadas nesse código?

Figure 2- Example of a problem to discuss different solution strategies (Source: 7th grade).

For the 8th grade, few combinatorial situations are proposed, in which the orientation given to the teacher in the guide is that they are basic examples so that the student can perceive differences between situations that involve the calculation of probability to those that involve the calculation of possibilities. In the students’ textbook, some examples are explored using the possibility tree and the double entry table, for the student to visualize all the possibilities. But there are no guidelines on how the teacher should address these issues. It is clear, then, that combinatorial situations, are used as tools for understanding probability (Figure 3), but one does not directly perceive orientations about this articulation along the student's textbook or the teacher's guide.

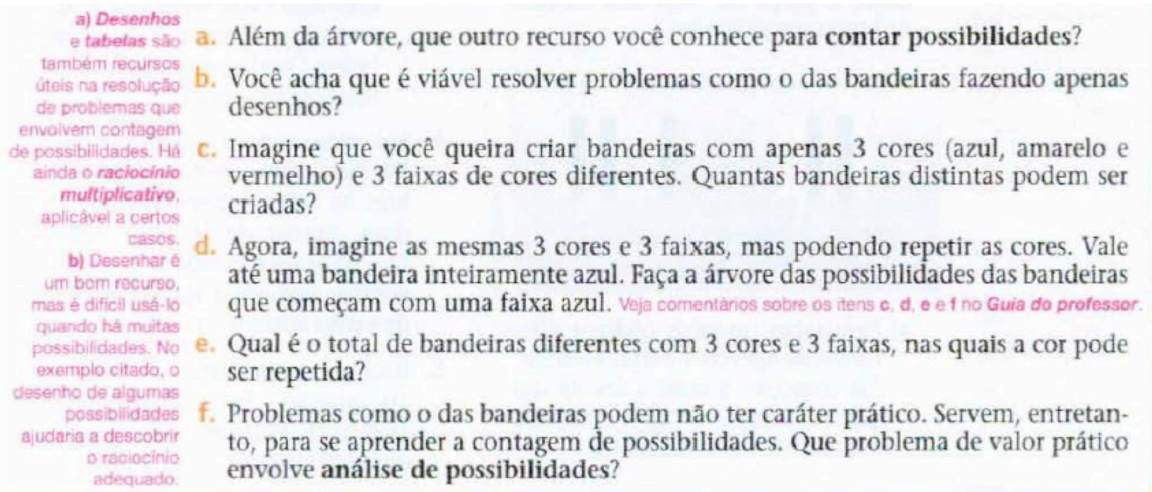

20. Vamos desenhar a árvore de possibilidades do lançamento de três moedas honestas. O começo da árvore é assim:

a) Copie e complete o desenho da árvore no caderno.

b) Ao todo, quantas são as possibilidades no lançamento de três moedas?

Figure 3- Example of problem situation to stimulate the visualization of all possibilities requested. (Source: 8th grade, p. 181).

In the 9th grade textbook, 19 problems involving combinatorial situations were identified. The proposal of this textbook, presented in the teacher's guide – TG, is an articulated work between the contents of statistics, probability and combinatorics, because, according to the authors of the collection analyzed, “the themes are interrelated: to determine the probability of certain events, it is necessary to solve counting problems; to ensure the validity of a statistical inference, we need to know the probability of the sample be trusted” (9th grade, TG, p. 49). Although affirming that this interrelation can be considered complex for a 9th grade class, the orientation for the teacher is that basic ideas be constructed between these contents. These orientations favor the development in the teacher of *Horizon Content Knowledge* when searching for this interrelation between these different fields of mathematics. In addition, there is also guidance on, starting from a simple approach, the teacher can help the student in developing more complex ideas that will be worked on during high school.


a) ***Desenhos*** e ***tabelas*** são também recursos úteis na resolução de problemas que envolvem contagem de possibilidades. Há ainda o ***raciocínio multiplicativo***, aplicável a certos casos.
b) Desenhar é um bom recurso, mas é difícil usá-lo quando há muitas possibilidades. No exemplo citado, o desenho de algumas possibilidades ajudaria a descobrir o raciocínio adequado.

a. Além da árvore, que outro recurso você conhece para **contar possibilidades**?

b. Você acha que é viável resolver problemas como o das bandeiras fazendo apenas desenhos?

c. Imagine que você queira criar bandeiras com apenas 3 cores (azul, amarelo e vermelho) e 3 faixas de cores diferentes. Quantas bandeiras distintas podem ser criadas?

d. Agora, imagine as mesmas 3 cores e 3 faixas, mas podendo repetir as cores. Vale até uma bandeira inteiramente azul. Faça a árvore das possibilidades das bandeiras que começam com uma faixa azul. Veja comentários sobre os itens **c**, **d**, **e** e **f** no ***Guia do professor***.

e. Qual é o total de bandeiras diferentes com 3 cores e 3 faixas, nas quais a cor pode ser repetida?

f. Problemas como o das bandeiras podem não ter caráter prático. Servem, entretanto, para se aprender a contagem de possibilidades. Que problema de valor prático envolve **análise de possibilidades**?

Figure 4 - Example of a sequence of activities to discuss different resolution strategies. (Source: 9th grade, p. 99).

The topic about combinatorics, according to teacher's guide (9th grade, TG p. 49) “can be considered as a collection of challenging problems” (Figure 4) and these problems can help in the development of combinatorial reasoning, in the perception of standards and regularities and in the generalization of resolution procedures.

Guidance for the teacher when planning this lesson, is that the basic ideas of counting are discussed with the students so that there is a recapture of what was seen about combinatorics in previous years. It is believed that guidelines such as these may favor the development of *Horizon Content Knowledge* of Mathematics’ teachers who make use of this material.

Some Considerations

At the end of the collection analysis, it is noticed that along the activities there are orientations made for the teacher on how to work with the different situations during each stage of middle school. The guidelines suggest that strategies be used gradually to solve the proposed situations, starting with double entry tables in the 6th grade and in the following years the use of the possibilities tree is incorporated so that, in this way, the student can perceive patterns in the resolution of the different complex combinatorial situations.

Throughout the activities there are guidelines made for the teacher on how to work with the different situations during each stage of the middle school. The guidelines suggest that strategies be used gradually to resolve the proposed situations, beginning with drawings and listings and multiplicative reasoning in the 6th grade and in the following years are incorporated the use of the possibilities tree so that students can see patterns in the resolution of various complex combinatorial situations.

This type of orientation and the presentation of combinatorial situations may favor the development of a collaborative work (although not explained in the guidelines presented in the teacher's guide) among the teachers of mathematics who teach in middle and high school.

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ONE-STEP MULTIPLICATION AND DIVISION WORD PROBLEMS IN THE 3RD GRADE TEXTBOOKS IN BOSNIA AND HERZEGOVINA

KARMELITA PJANIĆ

Abstract.
According to the curriculum for the 3rd grade in Bosnia and Herzegovina, concepts of multiplication and division are taught by modeling real world situations expressed in words and (or) a picture.
The main goal of this paper is to analyze the types of one-step multiplication and division word problems represented in mathematics textbooks for the third grade of primary school in Bosnia and Herzegovina. We examine distribution of one-step multiplication and division word problems that includes 5 multiplication structures (equal groups, rate, multiplicative comparison, rectangular array and Cartesian product) in four 3rd grade textbooks approved by relevant ministries of education.
The research results show that all types of one-step multiplication and division word problems are not adequately represented nor distributed in the third grade mathematics textbooks. Results of the qualitative analysis point to inconsistencies in division problems formulations and, in some cases, non-compliance of word problem formulation with the accompanying illustrations.
Since the textbook is the basic source of information to students as well as the basic, and sometimes the only source of ideas to teachers in lesson preparation, it is necessary to revise word problems given in textbooks and ensure that all types of multiplication and division word problems are presented in textbooks, which would contribute to a better understanding of multiplication and division concepts among the third grade pupils.
Keywords: one-step word problems, multiplication, division, textbook, Bosnia and Herzegovina

Introduction

Pepin and Haggarty (2001) emphasize that the textbook makes the connection between school and relevant knowledge. Stray (1994) defines a textbook as a book designed to give an authoritative pedagogical version of a particular area of knowledge and notes that textbooks are message carriers that are multiple encoded. Namely, the textbooks encode the meanings of some part of the knowledge i.e. provide what is supposed to be learned, and combine it with pedagogical meanings providing the way to learn content. Poljak (1980) defines a textbook as a basic school book written on the basis of a prescribed curriculum, a book that students use on a daily basis in their education, and which has been designed to make teaching and learning more rational, more optimal, more economical and more efficient.

The mathematical textbook can be described as a pedagogically formatted and officially authored mathematical book written to offer mathematical contents to students (Pjanić 2016). Mathematical textbooks play an important role in learning and teaching mathematics, especially in the process of teaching mathematics, as they are the primary teaching tool for both math teachers as well as students. Teachers use the textbook to prepare the class, and the students use it at the same time for independent work at home. The widespread use of mathematical textbooks in the process of teaching mathematics around the world draws the necessity of analyzing content and structure of the textbook. Pepin and Haggarty (2001) conclude that textbooks should be analyzed not only within the framework of content and structure but also in the teaching process.

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Numerous studies have focused on the way teachers use math textbooks in the teaching process. Researchers analyzed the way teachers teach mathematical content depending on the contents of textbooks (Freeman and Porter, 1989), the use and influence of textbooks in mathematics teaching, or they are focused on analyzing and comparing textbook contents (Fan and Kaeley, 2000, Fan and Zhu, 2007; Freeman and Porter, 1989; Boonlerts and Inprasitha, 2013).
As a math word problem we consider the problem situation described in the words, whose solution can be obtained by applying mathematical operations to the numerical data available in the problem statement. Typically, these problems are expressed in the form of a short text describing the relationship between the quantities. The student is required to translate the words to the numerical or algebraic expression. After the necessary computation, the numerical value obtained has to be interpreted in the sense of the problem context. Often, in a math class, a word problem is followed by an image. Since there is no simple or unique way of solving, it is important for a student to think and analyze the problem described in the words before trying to solve it.

Multiplication and division word problems

Multiplication situations can be classified according to the nature of the quantities involved and the relation between them (Nesher 1988; Vergnaud 1988, Schmidt & Weiser 1995). Greer (1992) suggested four categories that primarily apply to problems involving the multiplication of whole numbers: equal groups, multiplicative comparison, rectangular arrays and Cartesian product. Greer (1992) also highlights the range of different external representations for these situations. Equal groups and Cartesian product situations can be represented by diagrams of equal groups of objects and arrays respectively. Usiskin and Bell (1983) proposed thayt multiplication has three use classes: size change, acting across and rate factor, each of which is very rich in the breath of its applicability. These multiplication use classes are related to division use classes: size change divisor, recovering factor and rate divisor, respectively. Division has to other use classes: rate and ratio (Usiskin & Bell 1983).

According to previous findings, five multiplicative structures can be identified: equal groups, allocation/rate, multiplicative comparison, rectangular arrayy/area of rectangle and Cartesian product. Word problems indicating equal groups, rate and multiplicative comparison are asymmetric problems, while word problems indicating rectangular array/area of rectangle and Cartesian product are symmetric problems (Chapin & Johnson 2006).
Taking into account semantic structure of word problem we can differentiate 18 types of multiplication and division word problems that are presented in the Table 1.

Multiplicative structure	Multiplication word problems	Division word problems	
Equal groups	multiplication EGM	partition EGP	quotition EGQ
	A · B = __	A · __ = C	__ · B = C
	There are A boxes with B toys. How many toys are there in all?	C toys are shared equally in A boxes. How many toys will be in each box?	C toys are to be packed in boxes, B toys in each box. How many boxes are needed?
allocation/rate	multiplication ARM	partition ARP	quotition ARQ
	A · B = __	A · __ = C	__ · B = C

	One sticker costs B cents, how much would A stickers cost?	I spent C cents to buy A stickers. What is the cost of one sticker?	I have C cents to spend on stickers. If one sticker cost B cents how many stickers can I buy?
Comparison	multiplication $A \cdot B =$ __	partition $A \cdot$ __ $= C$	quotition __ $\cdot B = C$
	CMM	CMP	CMQ
	Sara is B years old. Her mother is A times older. How old is Sara's mother?	Sara's mother is C years old. She is A times older than Sara. How old is Sara?	Sara is B years old. Her mother is C years old. How many times is Sara's mother older than Sara?
	CLM	CLP	CLQ
	Sara is B years old. She is A times younger than her mother. How old is Sara's mother?	Sara's mother is C years old. Sara is A times younger than her mother. How old is Sara?	Sara is B years old. Her mother is C years old. How many times is Sara younger than her mother?
Rectangular array/ Area of rectangle	Multiplication RAM	Division RAD	
	$A \cdot B =$ __	$A \cdot$ __ $= C$	
	The trees are planted in *A* rows and *B* columns. How many trees are there?	If *C* trees are planted into an array with *A* rows, how many columns of trees are there?	
	Dimensions of rectangular carpet are 2m and 3m. What is the area of the carpet?	Area of the carpet is 6 m. Carpet is 3m long. What is the width of the carpet?	
Cartesian product/ combination	CPM	CPD	
	$A \cdot B =$ __	$A \cdot$ __ $= C$	
	Sara has A skirts, and B t-shirts, how many different outfits can she wear?	Sara has C different skirts and t-shirt outfits to wear. If she has A skirts. How many t-shirts does she have?	

Table 1. Classification of multiplication and division word problems

Boonlerts and Inprasitha (2013) analyzed multiplication topic in textbooks in Japan, Thailand and Singapore and concluded that Japanese textbooks include equal groups, multiplicative comparisons and rectangular array situations while Singapore textbooks provided equal groups and rectangular array situations. On the other hand, Thai textbooks focused only on equal groups situations.

Method

Taking into account the importance of teaching and learning mathematics, combined with the difficulties faced by students in addressing the word problems, starting with the problems related to learning the concepts of multiplication and division, this research is designed. The types of multiplication and division word problems in the third grade textbooks in Bosnia and Herzegovina were analyzed.

The issue of approving textbooks for primary and secondary schools in Bosnia and Herzegovina is under the jurisdiction of the Federal Ministry of Education and Science (FMON) and the Ministry of Education and Culture of the Republic of Srpska (MPKRS). In the school year 2016/2017 FMON approved 3 mathematics textbooks for the third grade, while MPKRS has one obligatory textbook for the third grade of primary school.

The aim of this research is to analyze the types of one-step multiplication and division word problems represented in mathematics textbooks for the third grade of primary school in Bosnia and Herzegovina. We define one-step multiplication and division word problem is one that answer to a problem can be obtained by applying one operation - multiplication or division. Those problems could be given in the mathematical and non-mathematical context.

Research is based on framework presented in the paper, i.e. classification of word problems in 5 categories: equal groups (EG), allocation / rate (AR), comparision (C), rectangular array / area of rectangle (RA), Cartesian product (CP) and identification of 18 types of one-step word problems. In this research we excluded two types of word problems related to area od rectangle as students do not learn about area od rectangle in the third grade od primary school.

The following research questions are formulated:

What is the distribution of 5 categories of multiplication and division one-step word problems in the 3rd grade textbooks approved for use in schools in Bosnia and Herzegovina?

- To what extent do multiplication and division word problems in the 3rd grade mathematics textbooks reflect the proposed 16 types?
- Compare distribution of 5 categories of problems in the 3rd grade textbooks.
- Compare distribution of 16 types of problems in the 3rd grade textbooks.
- Detect characteristic problems and possible misrepresentations.

The sample was consisted of 4 mathematics textbooks for the 3rd grade of primary school approved by FMON and MONRS as follows:

- T1 (publisher: Sarajevo Publishing (2014));
- T2 (publisher: Nam/ Vrijeme (2014));
- T3 (publisher: Bosanska knjiga/Bosanska riječ (2014));
- T4 (publisher: Zavod za udžbenike i nastavna sredstva (2016)).

Results

Analysis of one-step multiplication and division word problems presented in approved textbooks resulted in a response to the set research questions.

Let's first point out the approaches to introducing multiplication and sharing content. Each introductory example in observed textbooks is illustrated. Textbooks T1, T2, T3 that are approved by FMON show both, multiplication of a number and multiplication by number. The multiplication of the number is presented systematically and uniformed for all numbers up to 10 based on which a multiplication table (multiplication of the number) is compiled. There are suggestions to pupils to create multiplication tables (multiplication by number) by themselves, in each of textbooks T1, T2 and T3. In the textbook T4 which is approved by MPKRS, only multiplication of the number is processed and related multiplication tables are formed.

In textbooks T1, T2, and T3 multiplication and division are presented alongside, simultaneously. Examples of division are merely examples of partition division, and then we can find problems both

about partition and quotition division. In textbook T4 multiplication and division are introduced separately, consecutively, first multiplication, then division. Examples of division are merely examples of quotation division, and then pupils are asked to resolve both partition and quotition division problems.
The total number of tasks in the considered textbooks varies significantly as shown in Table 2.

	T1	T2	T3	T4
Multiplication	77	180	110	103
Division	82	82	113	105
Total	159	262	223	208

Table 2. Number of multiplication and division tasks (numerical and word problems)

The numbers of numerical and one-step word problems considering both multiplication and division in each of textbooks are given in the Table 3.

	T1	T2	T3	T4
Word	99	106	153	117
Numerical	60	156	170	91
Total	159	262	223	208

Table 3. Number of numerical and word problems

Numbers of asymmetrical and symmetrical one-step word problems are presented in the Table 4. It is obvious that asymmetrical problems dominate in textbooks. Symmetrical problems are present mostly in parts of textbooks dealing with commutative property of multiplication.

	T1	T2	T3	T4
Asymmetrical	92	96	145	116
Symmetrical	7	10	14	1
Total	99	106	159	117

Table 4. Asymmetrical and symmetrical one-step word problems

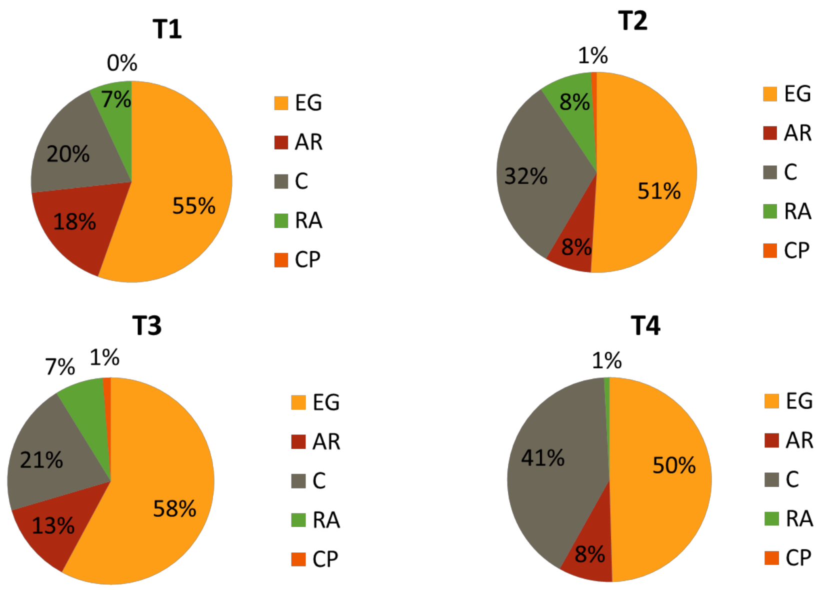

Chart 1. Distribution of 5 multiplicative structures in textbooks

The distribution of five multiplicative categories is not uniform in all textbooks as shown in Chart 1.

The equal groups problems are dominant in each textbook, followed by comparison problems. The absence of rectangular array/ area of rectangle problems as well as Cartesian product problems can be noticed. While the absence of area of rectangle problems can be justified by the fact that the area of rectangle and square is only taught in the fifth grade, there is no justification for the absence of rectangular array and Cartesian product problems. We compared the numbers of problems in each category in observed textbooks (Chart 2).

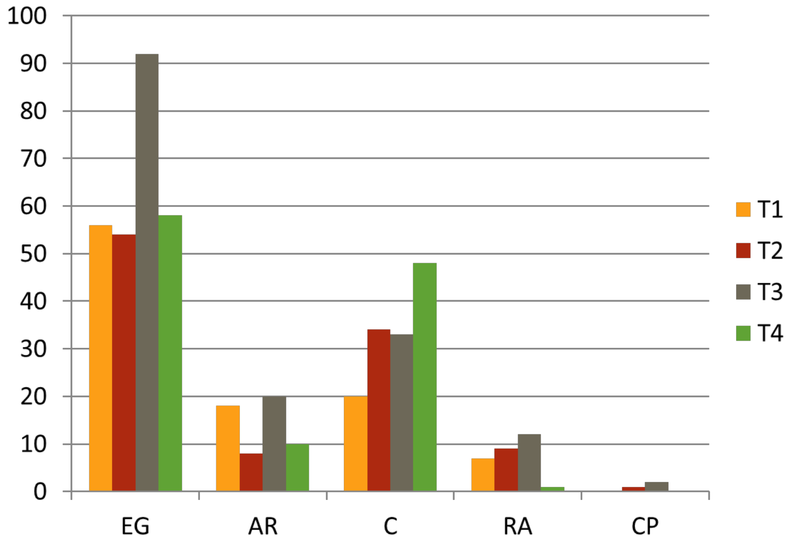

Chart 2. Numbers of problems in each category compared

Presence of 16 types of one-step multiplication and division word problems varies in textbooks (Table 5).

	EGM	EGP	EGQ	ARM	ARP	ARQ	CMM	CMP	CMQ	CLM	CLP	CLQ	RAM	RAD	CPM	CPD
T1	28	21	7	10	6	2	11	1	0	0	6	2	6	1	0	0
T2	34	14	6	5	0	3	20	0	1	0	20	3	9	0	1	0
T3	47	28	17	12	4	4	13	1	4	4	4	7	10	2	2	0
T4	27	17	14	8	1	1	23	0	1	2	20	4	1	0	0	0

Table 5. Distribution of 16 types of one-step word problems

In textbooks, the easiest types of word problems are the most common: equal groups multiplication (EGM) and partition division problems (EGP) and multiplicative comparison multiplication problems (CMM) followed with equal groups quotation division problems (EGQ) and multiplication comparison partition division problems (CLP). To form numerical expression students simply need to follow the text of problem. The verbs directly suggest which operation has to be applied to solve problem.

Conclusion

Both textbooks and word problems take an important place in the process of teaching and learning mathematics. Carefully created textbooks in which various examples and assignments are presented can enhance the teaching and learning process of mathematics. Namely, certain research has shown that the inability of students to solve certain difficult types of tasks (problems) is a consequence of the fact that they did not meet these types of tasks during their teaching (Freeman and Porter, 1989, Olkun and Toluk, 2003). Therefore, an analysis of the contents of the textbook is indispensable.

This research has shown that the most common types of one-step multiplication and division word problems (EGM, EGP, CMM followed by EGQ and CLP) in the considered math textbooks for the 3rd grade in Bosnia and Herzegovina are those that require the least mental effort of the students. The text, particularly the verbs used in those types of problems, directly indicates which operation to conduct in order to solve problems. On the other hand, the most difficult types of problems are under presented. Difficulty is reflected in the formation of a numerical expression using operation with counterintuitive meaning to the stated action in the text of the problem. Textbooks often contain only a few problems with unknown initial value, thus limiting students to meaningfully learn about multiplication and division.

A small number or a complete absence of certain types of multiplication and division word problems denies pupils the ability to create new problem-solving schemes and to detect links between opposing problem types. Incorporating pairs and triples of problem types within one multiplicative structure, in the same context and numeric values, should help to create and link a problem-solving scheme and meaningful learning about multiplication and division.

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THE TEXTBOOK IN MATHEMATICS: FINDINGS FROM A SYSTEMATIC REVIEW

NATASJA STEEN and MATILDE STENHØJ MADSEN

Abstract

In this paper we investigate the question: 'Which potentials and limitations of using textbooks in mathematics teaching can we identify?' We answer this through a qualitative systematic review in two parts. First we conducted a descriptive review of 458 references that resulted in nine analytical themes, sorted into four categories: 'the textbook', 'the textbook and the students', 'the textbook and the curriculum' and 'the textbook and the teacher'. The research is concerned with grade K-9. Second we conducted a qualitative synthesis of three of the analytical themes.

In summary we found that the research has mainly focused on the textbook in relation to the teachers or the students and to a lesser degree in relation to the curriculum and concept development within the field.

Introduction

The background for this review was a disparity, described in the research literature, between prevalent use of textbooks and a lack of research concerning the implications of the use of textbooks. Studies have first and foremost been concerned with the content of textbooks, comparing textbooks and use of textbooks both in international and Nordic contexts (Fan 2013; Fan, Zhu & Miao 2013; Rezat & Strässer 2013) whereas fewer studies have been conducted concerning the correlation between the textbook and other variables (Fan 2013).

It is primarily in the last three decades that the textbook has been the focus of research even though it is possible to find a few examples of research all the way back to the 1920s (Love & Pimm 1996; Fan 2013; Fan, Zhu & Miao 2013).

In this paper, which in its majority is based on edited excerpts from a master's thesis (Steen and Madsen 2016), we will provide an overview of the studies which have been carried out in the field and mention some of the results within the individual areas of focus.

The review process

In the following we describe the review process in general. We will hereafter elaborate on a few elements of this process. For a more complete description see the master's thesis (Steen and Madsen 2016).

The structure for the review is based on Fink (2014) and Gough, Oliver and Thomas (2012). This article describes our findings from a descriptive synthesis of the literature.

Search for relevant literature

In this review the search has been conducted in four databases; two Danish databases (bibliotek.dk and AU.library.dk) and two primarily English databases (ERIC and MathEduc). In Danish we searched for: 'læreb*' (textbook), 'matematikb*' (math book), 'grundb*' (course book) og 'lærem*' (instructional materials) in combination with 'matematikunderv*' (teaching math). The

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chosen English search terms were: 'textbook' or 'curriculum material' in combination with: 'mathematics' or 'mathematical' and 'education', 'teaching' or 'instruction'.

	Criteria for inclusion	Criteria for exclusion
Type of source	Published research article or book.	Description from conferences and so-called proceedings are excluded; furthermore textbooks in mathematics are also excluded. It must be possible to obtain the article.
Focus on textbooks	The textbook must be a central focus in the source.	The textbook is mentioned only in passing or in relation to other mathematical or mathematics educational topics.
Educational level	Must deal with grade level K-9.	The publication only deals with other educational levels than grade level K-9.
Focus on potentials or limitations	The publication must include general empirical or theoretical findings concerning the potentials and limitations of textbooks.	The publication only deals with textbook analyses of a few instructional materials.
Language	Available in Danish, English, Norwegian or Swedish.	Available in other languages than Danish, English, Norwegian or Swedish.
Quality of research	The publication must meet general standards for good scientific method.	The publication does not meet general standards for good scientific method.

Table 1. Overview of criteria for inclusion and exclusion. The 15 original criteria for exclusion are here consolidated into six overriding criteria

The search yielded a result of 404 sources including 22 duplicates, which resulted in a cumulative number of 382 sources. In addition to this we carried out a number of supplementary searches e.g. flicking through central mathematics education handbooks, which yielded an additional 76 sources. The search has most recently been repeated in May 2017 where we added a manual search of the journal *Nordic Studies in Mathematics Education* (NOMAD). The search yielded no new results compared to the first search in March 2016, while NOMAD led to the addition of 4 included and 3 excluded articles. From the 458 sources 61 are included and 396 are excluded based on the criteria described in Table 1.

We have calculated the geographical affiliation on the basis of where the research was conducted. Where this has not been possible we have checked what country the author's professional practice was associated with at the time of publication. The 61 included articles are distributed in the following way:

- 31 studies from the United States

- 21 studies from Europe (distributed between the countries Finland, the UK, Denmark, Sweden, the Netherlands, France, Norway, Belgium, Romania and Greece)
- 2 studies from Asia (distributed between the countries Bhutan, China and Indonesia)
- 1 study from Australia
- 2 studies from the United States and China
- 1 study from the United States and Australia
- 4 international studies (via Trends in International Mathematics and Science Studies TIMSS)

We decided that the search should not be limited by what year the research was conducted in. It turned out that the earliest source we found was from 1983. The included references are distributed over time in the following way: 1983-1989 (4 articles), 1990-1999 (8 articles), 2000-2009 (26 articles) and 2010-2017 (24 articles).

Developing the analytical themes

The next part of the analysis was carried out in three steps. The first step was the in-depth reading of the 61 included articles. Reading the articles yielded a number of different themes related to our original research question. E.g. 'What are the potentials of a textbook?' 'What are the limitations of a textbook?', 'What does the relationship between the textbook and the teacher signify for the teaching?' and 'What role does the textbook play in relation to the teacher and the teaching?' These questions and the answers gave rise to a list of terms and themes, which were dealt with in one or more texts.

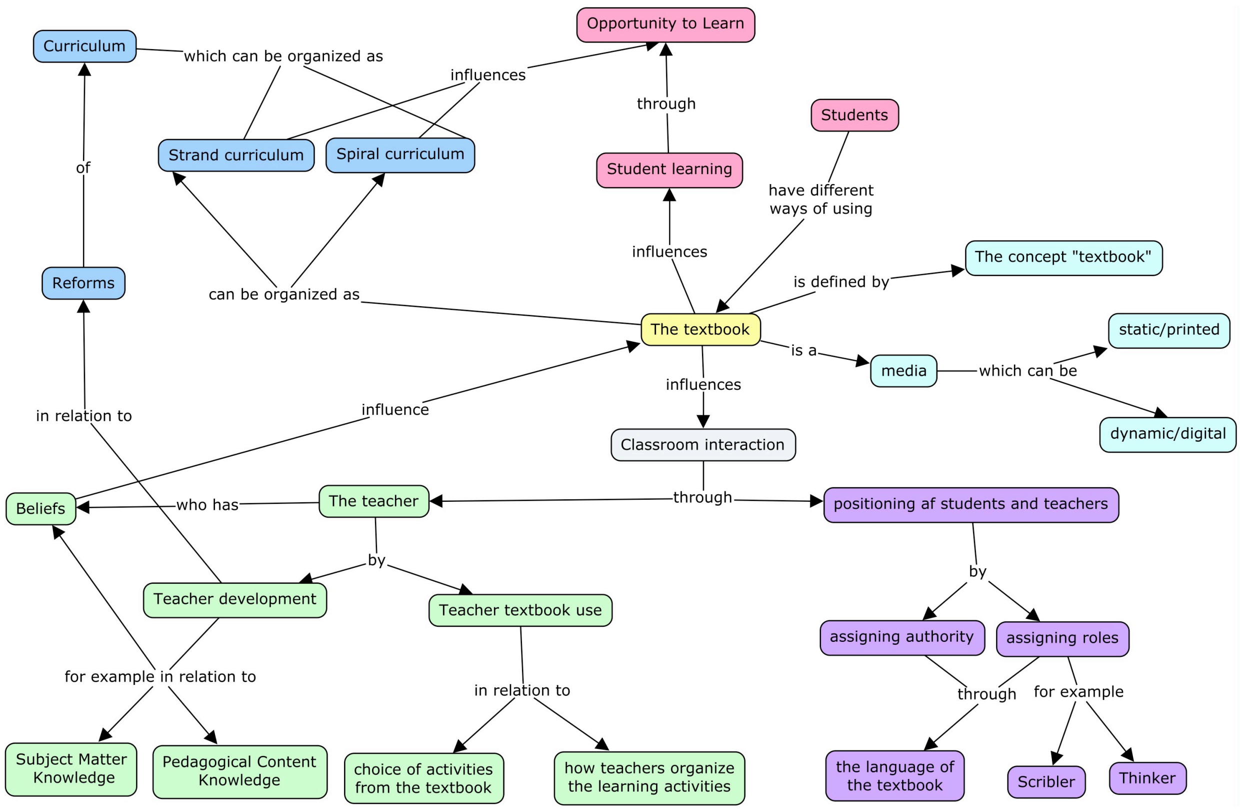

Figure 1. Concept map visualising categories of analysis

The second step was to process this list of terms and themes. We chose to draw up the concept map shown in Figure 1 because we perceived that the terms and themes had a great deal of correlation, which meant that we experienced difficulty in describing clearly separated categories of analysis.
The choice of a concept map as a way to visualise the categories makes it possible to show these mutual interrelations. In the centre of the concept map we placed the term 'The textbook'. Hereafter

we picked out the terms and themes from the list one by one and formulated their relation to the concept of the textbook.
Furthermore, in connection with the composition of the concept map, we found that some terms and themes needed to be consolidated or split up in order to obtain a more precise description. For example, one of the early analytical themes 'Reasons for over-reliance on the textbook' includes both cultural and social reasons which are not associated with the textbook and reasons which were already included somewhere else (i.e. in the theme concerning 'The teachers' beliefs about the textbook'), which is why it no longer made sense to include this theme as an independent 'branch' in the concept map. As a result, we chose to include it in the category 'the textbook and the teachers'.

The third and final step was to use the concept map to consolidate the many terms, themes and their interrelations in meaningful overarching categories and subcategories (see Table 2). By describing the four overarching categories in the following we will outline the analytical themes and describe how these developed as a result of the elaboration of the concept map.

MAIN CATEGORIES IN TEXTBOOK RESEARCH

The 61 included sources are distributed between 'categories and analytical themes' as shown in Table 2. In the following we will elaborate on the categories individually.

	The textbook is …		The textbook and the students			The textbook and the curriculum		The textbook and the teachers		
Reference	The concept 'the textbook'	The textbook as media	Student learning	Positioning of the students (and teachers)	Student textbook use	Strand/ spiral		Teacher textbook use	Teacher learning and the textbook	The textbook in relation to teacher beliefs
Total sources in the main category	13		28			15		40		
Total sources in the subcategory	11	4	21	5	2	2	14	24	17	11
Ahl, L.; Gunnarsdottír, G. H.; Koljonen, T. & Pálsdottír, G. (2015)								x	x	
Ball D. & Cohen D. (1996)							x			
Banilower,E.; Smith,S.; Weiss,I.; Malzahn,K.; Campbell,K. & Weis,A. (2013)								x		x
Barr, R. (1988)			x					x		
Behm, S.L. & Lloyd, G.M.(2009)								x		x
Brown, M. (2009)	x								x	
Chandler, D. G., & Brosnan, P. A (1995)			x							
Charalambous, C.Y.;Delaney, S.; Hsu, H.Y. & Mesa, V. (2010)			x							
Cheng, Q. & Wang, J. (2012)							x			
Collopy, R. (2003)									x	x
Cronberg, F.G. (2016)					x					
Davis, E. & Krajcik, J. (2005)							x		x	
de Kock, W. & Harskamp, E. (2014)		x								
Depaepe, F.; De Corte, E. & Verschaffel, L. (2007)				x			x			
Despina, D. & Harikleia, L. (2014)			x							
Forrester, T. (2009)										x
Foxman, D. (1999)	x		x					x		x

	The textbook is …		The textbook and the students			The textbook and the curriculum		The textbook and the teachers		
Reference	The concept 'the textbook'	The textbook as media	Student learning	Positioning of the students (and teachers)	Student textbook use	Strand/ spiral	The role of the textbook in implementing curriculum reform	Teacher textbook use	Teacher learning and the textbook	The textbook in relation to teacher beliefs
Freeman, D. & Porter, A. (1989)								x		
Grant, T.; Kline, K.; Crumbaugh, C.; Kim, O.; & Cengiz, N. (2009)								x	x	
Grave, I. &Pepin, B. (2015)								x		
Haggarty, L. & Pepin, B. (2002)			x					x		
Hansen, K. (1983)						x				
Herbel-Eisenmann, B. (2009)				x						
Herbel-Eisenmann, B.; Lubienski, S. & Id-Deen, L. (2006)							x		x	
Jablonka, E. & Johansson, M (2010)			x							
Johansson, M. (2003)	x							x		
Johansson, M. (2006a)								x		
Johansson, M. (2006b)	x			x				x		
Johansson, M. (2007)				x				x		
Kong, F. & Shi N. (2009)	x								x	
Kulm, G. & Capraro, R. (2008)			x							
Lepik, M.; Grevholm, B. & Viholainen, A. (2015)								x		
Lloyd, G. (1999)							x		x	
Lloyd, G. & Behm, S. (2005)									x	
Love, E. & Pimm, D. (1996)	x	x		x						
Manouchehri, A. & Goodman, T. (1998)							x		x	
Manouchehri, A. & Goodman, T. (2000)							x		x	
McDuffie, A. & Mather, M. (2006)									x	x

	The textbook is …		The textbook and the students			The textbook and the curriculum		The textbook and the teachers		
Reference	The concept 'the textbook'	The textbook as media	Student learning	Positioning of the students (and teachers)	Student textbook use	Strand/ spiral	The role of the textbook in implementing curriculum reform	Teacher textbook use	Teacher learning and the textbook	The textbook in relation to teacher beliefs
Nie, B.; Freedman, T.; Hwang, S.; Wang, N.; Moyer, J.; & Cai, J. (2013)			x				x			
Pearlman, L. (2011)			x							
Pehkonen, L. (2004)										x
Pepin, B., Gueudet, G. & Trouche, L. (2013)			x					x		
Pepin, B. & Haggarty, L. (2001)			x							x
Pepin, B.; Gueudet, G.; Yerushalmy, M.; Trouche, L. & Chazan, D. (2016)	x	x						x		
Remillard, J. (1999)								x		
Remillard, J. (2000)									x	
Remillard, J. (2005)	x							x		
Remillard, J.; Harris, B. & Agodini, R. (2014)			x				x	x	x	
Rezat, S. (2013)					x					
Schmidt, W.; McKnight, C. & Raizen, S. (1997)			x					x		x
Silver, E.; Ghousseini, H.; Charalambous, C. & Mills, V. (2009)							x		x	
Son, J. & Kim, O. (2015)								x	x	
Stein, M.; Remillard, J. & Smith, M. (2007)	x		x			x	x	x	x	
Tarr, James E., Chavez, Oscar, Reys, Robert E. and Reys, Barbara J. (2006)								x		
Tobin, K. (1987)			x							x
Törnroos, J. (2005)		x	x				x			
Tünde, B. & Gabriella, S. (2011)	x									

	The textbook is ...		The textbook and the students			The textbook and the curriculum		The textbook and the teachers		
Reference	The concept 'the textbook'	The textbook as media	Student learning	Positioning of the students (and teachers)	Student textbook use	Strand/ spiral	The role of the textbook in implementing curriculum reform	Teacher textbook use	Teacher learning and the textbook	The textbook in relation to teacher beliefs
Valverde, G.; Bianchi, L.; Wolfe, R.; Schmidt, W. & Houang, R. (2002)	x		x				x			
van Steenbrugge, H.; Valcke, M. & Desoete, A. (2013)			x					x		x
Wijaya, A.; van den Heuvel-Panhuizen, M. & Doorman, M. (2015)			x							
Xin, Y. (2007)			x							
Total sources in the subcategory	11	4	21	5	2	2	14	24	17	11
Total sources in the main category	13		28			15		40		

Table 2. The included publications distributed between the categories. A complete reference list of the included publications can be found at https://drive.google.com/drive/folders/1hEvzScqpIxXI2ttTnR5XLbHk4AggtJ_A

The textbook is…

In the concept map two themes begin with the formulation 'The textbook is…'. Both themes address attributes in the textbook itself without focusing on other influences on the teaching. We therefore found it meaningful to consolidate these two themes under the research category 'The textbook is…' This category contains 13 sources from a range of countries, with three from the United States and two from TIMSS with an international focus. Five sources are from Europe (two are from Sweden and are by the same author, one is from Finland, one is from the Netherlands and one is from Romania). One source is from China. One source has joint authors who together represent four countries (The Netherlands, France, Israel and the United States).

The publications were published between 1996 and 2016 and hence cover a time period of 20 years. In this time frame the sources are distributed so that we have two texts published between 1996 and 1999, six texts published between 2000 and 2009 and four texts published between 2010 and 2016.

Two analytical themes fall under this category: 'The concept the "textbook"' and 'The textbook as media'

In this presentation we will focus on one of the two analytical themes, i.e. the one concerned with defining the term 'textbook'. This part of the analysis is based on 11 publications, which were all published between 1996 and 2016. Our initial analysis of the sources showed that the descriptions of the textbook as a concept are distributed across nine approaches, which again can be divided into three different emphases when discussing the term 'textbook': 'the textbook as artefact', 'the textbook processes information' and 'the textbook affords interaction between students and teachers'.

In many of the definitions offered there is an explicit understanding of the textbook as a physical book. One example is Kong and Shi (2009, p. 270), who write, 'Textbooks are printed material that is the most standard and typical among all teaching materials'. Other definitions have a more implicit understanding of a textbook as a physical book, e.g. Valverde, Bianchi, Wolfe, Schmidt and Houang (2002, p. 1): 'They are fixed components providing an unchanging reference to the nature of these school subjects…' However, there are examples of definitions stating that electronic resources can also be perceived as textbooks, e.g. Stein, Remillard and Smith (2007, p. 323):

> While we use the terms *curriculum materials* and *textbook* (along with instructional resources and guides) somewhat interchangeably to refer to printed or electronic, often published, materials designed for use by teachers and students before, during and after mathematics instruction, many teachers and mathematics educators draw sharp delineations between the two.

This diversity in emphasis corresponds well with Hansen (2008), who argues that an agreement concerning the framing of the broader term 'instructional materials' has not yet been reached. In addition, many varying terms are used internationally to describe this field (Knudsen 2011). We therefore endorse Remillard (2005), who finds there is a need for conceptual development in the field.

The textbook and the students

In our concept map there are three themes, which address the interplay between the textbook and the students in different ways. The themes are 'the influence the textbook has on student learning', 'the function of the textbook in relation to positioning students' and 'the students' use of the textbook'. We have made these three themes part of the category 'the textbook and the students' in the mentioned wordings.

This part of the review used 28 publications. Geographically they are distributed with nine from the United States, ten from Europe (four from Sweden, two from Belgium, one from Greece, one from Finland, one from Germany and one from the UK), one from Indonesia, two which originate from international studies (TIMSS) and five studies involving several countries (two from the UK/France/Germany, one from the United States/Australia, one from the United States/China, one from France/Norway and one from Cyprus/Ireland/Taiwan).

The articles were published between 1987 and 2016. Of these six are from before 2000, eleven of them are from 2000-2009 and another eleven publications are from 2010-2016.

The analytical theme 'Student learning' emerged from the question 'Does the textbook make a difference?' The 21 publications used in this part of the review are predominantly in agreement: only one (van Steenbrugge, Valcke & Desoete 2013) concludes that the textbook does not make a difference. In the rest of the studies some effect on the content of the teaching or the achievements of the students is shown.

The analytical theme 'Positioning of students (and teachers)' might in principle also be found under the category 'The textbook and the teachers' since it includes both teachers and students and their interrelation. We have decided to keep them here, however, since the student perspective is the most prominent in the five publications in this subcategory. The texts are concerned with how one of the functions of the textbook is to stage or position the actors who interact with the book.

Herbel-Eisenmann (2009) for instance studies how the teacher's way of interacting with the book and the teacher's language affects where the authority is placed. Herbel-Eisenmann describes how, by reading from the textbook, the teacher places the authority with the textbook. However, if the teacher later comments on the content of the text, she or he thereby places herself or himself on the same level as the text.

'Student textbook use' is represented by the fewest number of publications in the review since only two publications are concerned with the students' use of textbooks (Rezat 2013 and Cronberg 2016). The articles are concerned with how the students use the textbook in relation to choosing content and which factors affect this choice. In a review of the collected data material we did not subsequently find other publications concerned with the students' use of textbooks. We therefore find it meaningful to maintain this theme since it contributes to an area of textbook research which there has only been shed a tiny bit of light on. We will return to this consideration at the end of the article.

The textbook and the curriculum

The textbook's relation to the mathematical curriculum is activated through two different analytical themes which give rise to the unified category 'The textbook and the curriculum'.
The two themes concern respectively the textbook's organisation as strand or spiral curricula and the curriculum's influence on the textbook in connection with reforms.
The category 'The textbook and the curriculum' contains 15 publications. These are geographically distributed with ten publications from the United States, three from Europe (one from Belgium, one from Denmark and one from Finland), one originating from an international study (TIMSS) and one is a comparison between circumstances in the United States and China.
The articles were published between 1983 and 2014 and are distributed with four from 1983-1999, eight publications from the period 2000-2009 and three from 2010 onwards.

The textbook and the teachers

As a final relation the concept map points to a number of subcategories, which focus on the relationship between the textbook and the teacher. We have compiled these in a fourth and final category which we call 'The textbook and the teachers'.
We have found 40 publications, which belong to this category. Of these twenty-two are from the United States, nine are from Europe (four from Sweden, one from Finland, one from Belgium, one from Norway, one from Estonia/Finland/Norway and one from Sweden/Iceland), one from Australia, one from China, two originate from international studies (TIMSS) and five are studies which involve several countries at the same time (two from the UK/France/Germany, one from the United States/Australia and one from the Netherlands/France/Israel/the United States/Israel).
They were published between 1988 and 2016 and are distributed in time with seven publications from the period 1988-1999, twenty-three publications from the period 2000-2009 and nine publications from 2010-2016.
Unlike the very limited interest in the students' use of textbooks, there is a greater research interest in the teachers' use of textbooks. The analytical theme 'Teacher textbook use' contains texts, which deal with how the teacher uses the textbook. This translates into how closely the textbook is followed, what activities are involved and to what extent the textbook functions as a guidance tool. One example is Freeman and Porter (1989), who study the ways teachers use textbooks and thus to what extent the textbook dictates the content of the teaching. There are 24 publications, which are concerned with this theme.
The analytical theme 'The textbook and teacher learning' consists of articles studying the opportunities of the textbook to affect the teacher learning. In the category are 17 publications published between 1998 and 2015. Of these 15 originate from the United States.
We found during the analysis that the teachers' experiences, attitudes and so on played an important role in the relation between teacher and textbook. Under the analytical theme 'The textbook in relation to teacher beliefs', texts, which reflect this perspective are collected. There are 11 publications connected to this theme. An example of the significance of teachers' beliefs in connection with the textbook is found in Collopy (2003), where the discrepancy between a teacher's beliefs and the approach in a particular textbook resulted in the teacher choosing not to use the textbook.

Concluding Remarks

When one considers our source material it is obvious that the screened literature geographically focuses on the United States and a number of European countries. Only ten sources do not fit within this frame and of these seven are either international or combined studies covering the United States and another country. There is therefore an obvious geographical disparity in the results in relation to covering a broad overview of the field. This might be because research on textbooks in countries outside this geographical area simply has not been carried out, or because the research does not fall within the methodological inclusion criteria we have established. The disparity might also be caused by the research being published in journals or in languages that have not been covered by the databases we consulted. It might therefore be relevant to repeat this review in other languages and in other parts of the world.

In this review we developed categories of analysis which connect the textbook to the teachers, the students and the curriculum. We found that the research has mainly focused on the textbook in relation to the teachers or the students and to a lesser degree on the textbook's relation to the curriculum and concept development within the field.

During the research it became clear to us that there is no consensus about a definition of the concept of the textbook, which makes it harder to compare studies and extract general conclusions across studies. We therefore agree with Fan, Zhu and Miao (2013) that further concept development in the field and further study of the textbook in relation to other variables is necessary.

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TEXTBOOK ANALYSIS IN UNIVERSITY TEACHER EDUCATION

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Textbooks as mirrors for modern educational reforms

In German middle and secondary schools, mathematical textbooks are extensively used in lesson preparation, in the classroom as well as for students' homework. Lesson planning for mathematics was and is generally done by mathematics teachers at school based on the mathematics textbook series used in that particular school. Also in preparation of the exercises for homework, tests and tasks for individual learning, mathematics textbooks still serve as an essential basis (Rezat 2009). An internal commission at school, consisting of all mathematics teachers, a parent and a student representing their respective groups, makes the selection of the mathematics schoolbook. In the last two decades, publishers and authors of textbooks had to adapt textbooks to competency models, reduced curricula and output orientation. The mathematics teachers at each school in particular were asked to prepare concretised school curricula based on very general educational standards and competency models. That way since the so-called Pisa shock, the German education system has undergone subtly comprehensive restructuring, the concept of "Bildung" (usually translated as "education") being replaced by the notion of "Ausbildung" (training) accompanied by a gradual economisation of the educational system during the last decade. All this had and has implications on language, approaches to problems as well as on the knowledge relevant to action, prognosis and orientation of our student mathematics teachers. Mathematics education is a reflective science. An important goal of university courses in this area should support the discourse about educational reforms and related changes to educational values. The analysis of the last editions of various modern mathematics textbooks is an excellent way to understand the implications of these reforms on general education (Allgemeinbildung) and expertise, it supports a comparative view on the everyday world and allows to disturb widespread routines.

Mathematics Textbooks as the continuous path connecting the former pupil's life with the life to come as a teacher

Taking into consideration the long German tradition of textbook development and the use of schoolbooks in mathematics classes (cf. Otte 1981), it is even more astonishing that there are hardly any canonical subjects in mathematics teacher education at university related to mathematics textbook analysis. A major problem of teacher training is the often cited double discontinuity:

> The young university student finds himself, at the outset, confronted with problems, which do not remember, in any particular, the things with which he had been concerned at school. […] When, after finishing his course of study, he becomes a teacher […] he will be scarcely able, unaided, to discern any connection between this task and his university mathematics (Klein 2016, p. 1).

As an approach to deal with the double discontinuity we look at mathematics textbooks as the continuous path connecting the former pupil's life with the life to come as a teacher. Because of the long tradition of some very wide used textbook series like "Die Elemente der Mathematik" and "Lambacher Schweizer" students can work on different editions of one textbook they were learning with as a pupils in school and they are likely to work with as a teacher. The study of "familiar" textbooks from the perspective of a teacher or author enables a direct approach to discontinuity and makes available various tools to support the maturation from student to teacher.

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Mathematical Textbooks as historical artefacts

Mathematical textbook series are also historical sources to study the history of education. Of the textbook series "Elemente der Mathematik", one can easily find hard cover versions of different editions tracing back to the middle of the 19th century (Reidt 1868) as well a digitalised version of Reidt's exercise book (Reidt 1884). A comparative analysis of the different editions related to important periods and reforms in mathematics education, like the "New Geometry (Neue Geometrie)", "reform pedagogy (Reformpädagogik)", "transformation geometry (Abbildungsgeometrie)" "Algebraic Analysis (Algebraische Analysis)", "functional thinking (Erziehung zum Funktionalen Denken)", "the introduction of differential and integral calculus", "Mathematical applications", "New Maths and set theory", "Modeling and Realistic Maths" and "Output and Compentcy orientation in mathematics". It is also quite instructive, from a cultural-historical perspective, to compare the design of the exercises and applications from different periods. Moreover, comparative studies between reforms and traditions in different national educational systems as done by Gispert and Schubring (Gispert & Schubring 2011) can inspire the search for traces and implementation of described reforms in the textbooks of the studied countries.

Textbook Analysis as preparation for teacher practice

Teacher education for German Gymnasia consists of two parts in the federal state Rhineland Palatinate: a university study with a Bachelor's and a Master's degree (5 years) and a teacher training as interns (18 month). University teacher education in Germany has to deal with two basic principles at once, which partly exclude each other: On the one hand, courses in mathematics and mathematics education introduce students to scientific disciplines according to the concept of unity of research and teaching, whereby research should aim at insight and not at profit and usefulness. On the other hand, instantly after obtaining the university degree teachers have to teach at own responsibility because of cuttings in education and the reduction of teacher training from 2 years to 1.5 years. No wonder students expect to be trained in practical matters as lesson planning and primarily consider everything related to teacher training as useful.

Textbook analysis gives the possibility to combine the training of practical skills with the study of history of education, concept development and research design. Nevertheless, from our experiences related to reflection in lesson planning as well as on mathematical concept development only very few prerequisites can be assumed. Assessments of the capabilities of our students show that they master the reproduction of information and texts very well. They work hard on the perfection of presentational skills. Their strengths also include the use of modern media to access information and pattern recognition skills. Their weaknesses lie in their conceptual understanding.

Volker Ladenthin's description of contemporary students' problems confirms our experience:

> Students are barely able to use abstractions. One has to speak in examples - and they will be happy to discuss on the level of examples. However, generalization and transfer of expertise hardly succeed. To transmit the statements of ancient authors (Aristotle) in contemporary parlance fails less due to fragmentary historical knowledge than to the lack of transferability. Textual analysis is done very vaguely and always very generally, ("Comenius says that school is good for the people"). Syntheses are created additively and is by no means nuanced. Judgments are linear (not multi-perspective) (Ladenthin, 2014, p. 17).

The analysis of the design of a mathematics textbook, its presentation of basic information, classification of exercises, implicit or explicit concept development, the choice of examples to work exemplary as well as the choice of contexts and applications support conceptual understanding of mathematical notions. The latter can be guided and adapted to different prerequisites by the choice of material, questions for reflection, tools for structuring, and the provision of solution schemes.

Mathematical textbooks as a tool to change perspectives from student to author

Textbook analysis is also an excellent way to change perspectives from that of a student (solving tasks) to that of a teacher (lesson planning with tasks) to that of an author: what is the conceptual understanding of the notion? Which tasks are in the zone of proximal development of a possible student?

In our experience, this change of perspective does not happen automatically. Quite often students do the didactical analysis of a mathematical concept development not from the perspective of the teacher, but from the pupil's perspective. A typical approach of our students to lesson planning is the search for an "activating" introductional problem in textbooks or the Internet. "Activating" means here "making the pupils active in a psychological sense" and is related to methodological criteria, such as the form of cooperation or the integration of material tools. From a pedagogical point of view the students act from a teacher's perspective, since they try to organize and arrange activities and cooperation of pupils in the classroom. Thereby the search for methodological suitable "activating" tasks and exercises is often accompanied by the search for their solutions, also in textbooks or the Internet. The level of difficulty and time necessary for the solutions of the activating exercises are often assessed intuitively and not on the base of a self-conducted detailed solution. The work with mathematical textbooks provides possibilities to make the different perspectives explicit and to differentiate them from each through different tasks and activities with the textbook. Pupil's views are taken up by solving tasks and exercises and approaches. The perspective of a teacher comes with the comparison of different solutions, their systematization and their study as part of the concept development of the involved mathematical notion. The perspective of an author is taken when comparing different textbooks and analysing the roles of the different examples and tasks.

Explicit demarcation of the different perspectives and assistance in the form of reflection questions help students to assume the role of the teacher not only in the context of classroom management but also as experts in school mathematics and its suitable presentation as well as to feel responsible for the latter.

Development of a Concept of a seminar on textbook analysis

In the following, we present the concept of a seminar on textbook analysis, which has been held and developed in action research over seven years in 17 different groups. It is part of master's degree studies at the university Mainz in Germany. The aim of the seminar is to get acquainted with different mathematics textbook designs with regard to implicit or explicit introduction of mathematical concepts, different contextualisation, systematisation and formalisation of the contextualised concepts, differentiation in exercise tasks, as well as the comparative analyses on specific topics. The sessions of the seminar are held by one or two students in form of workshops.

During the semester (6 months) before the seminar, there is a lecture course about chosen mathematical concepts from the curriculum of secondary school maths. In this lecture course research methods of mathematical education as well as "Stoffdidaktik" and task-related aspects of mathematical concept development are presented and discussed exemplary. The seminar deepens the topics of the lecture course, so the subjects of the 14 sessions are close to those of the lectures. The present choice of topics and examples in the lecture course is one of the results of an action research in the seminars and an answer to the question which mathematical concepts of secondary school mathematics are suitable for a comparative analysis of textbooks. In the first seminars 7 years ago we primarily tried to find subjects and topics combing goals of mathematics teacher training like lesson planning with educational objectives of the scientific discipline mathematics education. Therefore the first seminars studied a wide range of topics concerning analysis, linear algebra/analytic geometry as well as stochastic and concentrated on suitable topics for analysis and variation mathematical problems and tasks from actual mathematics schoolbooks. In the following reflection of the seminars it became obvious, that most of the students had problems to analyse even canonical exercises, prerequisites, learning goals and methodology a fortiori to vary them. A first step was to limit ourselves to the analysis of tasks and to try variation of tasks only for chosen suitable examples.

Another result of the reflections was the study not only of exercises and practice but also other parts of concept development of a mathematical notion in mathematics schoolbooks.

The typical design of actual German mathematics textbook is: a) repetition b) introductive and propaedeutic problems and tasks, c) unification, systematisation, formalisation of the introductive problems, d) standard solutions, e) presentation of basic knowledge f) exercises, d) excursions.

Therefore, in the changed concept students could choose which part of concept development they wanted to explore. Most of them decided to study introductory and propaedeutic problems. As a result of the analysis and discussions of the student's essays we decided also to limit the topics to analysis and to discuss only basic notions which were preliminary discussed in detail during the lecture course: real number, function, limit of series, differential and integral. The current concept of the seminar students compares - for a given mathematical notion, like limit - its presentation in three different modern textbooks. This analysis is guided by general questions about introductory problems (implicit or explicit concept development, inductive or deductive approach), tasks and solution schemes, tools for generalisation and unification, categories and differentiation of exercises, excursions. The analyses of this common subject are discussed during the seminar and are part of an essay made by every student. In addition, in every session a student presents her or his textbook analysis related to one of the mentioned mathematical concepts and on its basis a plan of its concept exposition for the classroom. Additional materials are older textbooks, didactic articles about the mathematical notion and the material from the lecture course. The comparative analysis of the concept development in the schoolbooks with the alternative developed by the student constitutes the second part of the essay. The earlier described student's, teacher's and author's points of views are structuring tools to support the change of perspective from pupil to author.

The work from a pupil's perspective, i.e. detailed solution schemes of exercises to the concept to be discussed in the seminar are done in advance of the sessions as homework of all students.

Conclusions

Working with a textbook also provides tools to deal with different levels of awareness (Mason 1998). It takes a longer time and accompanying guidance until the students themselves ask questions concerning not only the solution of a task but also about the educational value, the development of meaning and about the existence of mathematical objects. The continuous reflection on the handling of textbooks in the frame of action research supported this maturing process. The developed criteria and categories for textbook analysis take into account the expertise and skills of the students and support the transition between using textbooks as a student and using the textbook as a teaching tool.

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SECTION ANALYSIS OF HISTORICAL TEXTBOOKS

RATIOS AND PROPORTIONS IN ICELAND 1716–2016

KRISTÍN BJARNADÓTTIR

Abstract

The topics of ratios and proportions are investigated in the oldest Icelandic arithmetic textbook, Arithmetica Islandica of 1716, preserved in the manuscript Lbs. 1694 8vo. The conjecture that it is a translation of the printed Danish Arithmetica Danica of 1649 is rejected, while there may be some transmissions of ideas. In continuation, the history of teaching ratio and proportions from Arithmetica Danica until the latest textbook in Icelandic, Skali of 2016, is investigated and related to recent research on obstacles in proportional reasoning.

Introduction

Arithmetic textbooks, written in the Icelandic language, have a history from 1716 until present. This study concerns both ends of this sequence. The content of *Arithmetica Islandica* of 1716, as preserved in a manuscript, is analysed with respect to works influencing it. Certain sources suggest that *Arithmetica Danica*, written in Latin by Geo Frommius (1649), was a model for *Arithmetica Islandica,* which in turn might be considered as an abridged version, comparing the sizes of the textbooks. *Arithmetica Islandica* may, however, have more models, and exercises in *Arithmetica Islandica* are also found in other younger arithmetic manuscripts.

The study focuses on the history of teaching ratios and proportions, and a comparison of its teaching in arithmetic textbooks. The books range from *Arithmetica Danica* and *Arithmetica Islandica* to *Skali*, an adapted translation of a Norwegian textbook series (Tofteberg, Tangen, Stedøy-Johansen & Alseth 2017) into Icelandic, the latest textbook series for the lower secondary level.

Textbook writing in Icelandic

During seven centuries from around 1100 until 1800 there were two episcopal seats in Iceland, with cathedrals and cathedral schools, one of each in Northern Iceland and in the South. The schools served to educate priests and officials. As Iceland belonged to the Danish Realm, university education had to be sought in Copenhagen, Denmark. A printing press, imported in the mid-16th century, mostly printed religious books, while secular works were printed in Copenhagen. After around year 1800, the printing press was no longer in the charge of the Church, and gradually, education literature became produced domestically.

The population of Iceland was 50,000 in 1703 and did not rise until the 19th century. The population in 2017 was 340,000. Persons, knowledgeable in Latin in the early 18th century, were 245 clergymen, 7 headmasters and teachers, and about 60 other graduates from one of the two schools and even from a university abroad, as recorded in the 1703 census (Statistics Iceland, Table 3.2, Occupations in 1703). In spite of a great distance from the mainland of Europe, which made communication by sailing only possible during summer, a considerable collection of European books existed in Iceland in possession of the learned elite, the clergy or the cathedrals. Among them were the mathematics books *Arithmeticae practicae methodus facilis* by Gemma Frisius, published in Antwerpen in 1540; *Arithmeticae libri Duo* by Petrus Ramus, Basil 1569; *Arithmetica Danica* by Jørgen From alias Geo Frommius, Copenhagen 1649; and *Compendium Arithmeticum eller vejviser* by Søren Matthisen, Copenhagen 1680 (Bjarnadóttir 2007, 58, 61–62; Ulff-Møller 2008).

Foreign literature was often translated, frequently in extracts. German and Danish books were the main foreign literary sources in Iceland after its Lutheran Reformation in 1550. The educated public, mainly clergymen, prepared the first copies of manuscripts by translating and/or adapting

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foreign texts, or even composing own texts. The manuscripts circulated from parish to parish. At least three early 18th century arithmetic texts in manuscripts exist presently in libraries. The oldest of them is *Arithmetica Islandica* in Lbs. 1694 8vo, dated 1716 on its front page. The text itself suggests the date 1733 which may be the date of the extant manuscript.

The first substantial printed arithmetic textbook in Icelandic was published in 1780. Up to 1910s, textbooks began teaching arithmetic from scratch as schools were scarce. During 1920s–1960s, the arithmetic textbook tradition continued in two phases, each intended for primary and lower secondary education. Since the 1970s, featuring a series highly influenced by the New Math, five textbook series have been published for the lower secondary school level, all by a state textbook publishing house. The latest of them is a Norwegian arithmetic textbook series, recently translated into Icelandic, *Skali* (Tofteberg et al. 2015; 2017).

Learning Ratios and Proportions

For centuries, proportional problems were solved by a method called in Latin *Regula Trium*, the Rule of Three. The method consists of finding the fourth proportional to three known quantities. It is traced back to Italian merchants in late medieval times, described in arithmetic books, the *libri d'abbaco* (Van Egmond 1980), while the way of thinking in the Rule of Three can be found in ancient Indian works by Brahmagupta (597–668) and Bhaskara II (1114–1185) (Tropfke 1980).

Proportional reasoning is considered a unifying theme in mathematics. It involves a sense of co-variation and the ability to make multiple comparisons in relative terms. The skills needed for proportional reasoning include multiplicative and relational thinking; and a highly developed understanding of foundational concepts, including fractions, decimals, multiplication, division, and scaling (Van de Walle, Karp & Bay-Williams 2010).

Many researchers have elaborated on students' understanding of proportions and proportional concepts. Keranto (1994) lead a teaching experiment focussing on developing proportional reasoning and ratio concept in the eighth grade where the problems were first learned to be solved mentally, emphasizing the unit-rate method, then in writing, using proportions. In this way, the pupils' real-world experiences and spontaneous models of solution were utilized naturally in teaching.

De Bock, Van Dooren and Verschaffel (2013) conducted two studies on students' ability to model textual description of situations with different kinds of representations of functions:

- proportional, $y = \mathrm{k}x$
- inverse proportional, $y = \mathrm{k}/x$, $x \neq 0$, and
- affine, $y = \mathrm{k}x + \mathrm{b}$, $\mathrm{b} \neq 0$.

Their results indicate that

- students tend to confuse these models, and
- the representational mode has an impact on this confusion.

When investigating students' ability to link representations of proportional, inverse proportional, and affine functions to other representations of the same functions, results indicated that students make most errors for decreasing functions. The number and nature of the errors also strongly depended on the kind of representational connection to be made. In both studies, mutual confusion between two increasing, and between two decreasing functions was reported. Both studies provided evidence for a strong impact of representations in students' thinking about these different types of functions. In a mathematical modelling context, graphical representations were helpful in most cases to detect the model underlying a realistic situation. For mutually connecting representations, tabular representations, providing concrete function values, proved to be most supportive.

The study

The following study is divided into two parts:

1. Comparison of *Arithmetica Danica* (Frommius 1649) and *Arithmetica Islandica*, contained in manuscript Lbs. 1694 8vo (1716/1733).

2. Presentations of ratio and proportions in 18th, 19th and 20th century books, up to *Skali* (Tofteberg, et al.2015; 2017),

The questions are:

1. Is Arithmetica Islandica an adapted extract of Arithmetica Danica?
2. How has the presentation of ratio and proportions developed during the 300-year period between the two arithmetic works, *Arithmetica Islandica* and *Skali*?

The contents of *Arithmetica Danica* and *Arithmetica Islandica* will be listed side by side, thus comparing their order and length of content. The *Regula Trium Directa* and *Regula Trium Inversa*, the ancient rules to solve directly proportional and inversely proportional tasks, will be examined and contrasted by other methods, such as verbal, tabular, algebraic, geometrical and graphical representations.

The two arithmeticas

Arithmetica Islandica

The manuscript *Arithmetica Islandica* of 1716 is the oldest arithmetic textbook in the Icelandic language from modern times. Earlier works are encyclopaedic. This arithmetic treatise on 73 handwritten sheets, 146 pages, is a part of a larger manuscript, Lbs. 1694 8vo, contained in pages 37r–109v. A senior enforcement officer of the 18th century Iceland, Skúli Magnússon (1947), recounts in his biography that his father, clergyman Magnús Einarsson (1675–1728), had made a free translation of *Arithmetica Danica*. The time period and name of the treatise suggests that *Arithmetica Islandica* could be its free and adapted translation.

Figure 1: The title: "*Arithmetica Islandica* Skrifað Anno MDCCXVI" (*Lbs. 1694 8vo*, 37r).

On its front page, it says "*Arithmetica Islandica*, Skrifað Anno MDCCXVI" [… written in 1716]. Later in the work, the year 1733 is mentioned as the current year's datum, suggesting that the date of the extant copy is 1733.

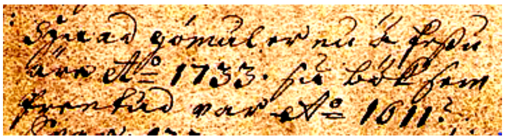

Figure 2. "How old is now in this year, A° 1733, the book which was printed in A° 1611?" (*Lbs. 1694 8vo*, 54v).

After discussing the meaning of the terms *arithmetica*, *geometric*a, *astronomia* etc., the text continues into *Numeratio*, numeration, reading large numbers in Hindu-Arabic number notation as well as Roman notation, similar to *Arithmetica Danica* without any identical examples. The text goes further into monetary, measuring and time units. These topics were important for the Icelandic public in their trade with foreign merchants while these matters are not mentioned in *Arithmetica Danica*. The treatise continues through the four arithmetic operations in whole numbers and the various units. Both works explain *Probatio*, that multiplication and division can probate each other. The second section, *Progressio*, is about sequences, such as the odd numbers: 1, 3, 5, 7, …; and every third number: 1, 4, 7, 10, …; the even numbers: 2, 4, 6, 8, 10, … and then further into decreasing sequences, followed by geometric sequences: 1, 2, 4, 8, 16, …; 1, 3, 9, 27, … The third section concerns fractions; first a definition, then reduction, and the four arithmetic operations. The fourth section is called *Regula Trium*, the Rule of Three, which is claimed to be most useful and indispensable to all those who exercise the art of reckoning.

Arithmetica Danica

The title of the book is *Arithmetica Danica seu brevis ac perspicua institutio arithmeticae vulgaris* (Frommius 1649). It is a total of 164 pages, written in Latin by the Dane Jørgen From whose Latinised name is Geo Frommius. Book one is named *De Arithmetica Simplici*, on simple arithmetic: numeration, including Roman numerals, and the four arithmetic operations in whole numbers; notation of fractions, their reduction and least common multiple, and the four operations; and extraction of quadratic and cubic roots.

Book two is named *De Arithmetica Comparata*, on comparative arithmetic. It contains ratios between numbers, proportions and progressions, i.e. sequences; *Regula Trium*, i.e. Rule of Three; the inverse Rule of Three, the composite Rule of Three, *Regula Societatum* and *Regula Falsi* together with further elaboration on proportions.

Arithmetica Danica was a registered property of the South Cathedral Skálholt in 1744 (Ágústsson & Eldjárn 1992). Concerning the conjecture that clergyman Magnús Einarsson wrote the *Arithmetica Islandica* one must consider that a clergyman in Northern Iceland may only have seen *Arithmetica Danica* but not had it by hand. Einarsson, however, had a young mathematically-inclined teacher, Jón Árnason, at the North Cathedral School, Hólar, from 1692. Árnason became bishop and served at the south episcopal seat Skálholt during 1722–1743 (Ólason 1950). The estate at his death, dated May 4, 1743, reveals that he possessed foreign mathematical books, among them *Arithmetica Danica*, the aforementioned Sören Matthisen's *Arithmetica*, and a biblical *Arithmetica* by Jacob Borrebye (The National Archives of Iceland). This leads to the conjecture that Árnason brought the book with him from his studies in Copenhagen in 1722 when he began teaching at the Hólar North Cathedral School. Einarsson, then 17-year-old, may have made his own copy from his teacher's notes, tailored after *Arithmetica Danica*. Bishop Árnason may have bequeathed the book to Skálholt cathedral later.

We know that Bishop Árnason was the best mathematician in Iceland of his time, but indeed there was not much competition. He published a book in 1739, *Dactylismus Ecclesiasticus eður Fingra-Rím,* (Árnason 1838), to present the Gregorian Calendar that was adopted in year 1700 in the Danish Realm. It also introduced for the first time in print the domestic farmers' calendar, an ancient week-based calendar (Bjarnadóttir 2016).

Comparison of contents of Arithmetica Danica and Arithmetica Islandica

In *Table 1*, the length, size and overview of the contents of *Arithmetica Danica* and *Arithmetica Islandica*, are presented. The reader should keep in mind that *Arithmetica Islandica* was a manuscript in a smaller size than the printed *Arithmetica Danica*.

Arithmetica Danica (108 pp. in 4°)		Arithmetica Islandica (146 pp. in 8°)	
	Book 1, De Arithmetica Simplici	[I]	
Cap. 1–2 (17. p.)	Definitions, number notation, literature examples	Introduction (19 pp.)	Translation of terms, number notation, measuring and monetary units
Cap. 3–6 (20 pp.)	Addition, subtraction, multiplication, division	(56 pp.)	Addition, subtraction, multiplication, division
Cap. 7–13 (15 pp.)	Fractions: notation, reduction, the four operations, compound fractions		

Cap. 14–15 (13 pp.)	Extraction of square root and cubic root		
	Book 2, De Arithmetica Comparata		
Cap. 1–2 (11 pp.)	Ratios, proportions and progressions	II Progressio (14 pp.)	Progressions
		III Fractions (25 pp.)	Fractions: notation, reduction, the four operations
Cap. 3–4 (9 pp.)	Regula Trium Directa, Regula Trium Inversa	IV Regula Trium (16 pp.)	Regula Trium Directa, Regula Trium Inversa aut Obliqua
Cap. 5–7 (19 pp.)	Regula Trium Composita, Regula Societatum, Regula falsi	(16 pp.)	Regula Trium Dupla, Regula Duple Reciproca, Regula Alligationes, Regula Consortio

Table 1: Comparison of the contents of Arithmetica Danica and Arithmetica Islandica

One notices that the order of the contents is different. Progressions are presented before fractions in *Arithmetica Islandica*. Furthermore, the examples given are completely different. The Icelandic examples refer to Icelandic environment and circumstances, while in *Arithmetica Danica* many of them are historical.

Comparing Regula Trium Directa

Arithmetica Danica

The *Regula Trium*, the Rule of Three, is introduced by the following text (in a crude translation):

> On account of its immeasurable usefulness, this proportional rule deservedly is highly valued as Aurea [Golden]: which, given that out of three known parts, with definite calculation of arranged numbers, leads the way to the fourth (Frommius 1649, 76).

In continuation, the author refers to Euclid's *Elements*, proposition 19, book 7:

> If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and the third; and if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional (Euclid 1956, 318).

The author takes an example of that 2 times 9 is equal to 3 times 6:

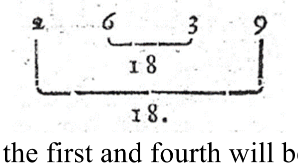

Figure 3: "… the number produced from the first and fourth will be equal to the number produced from the second and the third …", (Frommius 1649, 77).

The author then describes the method of the Rule of Three: Four numbers are to be arranged so that the first and the third are of the same kind and the second of the same kind as the fourth. When one wanted to know the fourth number, the second and the third should be multiplied together. The product should be divided by the first number to give the sought after fourth number (Frommius 1649, 77–78). Example:

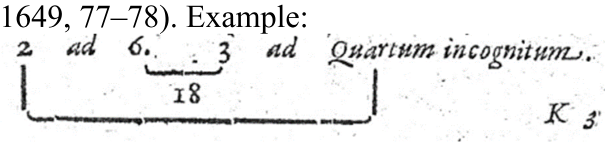

Figure 4: Finding the fourth unknown proportional (Frommius 1649, 77).

Arithmetica Islandica

The introduction to the direct Rule of Three tells the reader that the rule is indispensable for all those who practice the art of reckoning and can rightly be called [*Regula*] *Aurea*, that is the golden rule, because as gold stands out of other metals, so it surpasses other rules. The method presented is similar to that in *Arithmetica Danica*: The first and the third numbers are to be of the same kind, and the second of the same kind as the unknown fourth. The second and third numbers are to be multiplied together and divided by the first to gain the fourth (Lbs. 1694 8vo, 94r–95r). The problems are domestic. A typical question is: If 30 eiderduck-eggs weigh 12 marks, what then weigh 135 [eggs]?

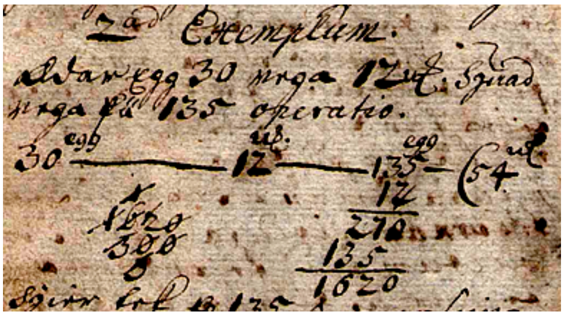

Figure 5: The weight of 135 eiderduck-eggs by the Rule of Three (Lbs. 1694 8vo, 95v).

Comparing Regula Trium Inversa

Both books explain the inverse rule such that the less the first number is to the third, the more is the second to the fourth unknown, and vice versa, assuming the same sequence as before.

An example from Arithmetica Islandica:

> To complete a work during 16 weeks, 9 men are needed. How many men are needed for the same work during 24 weeks? (Lbs. 1694 8vo, 100v).

Solution: Multiply the first and second numbers and divide the product by the third to gain 6 men, see Figure 6.

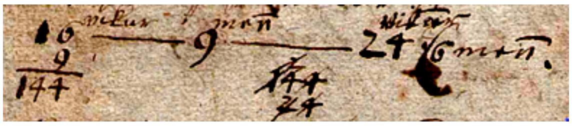

Figure 6: Finding the number of men to complete a work by the inverse Rule of Three (Lbs.1694 8vo, 100v).

Examples in Arithmetica Islandica also found in other works

Several arithmetic examples have circulated in Protestant Europe. A textbook by the Protestant Sigismund Suevus (1593), *Arithmetica Historica. Die löbliche Rechenkunst*, contained a number of arithmetic examples, disguised in biblical dress. These examples showed up in later textbooks, such as Euler's (1738) *Einleitung zur Rechenkunst*, and at least two Icelandic textbooks in manuscripts, one of them *Arithmetica Islandica* and the other manuscript *ÍB 217 4to Arithmetica – Það er reikningslist* [*That is Reckoning Art*], presumably written in 1721 (Bjarnadóttir, 2011).

Among those examples is a story about the age of Methusalem. Examples about the number of hours in a year, and the circumference of the Earth, both appear by Suevus (1593), *ÍB217 4to* of 1721, and Euler (1738), illustrating how examples were copied from one textbook to another in early modern times. Neither of these examples, however, appear in *Arithmetica Danica*, which is entirely void of practical examples, nor in *Compendium Arithmeticum* by Matthiesen (1680) which is known to have been in the possession of Bishop Árnason (The National Archives of Iceland).

Presentations of ratio and proportions in recent centuries

Iceland in 1780–1900

A number of calamities fell upon Icelanders in the 18th century. Among Danish subsidies after mid-18th century, were grants to printing the first substantial arithmetic textbooks, written in Icelandic by Olavius (1780) and Stefánsson (1785), both printed in Copenhagen. No further mathematics textbooks were printed until 1841. Olavius (1780, 172–178, 290–294) explained that the task in the Rule of Three, the Golden Rule (*regula aurea*), was to find the fourth term in a geometric equality. The numbers were to be arranged with the fourth term missing, such as 4 – 12 – 6 – *X*. Then the working method was: Multiply *middle term* and *rear term* and divide by the *front term*.

Stefánsson's (1785, 132–137) method to find the fourth proportional number is the same as in *Arithmetica Danica*: multiply the second and the third term and divide by the first term, assuming the same order. Stefánsson's son edited the book when preparing it for print in Copenhagen. In that new version, the phrase is found: "it is also called the golden rule or *Regula aurea* for its supremacy, because as much superiority as gold has over other metals, so much it surpasses other arithmetic rules" (Stefánsson, 1785, 132). The similarity of this phrase to that of *Arithmetica Islandica*, especially the word "yfirgengur" [surpasses] suggests that the son knew that work. However, the term *Regula aurea* may be found in many, or most, arithmetic textbooks of early modern age.

Briem (1869) wrote an influential textbook, used in the emerging lower secondary schools from the 1870s. The search for the unknown in direct Rule of Three was to find a number that is as many times greater or less than the *middle term*, as the *rear term* is greater or less than the *front term* in the sequence of the three known proportional numbers.

England in the 19th century

J. Stedall (2012, 56) said in The history of mathematics – A very short introduction:

> The Rule of Three was a rule that enabled countless generations of students to answer questions like: A men dig a ditch in B days, how long would it take C men to do the same job?
>
> A 19th century English century school boy was not expected to start doing anything on his own initiative. He would be taught that he must multiply A by B and divide by C.

Problems of that kind, digging ditches, were still taught in Iceland in the 1970s (Gíslason 1962, 52).

Iceland in 1920–1960

Mathematician Ó. Daníelsson was an undisputed leader of secondary mathematics education during 1920s–1960s. He presented by examples two cases, the direct and the inverse Rule of Three:

Case I:	Case II:
4 meters cost 3 crowns	4 men complete a work in 3 days
6 meters cost *x* crowns	6 men complete the work in *x* days

In the two cases the student is to consider the ratios 6/4 and 4/6. Which to choose depends on whether the outcome should be greater or less than the known number 3.

In case I, one should choose to multiply 3 by 6/4 to gain the answer 4 ½ crowns.

In case II, one is to multiply 3 by 4/6 and the answer is 2 days (Daníelsson 1938, 45–46).

Gíslason (1962, 45–52) tried to modify the method by inserting a unit sentence:

1 meter costs ¾ crowns	1 man completes the work in $3 \cdot 4$ days

This procedure was expected to make it easier for the reader to decide if to continue by multiplying or dividing.

Present times

Skali 2015–2017

By the introduction of the New Math, methods and procedures became objects of scrutiny. The term Rule of Three disappeared. Proportions, however, continued in the curriculum. With improved printing technology, they became represented in a great variety of ways: verbally, tabular, geometrically, graphically, and algebraically in functions. The latest arithmetic textbook series for the lower secondary school level is a Norwegian series in six volumes, *Maximum*, termed *Skali* in Icelandic. In *Skali 2A* (Tofteberg et al. 2015) for the 14-year age, the ratio of mass to volume is examined by the students by drawing a double number line, see Figure 7.

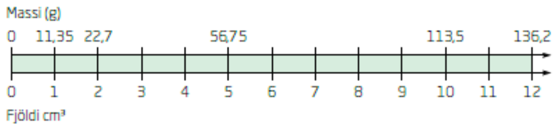

Figure 7: The ratio of mass in kg to volume in cm^3 (Tofteberg et al. 2015, 172).

Students are also trained in recognizing direct proportionality as a linear function including the pair of coordinates of Origo, (0,0). In continuation, in *Skali 3B* (Tofteberg et al. 2017), they learn about the characteristics of the graph of inversely proportional quantities, see Figure 8.

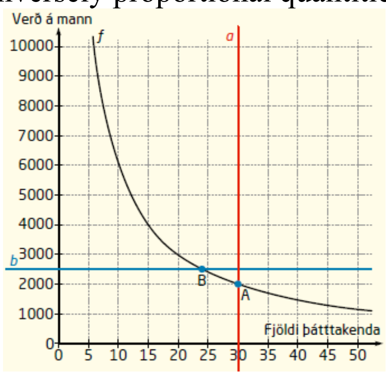

Figure 8: Sharing cost between varying number of participants (Tofteberg et al. 2017, 30).

Students also practice distinguishing between the various types of functions, such as:

- proportional (4): $y = kx$
- inversely proportional (3): $y = k/x$,
- quadratic (1): $y = k \cdot x^2$
- affine (2): $y = kx + b$, see Figure 9.

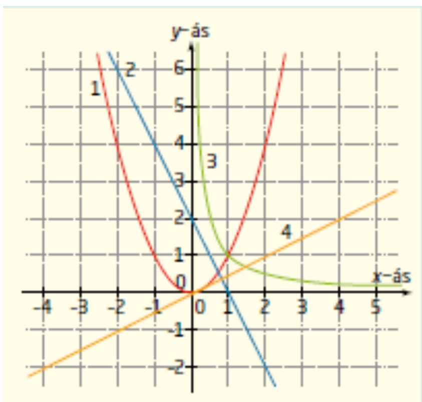

Figure 9: Students practice distinguishing between different types of functions (Tofteberg et al. 2017, 28).

They also practice reading tables, and decide whether the relation between two variables, *x* and *y*, adheres to $y/x = k$, $(x \neq 0)$ or to $x{\cdot}y = k$, see Figures 10 and 11:

a

x	3	4	8	12
y	9	12	24	36

b

x	0	5	7	15
y	0	10	21	60

Figure 10: Exercises in reading tables (Tofteberg et. al. 2017, 28).

x	3	10	12
y	20	6	5
y · x	60	60	60

Figure 11: Quantities in inverse proportions (Tofteberg et al. 2017, 41).

Discussion

The question if *Arithmetica Islandica* was a translation or an adapted extract of *Arithmetica Danica* can be answered negatively. We see that the structures of the works are similar, but both of them also adhere to a pattern of arithmetic textbooks that had been developed since the early modern age. No examples are the same, not one, and even the phrases about the usefulness and superiority of the Rule of Three, the Golden Rule, are general and can be found in so many books that they cannot be claimed to be related. However, the author of *Arithmetica Islandica*, who most likely was a scholar that graduated from one of the two cathedral schools, must have had access to foreign books and a knowledgeable teacher, who actually was available in the northern Hólar Cathedral School. We may conjecture that he knew *Arithmetica Danica* and wanted Icelanders to have a similar work.

Neither can we confirm that the Reverend Magnús Einarsson (1675–1728) wrote *Arithmetica Islandica*. But having been a student of the young scholar Jón Árnason, later Bishop Árnason (1665–1743), makes it quite likely that he had acquired knowledge to complete such a work. In 1716, Einarsson was in his early forties and could have had time to collect examples from his experience and from books that he may have seen or possessed. In the thinly populated country, there are not many candidates for such an enterprise. Another candidate is the Bishop himself. The facts that weigh against him are that his works are well documented and available in archives. Besides he had more possibilities than others to have his works printed, at least in Copenhagen. The third candidate is named Magnús Arason Thorkelin (–1728), titled a sea captain. He served with the mathematician Ole Römer in Copenhagen and later in the Danish army on geodesy. He lived abroad for 16 years, spoke a number of European languages, and returned to Iceland first in 1721 (Ólason 1950), after the assumed date of the original manuscript of *Arithmetica Islandica* of 1716. He may, however, rather have been the author of *Arithmetica – That is reckoning Art* of 1721, contained in manuscript ÍB 217 4to, which does not refer to Icelandic environment (Bjarnadóttir 2011). Einarsson is therefore the most promising known candidate for authoring *Arithmetica Islandica*.

The Rule of Three, the Golden Rule, has been useful through the centuries, especially in trade. Proportional reasoning is still important, for example in the task of converting from one currency to another. The methods practiced are more debatable. The old versions of arguments supporting methods for solving direct proportionality or inverse proportionality, can also apply to other kinds of functions, such as quadratic functions or affine functions and are therefore insufficient and confusing. The unit-rate method, used by Gíslason (1962) and emphasized by Keranto (1994) was an effort to improve the Rule of Three and is still in use where applicable.

The context is also important. Many generations have planned works such as digging ditches even if few workers and still fewer teenagers are occupied with that kind of work anymore. The contexts in examples on proportionality have only lately approached the environment of contemporary youth, such as enlarging pictures, or planning activities and sharing cost. New topics, such as statistics and probability, have brought opportunities and needs for applying proportional reasoning.

Presently, other kinds of functions than proportionality appear frequently. The difficulty is to decide if proportional reasoning is applicable or not. Referring to the research of De Bock et al. (2015), students tend to confuse situations with different kinds of representations of proportional, inverse proportional, and affine functions, and the representational mode has an impact on this confusion. In a mathematical modelling context, graphical representations were helpful in most cases to detect the model underlying a realistic situation, while for mutually connecting representations, tabular representations providing concrete function values, proved to be most supportive.

We see that *Skali*, as a representative for contemporary textbooks for teenagers, has adhered to trends that are discussed by De Bock et al. (2015) in considering that representation is important, such as practice in tabular representation, and practice in distinguishing between the various types of increasing and decreasing functions. *Skali* also offers a variety of contexts that concern contemporary teenagers. The presently available versatile representations – verbal, tabular, algebraic, geometrical and graphical – hopefully will serve as aids in reducing the number of pitfalls meeting students.

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RUSSIAN POST-REVOLUTIONARY MATHEMATICS TEXTBOOKS: A SHORT-LIVED HISTORY

ALEXANDER KARP

Abstract

This study is devoted to Russian mathematics textbooks during the years 1918-1931. The period under examination was a time of reforms, when old schools with all their methodological riches were rejected (at least, in declarations), and educators were tasked with teaching in a new way, by fostering and developing students' independence. Old textbooks, and to a certain extent textbooks in general, turned out to be discordant with the new demands (again, at least in declarations), and new textbooks had to be prepared. The present study examines some of them: some were new versions of textbooks that had already been prepared before the revolution; others were written only after the revolution. The ideas that guided their authors were (and remain) popular outside of Russia as well, for which reason the history described here is of general interest.

Introduction

This paper is devoted to Russian textbooks from 1918-1931 and constitutes a continuation of studies by Karp, 2009, 2010, 2012, which focused mainly on what occurred during these years in mathematics education as a whole, without analyzing specific textbooks.

After the revolution of 1917, an attempt was made to establish a new socialist school, which rejected the traditions of drill and rote memorization, characteristic of the old pre-revolutionary approach. The theoretically "new" approach was in fact a strange blend of American progressive education; ideas developed by the international reform movement, which arose from the International Commission on Mathematics Instruction; certain traditions of Russian pre-revolutionary democratic pedagogy; and Soviet phraseology, often in a rather primitive form. In the early 1930s, schools were redirected toward the old ways, and all of these experiments were rejected as left-wing perversions, and their authors were denounced as schemers (Karp, 2010).

Surviving textbooks from this brief period of reforms are of considerable interest, if only as records of an attempt at a new approach to the problem of educational literature. Although many ideas promoted in Russia at that time were also popular in other countries, and frequently had even come to Russia from abroad, nowhere, probably, did they triumph as much as they did in Russia. For all the popularity of the laboratory method or the Dalton Plan during the first third of the twentieth century in other countries, it is impossible to imagine state-organized meetings at which teachers would be urged to teach in such a way and no other, which was quite an ordinary occurrence in the USSR. For a little over a decade, the power of the centralized totalitarian state was aimed at implementing the tenets of progressive education, which had been formulated under entirely different conditions.[1]

Young Soviet pedagogy saw a need to fight against teacher-centered education (although it used different terminology). It encouraged independent work by students (the Dalton Plan consisted precisely in giving a leading role to such work); insisted on maintaining a constant connection with the real world, which was to be studied in the school-laboratory through various types of

[1] It is noteworthy that the founders of American liberal pedagogy initially had a very positive attitude toward what was taking place in the USSR. Counts's (1930) account is representative: it follows the author as he travels around the country and, though admitting certain problems, nonetheless mainly points out notable achievements.

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measurements and experiments (the laboratory method); opposed the study of separate subjects and, instead, promoted "complexes," that is, the study of a given phenomenon from different sides, with a view to combining knowledge from different fields (sometimes, although far less frequently, such a system was also called the "accord system") (Karp, 2010, 2012).

Many of these ideas and approaches have remained popular to this day. In a certain sense, it is these ideas that should today be considered traditional for the United States, although this word is usually used with reference to a completely different kind of pedagogy (Klein, 2007). Beginning in 1931, the resolutions of the Central Committee of the Communist Party seemingly exterminated these ideas in the USSR; but in recent times they have started to come back, for which reason the assessment of what took place in the 1920s, too, began to change from an unequivocally negative one to a more cautious one.

It should be noted that, despite the whole force of the state apparatus and its educational agencies (first and foremost, the so-called GUS – Gosudarstvennyi Uchenyi Sovet – the State Academic Council of the Ministry of Education – People's Commissariat for Education, which developed and approved educational programs), the leadership was not able to achieve everything that it desired. "Complexes" may have been implemented in one way or another in elementary schools – so-called stage-one schools – but subsequently, in stage-two schools for older students, their implementation proved far more problematic. Teaching a serious course in mathematics (and the course was quite intensive and substantive in its design) as part of a complex with something else turned out to be unfeasible (Karp, 2010, 2012).

The Russian fate of ideas that have remained influential in the world by now for at least a century is undoubtedly worthy of study; and it is important to examine both what was sought and what proved feasible. Textbooks offer a certain (although, of course, not a complete) opportunity to form a picture of both the former and the latter.

On Educational Literature

Initially, after 1917, Narkompros (Ministry of Education) officials believed that textbooks should be eliminated from schools altogether (Glushkov, 1951); but things turned out differently in reality. Vol'berg (1918), a leading figure in the reform of mathematics education during its first phase, commented ironically that, after humoring themselves with talk of new schools based on "individual initiative, creativity, and labor" (p. 35), teachers were forced to resort to the old, pre-Revolutionary textbooks in their teaching. Analyzing records pertaining to the publication of educational literature in mathematics during the years 1917-1920, Glushkov (1951) pointed out that, for economic reasons, much less literature in general was published than had been done previously, but that, no less significantly, the amount of new literature that appeared was negligible. Naturally, some books (above all, new problem books in arithmetic) did get published, and over the years (looking beyond 1920) their numbers grew. Nonetheless, opening a survey of educational literature in any pedagogical journal in those years, we will see names that had already appeared before the Revolution. For example, in the "Criticism and Bibliography" section of the "Siberian Pedagogical Journal" for 1925, we see works by Shokhor-Trotsky and Goldenberg, Arzhennikov and Volkovsky, all repeatedly republished before the Revolution – although alongside of them, for example, we also find the post-Revolutionary "Problem Book in Arithmetic on a Basis of Social Sciences for Stage-One Schools" by Lankov (Gurinovich, 1925).

Furthermore, if for stage-one schools the number of new collections of problems, so-called *workbooks* or works on the methodology of teaching in labor schools was not that small (and to repeat, systematic textbooks, in principle, went against the reigning tendency), then things were worse in this respect with stage-two schools. Tamarin (1930) not without irony quotes an RSFSR State Publishing House advertisement, which appeared in May 1930 in *Pravda,* the country's leading newspaper: the advertisement listed 12 books in mathematics, almost all of them by respected, pre-Revolutionary authors – Kiselev, Rybkin, Shaposhnikov, and Val'tsev, Rashevsky.

"You are surprised, reader?" Tamarin asks, ironically suggesting that "this reviewer is being deceitful, reproducing an advertisement that was probably written 15 years ago." Later, however, he explains that the aforementioned books were advertised not as textbooks for stage-two schools, but only as a list of preparation manuals for colleges and technical colleges. But, as Tamarin also notes, the main consumers of these books were stage-two schools.

Consequently, one can identify two questions for historical research, which are still a long way from being answered (and which, naturally, cannot be answered fully within the bounds of this short article): how did pre-Revolutionary textbooks change in later editions, and which new instruction books did in fact appear after the Revolution? With respect to the first question, the same Tamarin wrote as follows:

> The textbook has been preserved in all of its former inviolability. My mistake: some books have been subjected to a decisive "rejuvenation" procedure. For example, in the "Collection of Problems in Algebra" by Shaposhnikov and Val'tsev, the word "merchant" has everywhere been replaced by the word "cooperative," and "landowner" has been replaced by "state farm." (p. 108)

But even such a "rejuvenation" merits attention. As for new books, while it remains unclear to what extent they were used in schools, the number of them published in the late 1920s even for stage-two schools was relatively substantial. Tamarin himself names a number of problem books and workbooks that came out in 1929-1930. Below, we briefly describe some books published during the years 1917-1931, without attempting to offer an exhaustive survey of the existing literature, but examining texts that we consider sufficiently representative of the given period, since they exemplify various approaches that existed at that time.

"Workbook in Mathematics for the Third Year of Stage-One Schools" by A. M. Voronets

This book, which came out in 1926, was one of a series of workbooks for different grades published by A. M. Voronets. The author notes in the introduction that the sequence of topics in the book corresponds to the State Academic Council's curricula. The book is divided into three parts (trimesters). The first of them – the fall trimester – is entirely devoted to the topic "Human being." It includes the following sequence of sections: (1) The Labor of the Worker and the Peasant; (2) Skills; (3) Work and Nourishment; (4) Skills; (5) The Growth and Weight of Human being; (6) Skills, etc. The winter semester contains the following sections: (13) The October Revolution; (14) Skills, Square Measures, and also a number of sections united under the heading "Our City" – (15) The Geographic Location of Our City; (16) Meteorological Observations, etc. Finally, the spring trimester includes the following sections: (27) Measuring a Kilometer; (28) Territorial Measures; (29) The District, etc. There is also an appendix, which contains various puzzles under the heading "A Time to Work, A Time to Play."

As can be seen, the sections are dissimilar. For example, the section on "The October Revolution" begins with a problem:

The Revolution won an eight-hour work day for workers. Formerly, work in factories took place in shifts of not fewer than 10 hours, and usually 12 hours. How many hours of freedom did workers acquire every day? How many hours and days would this add up to over the course of a year? (p. 38)

This is followed by various word problems with a political content. The next section, "Skills," while containing real world problems about finding the area of a room or a postage stamp, consists for the most part of far more purely mathematical assignments. Here, students are given the definition of a rectangle, along with a touching warning not to confuse it with a right triangle; they are asked to draw perpendicular lines using a ruler and set square; the formula for the area of a triangle is discussed (but not proved), etc. In general, this "workbook" is in effect a problem book whose content may be studied in ways that are quite different. It is clear, however, that the author envisions exercises "outside the complex" devoted to developing students' skills.

"Textbook in Geometry" by A. R. Kulisher (1922)

A. R. Kulisher was a very active participant in pre-Revolutionary congresses of mathematics teachers, where he promoted the idea of introducing a preparatory visual course in geometry. He published a textbook for such a course in 1913. Subsequently, this textbook was reissued for use in stage-one unified labor schools.

As the introduction to the first edition explains, the course was intended to span three years or 75 classroom hours. It included an introduction to the basic plane and spatial figures and the basic kinds of configurations of lines and planes (parallelism and perpendicularity), the measurement of areas and volumes, and such concepts as symmetry and similarity. The book contains many pictures. Its main part contains almost no expository writing: only problems are given. Definitions and arguments are given in an appendix, which begins with a specially marked warning to the effect that “All of the definitions and arguments provided below are to be studied by the class only after they have been discovered by the students themselves through corresponding classroom exercises” (p. 103). Examples of problems that appear in the text include the following: “Run your hand along the top of the desk at which you are sitting. What shape is this desk top (by feel)?” (p. 13); “Draw two parallel planes and three perpendiculars to one of them. Will the extensions of these perpendiculars intersect the other plane?” (pp. 48-49); “With respect to which straight lines is an ellipse symmetrical?” (p. 88).

Kulisher notes rather severely that the success of teaching (and teachers using this textbook are left with a great deal to think through on their own) “depends on the extent to which basic general educational principles are implemented in practice” (p. 10). Glushkov (1951) no less severely concludes that “Kulisher fills his textbook with material that is interesting, but too difficult for the students” (p. 312).

“Mathematics. Workbook for the Sixth Year of School,” general editor E. S. Berezanskaya

This book went through several editions (quotes below are from the third, 1931). Its introduction notes that since “a large part of the sixth-year curriculum in mathematics has a formal character, the connection with curricula in other disciplines during this year is, as is known, weak” (p. 3). The amount of material in the book is large; it is divided into ten sections: Relative Numbers; Equalities and Equations; Parallel Lines; Monomial Expressions; the Relative Positions of a Straight Line and a Circle; the Relative Positions of Two Circles; Triangles and Axial Symmetry; Polynomial Expressions; Quadrilaterals and Polygons; First-Order Functions; Lines and Angles in the Circle; Elementary Land Surveying Assignments.

As we can see, the course includes both geometric and algebraic material. In style, the textbook is noticeably different from, say, Kiselev’s pre-Revolutionary textbook. For example, the section “Parallelogram” begins with a problem in which students are asked to draw parallel straight lines and to intersect them with another pair of parallel straight lines; they are then told that the quadrilateral that has been formed is called a parallelogram; and only after this is a formal definition of a parallelogram given. In the next section, a formal proof of the fact that a parallelogram is divided into two congruent triangles by either of its diagonals is preceded by the suggestion that students cut a parallelogram out of a piece of paper and become convinced of this assertion experimentally. And the formal proof itself is composed of questions (“What can be said about the triangles...?” “According to which statement on congruence will the triangles be congruent...?”), which the students must use to carry out the proof on their own. No complete proof is given in the book (pp. 156-157).

In other words, the textbook to a much greater extent is conceived of not as a collected body of knowledge, but as a manual with whose help such knowledge may be established. One can point to other differences from pre-Revolutionary textbooks as well. Nonetheless, it is impossible not to agree with the already-cited Tamarin (1930), who acknowledged that this book and its continuation for the seventh year of schooling are “literate and sound,” but who remarked that “in their novelty one senses the old days,” and who complained that in these books “one does not feel the beating of the ‘socialist’ pulse, [that] there is no connection with production, no polytechnism” (p. 111).

"Concentric Textbook in Algebra" by V. G. Fridman
This textbook was first published before the Revolution, in 1912. In 1922, it was reissued with substantial abridgements, and subsequently it was revised once again, so that, for example, as the author explains in the introduction to the 1924 edition, "all problems and exercises which in previous editions had been taken from Shaposhnikov and Val'tsev's problem book and thus were abstract in nature have been replaced with real-world problems." The textbook owes its "real-world relevance," for example, to such problems:

> Determine the number of children (out of 1000) who die before the age of one in the USSR and in Norway based on the following information: if the number of children dying in the USSR is increased by 8, and the number of those dying in Norway is increased by 3, then the first number will be 4 times greater than the second number. If the number for the USSR is decreased by 22, and the number for Norway is decreased by 17, then the first number will be 5 times greater than the second. (p. 65)[2]

In essence, this textbook (like the problem just quoted) is rather traditional (at the very least because it involves no "complexes," etc.). Its unusual characteristic consists in the fact that the author returns again and again to certain topics (the concentric approach). For example, in the first "concentration" (stage), the author examines the concept of the square root and finding the square roots of perfect squares; in the second "concentration," students learn about extracting square roots of integers in general; and in the third "concentration," they are taught to extract square roots of fractions (p. 6). The author takes credit for his attention to the concept of functional dependency, for the presence of historical commentary in his text, and above all, for the fact that he "tried to construct the presentation of the educational material in such a way that examples would precede the derivation of rules, serving as the foundation for this very derivation" (p. 7).
Glushkov (1951) describes this textbook as being one of the most widely used textbooks of the 1920s, along with the textbooks of Kiselev, Lebedintsev, and Rashesvky (while criticizing it for the straightforwardness with which it implements the concentric approach and even for its "anti-scientific bias.") But the textbooks just mentioned were clearly more popular than Fridman's before the Revolution, although their popularity was also not equal (one need only notice the gap of many years that passed between the first edition of Fridman's book, in 1912, and later editions).

"Workbook in Mathematics," Edited by G. A. Popperek (1924-1925)
This book came out in three parts, the first of which, as the book indicated, was intended for 5-7 years of study at labor-based seven-year-schools and for the first year of workers' faculties (educational institutions that prepared individuals typically from working-class backgrounds, without a secondary education, for higher educational institutions); the second part was intended for the second and third years of workers' faculties; and finally, the third part contained the study of logarithmic and trigonometric functions, as well as basic advanced mathematics, in other words, matched the curriculum of institutions of higher learning. The book was apparently oriented first and foremost toward workers' faculties and went through no fewer than 18 editions (the first parts).
The authors designate their book as a handbook for the study of mathematics based on the Dalton Plan and the accord system. The introduction to the first part opens with a statement, printed in italics, to the effect that the present book constitutes a "handbook for the laboratory method of instruction, in which students work through the material independently" (Popperek, 1924, p.1). The authors go on to note:
However, the laboratory method, which is today universally recognized as the latest and most advanced achievement of pedagogical thought, has remained until now a *pium desiderium* [pious

[2] It must be noted that the presence of a problem that makes it clear that something in the USSR is worse than abroad would be unthinkable in Soviet textbooks from a later time.

wish] as far as productively applying it on a mass scale is concerned, since adequate educational literature for its dissemination and practical application on a mass scale has been lacking (p.1).

The authors then write that their handbook is not a handbook in arithmetic or algebra, but a handbook specifically in mathematics, which fact itself in their view exemplifies a complex-based approach, which they also variously extol, but which, in their view, is not promoted by a sufficient number of books. The authors expect that their own book will alleviate these "pressing needs."

The book consists of assignments, which are divided into sections. In the sections, we find problems (often with solutions), questions, and exercises. The questions are often organized sequentially in a way that makes it possible, while answering them, to arrive at certain conclusions. The conclusions are explicitly formulated. Sometimes, the word "theorem" also appears. The authors' assertions are broken up into questions, which the students must answer independently. The following sections, for example, comprise the sixth chapter, "Regular Polygons" (their sequence makes it clear what the authors mean by a "complex-based approach").

- A circle circumscribed around a triangle.
- A circle inscribed in a given triangle.
- Regular inscribed and circumscribed polygons.
- Finding the length of the side of an equilateral triangle inscribed in a circle, in terms of the radius of this circle.
- The area of an equilateral triangle. The regular prism and the regular pyramid.
- Trigonometric functions of 30^0 and 60^0 angles.
- Problem. Inscribe a regular hexagon in a circle and find the length of its side [in terms of the radius].
- Problem. Inscribe a regular quadrilateral in a circle and find the length of its side in terms of the radius.
- Trigonometric functions of a 45^0 angle.
- Theorem. One and only one circle can be inscribed in any regular polygon and a one and only one circle can be circumscribed around any regular polygon.
- The area of a regular polygon.
- Finding the length of the side of an equilateral triangle circumscribed around a circle, in terms of the radius of this circle.
- Finding a_n [the side of an inscribed regular polygon] using trigonometry.
- Finding b_n [the side of a circumscribed regular polygon] using trigonometry.
- Incomplete quadratic equations (Popperek, 1925)

It is easy to see that very different sections of school mathematics are indeed represented here. It is hardly likely, however, that subject matter that appeared out of nowhere – introduced "by the way," as it were, since it happened to come to mind (the clearest example of this being the regular prism and the pyramid) – was assimilated well by the students (it must be remembered, too, that the students at workers' faculties did not come from academic environments).

Discussion and Conclusion

None of the examined handbooks outlived the period discussed above. After a series of resolutions by the Central Committee, A. P. Kiselev's pre-revolutionary textbooks, which had become popular before the Revolution, came back into the schools. Indeed, even the books discussed above reflected pursuits that dated to before the Revolution. This can be felt in the style used by their authors: Latin expressions in handbooks for workers' faculties look somewhat surprising and at once reveal a former gymnasium teacher (which G. Popperek had in fact been).

On the other hand, Popperek and Kulisher and Fridman were all fighters for a transformation of the schools; the most active of them was probably A. R. Kulisher, but G. A. Popperek, for example, long before writing the handbook discussed above, had also made presentations at teachers' congresses (see Sinitsky and Popperek, 1906) and on other similar occasions.

It is easy to see the differences between the textbooks of the 1920s and later ones (or, which is almost the same thing, earlier ones). The former indeed required the students to work independently. Consequently, these textbooks were structured differently: instead of theorems-examples-problems and exercises, they contained questions and problems, out of which theoretical knowledge was supposed to grow. At different times, certain wonderful books in mathematics (for example, Pólya and Szegő, 1998, to mention just one) have been structured on this principle, but just how useful this method proved in mass-scale schools (or workers' faculties) is open to debate. We have no access to any statistics (and likely will never access to any statistics) that might tell us what share of students proved capable of studying in this way, or how often teachers themselves were forced to explain, in the old-fashioned style, how to answer the assigned questions. Judging by the fact that E. Berezanskaya was subsequently (after 1931) one of the most active figures in mathematics education, it may be supposed that prior to this, too, she did not rule out the possibility of using her book as an ordinary traditional textbook.

Clearly, the radical version of the complex-based approach ("accordness"), understood as the simultaneous study of different subjects, was not applied in the upper grades. The complex-based approach that was used in these grades was one that eliminated divisions between mathematical subjects. Demonstrating the unity of mathematics is undoubtedly important, but once again, it is not obvious to what extent educators took into account the fact that not all students could immediately connect different sections and different ideas.

What appears indisputable is the fact that the books analyzed above contained many interesting assignments (for both stage-one and stage-two schools), many of which were in one way or another later made use of. Moreover, we have noted earlier (Karp, 2009) that the creators of the Sputnik, contrary to what is sometimes thought, attended and graduated from Soviet schools that were not those which took shape after 1931, but in fact post-revolutionary schools. Without taking this as proof for the high quality of post-revolutionary schools, we might nonetheless ask toward whom in reality (and not in political declarations) these schools were oriented and to what extent they proved capable of educating the average (rather than the exceptionally gifted) factory worker or peasants' child in mathematics. Likely the only remaining way of investigating this that is available to us is by analyzing the recollections of the former students of that period (which the author hopes to undertake in the future).

One last thing that must be mentioned is the politicization of the assignments, which is so conspicuous in many of the handbooks, in which even in a class in mathematics a child is constantly being told about socialism's achievements and capitalism's defects. The politicization of assignments during this period was indeed very considerable. Nor did it disappear, of course, in later years; but it changed. Changes in its orientation and intensity also deserve to be investigated.

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DESCRIPTIVE GEOMETRY TEXTBOOKS TRANSMITTED TO BRAZIL: HOW THEY WERE RECEIVED AND DIFFUSED

THIAGO MACIEL DE OLIVEIRA and VINÍCIUS MENDES COUTO PEREIRA

Abstract

The article presents how descriptive geometry textbooks were received in Brazil. The historiography of this field of study focuses on the Brazilian translation of Gaspard Monge's book and its use at the Royal Military Academy of Rio de Janeiro. Documents that were recently found in public archives show the presence of other textbooks, like Silvestre-Françoix Lacroix's "Descriptive Geometry" as well. The article will also address Brazilian textbooks on this subject that were published during the 19th and 20th centuries by professors at several scientific institutions, including the Brazilian translation of the "Éléments de Géométrie Descriptive avec nombreux éxercices par F.I.C". Brazilian publications about descriptive geometry show the reception of new developments on the subject developed within the French community. The present article will also highlight Alvaro José Rodrigues' work. He was a former professor at the National School of Fine Arts and he published three volumes on descriptive and projective geometry during the first half of the 20th century.

Introduction

The advance of Napoleon's troops towards the Iberian Peninsula forced the Portuguese royal family to escape to Brazil, which was still a colony of Portugal at the time. The Emperor Dom João VI settled in Rio de Janeiro in 1808. Saraiva (2007, p. 24) affirms that the war brought unexpectedly good consequences with regard to the diffusion of ideas. Among the main acts of the emperor, which triggered profound economic and cultural changes in Brazil, was the creation of the Royal Military Academy, which began in 1811. "A regular course in the area of exact sciences that had its application in military and practical studies, and which explicitly aimed to train more qualified officers for the exercise of their profession" was thus established in Rio de Janeiro (Oliveira 2005, p.159).

The professors assigned to teach these subjects at the Academy were strongly encouraged to produce a textbook or translate a renowned text on the subject. The texts indicated were "Elements of Geometry" by Adrien-Marie Legendre, Sylvester-François Lacroix's "Differential and Integral Calculus" and Gaspard Monge's "Elements of Descriptive Geometry". The texts written by Étienne Bézout, Benjamin Robins and the memories of Leonhard Euler were also suggested for use (Oliveira 2005, p.175-176).

Mormêllo (2010) states that descriptive geometry was a novelty for the time, considering the educational institutions established in Brazil before 1810. Furthermore, the author explains that "descriptive geometry was most certainly introduced into the Royal Military Academy under the influence of the *École Polytechnique* of France", since the country was a model for Portugal and the literature used in Portuguese military education was almost entirely French (Mormêllo, 2010, p.73).

The first professor of descriptive geometry at the Academy was José Vitorino dos Santos e Sousa, a Brazilian lieutenant of the Royal Corps of Engineers that had graduated from the Faculty of Mathematics at the University of Coimbra. His translation of Monge's "Elements of Descriptive Geometry" is based on the first edition and was published in 1812 by the newly created Royal Press. Vitorino affirms that he produced such a text in order to "contribute to the raising of the empire of sciences and the fine arts in a new world, which offers many natural resources for their

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application in industrial efforts, and furthermore, has the potential to improve the arts" (Sousa 1812, p. xix).

French publications on descriptive geometry: A brief rationale

For the period from 1812 to 1843, Barbin (2015) has listed the French publications on descriptive geometry. For the author, the dissemination of descriptive geometry corresponds to new *élémentations* in relation to Monge's text. Four important changes were identified. The first one concerns preliminary ideas, in which "more and more considerations are introduced to help the students to solve problems" (Barbin 2015, p.63). With this in mind, the author highlights the approaches of Lacroix (1795) and Hachette (1822). While the former began with the projections of straight lines and planes, the latter began with the projections of points and lines. Vallée (1819) and Adhémar (1832) proposed a complete decomposition of point, straight line and plane projections. The second change brought an end to Monge's "beautiful lesson in geometry", since "the presentation of descriptive geometry in the rich context of all the figures of space and the inaugural problem to motivate the theory were given up." (Barbin 2015, p.63). The third change concerned curved surfaces. While Monge started from the more general and went to the more specific, the reverse was in fact adopted. Thus, "the projections of a curve and of its tangent in one plane became a new element of descriptive geometry" (Barbin 2015, p. 63). The fourth change was linked to the search for tools used to solve problems. The *rabattement* method was introduced, and it was considered to be a concept with the properties of an operation, and not just a practice of presenting drawings (Barbin 2015, p. 63). Barbin (2015) shows that Olivier's text from 1843 introduced a new *élémentation*: a change in the projection plane method. Olivier introduced, in chapter II of his *Cours de géométrie descriptive*, the idea of overcoming difficulties in solving problems from the proper choice of a new projection plan. Barbin (2015, p.67) states that, in order to solve a problem,

> it can be necessary to change the vertical plane in relation to a point, or the horizontal plane, to change the planes of projection in relation to a line or to lead a plane in a parallel position to one plane of projection. These four changes constituted "fundamental problems", and 76 of his 80 problems only concern motions of straight lines or planes. Olivier began to use the word "rotation" and its properties from problem 10, which asks to make a plane parallel to the ground line. Thus, the "method of changes" implies a new *élémentation*, with the notion of new "elementary" problems, which are the "fundamental problems", and with transformations of the planes of projection.

Lacroix (1795), upon solving problems regarding straight lines and planes, in fact solved a larger set of problems concerning spheres. And this is all before he got into a discussion about the generation of surfaces in the second part of his book. His concept, as Barbin (2015) states, was based on the construction of a sequence of problems, so that the solution of each one depends on the previous. Lacroix introduced the general concept of a surface in a similar way to Monge[1], and then focused on the study of conical, cylindrical, double-curved, and revolution surfaces in addition to the problems of intersections between surfaces. Before talking about the projection of surfaces, Vallée (1819) introduced a study about the projections of curves by means of their traces, and the projections of tangents to the curves. Vallée gave a general idea regarding Monge's surfaces, but his study was based on particular cases: cylindrical, conical, revolution surfaces and envelopes. Adhémar (1832) followed in the same footsteps as Vallée, giving particular attention to the tangent theorem (the projection of the tangent to a curve is tangent to the projection of the curve). First the author asserted that the preliminary study of the projection of a curve precedes that of the surface and then went on to explore cylindrical surfaces, which he considered the most simple and useful.

Before solving various problems, points, lines and planes are represented at the most remarkable positions in Vallée's text. The author showed, for example, how straight lines, horizontal lines,

[1] It is possible to see different ideas among the French authors who succeeded Monge. Monge (1799) started from a general concept of a surface to, later, analysing particular cases.

parallel lines and perpendicular lines to the ground line are represented. In the introductory part of Lacroix's text, there is a study about the several ways of representing points and lines, as well as the theorems related to how these geometric entities are determined. Monge did not explain these geometrical entities in their several particular positions. However, in this regard, the author stated that “it will be by numerous examples and by the use of the rule and the compass that we will acquire the habit of constructions, and that we will accustom ourselves to choose the most simple and most elegant methods in each particular case” (Monge 1799, p. 16).

Descriptive geometry textbooks in the Brazilian context

During this research, evidence was found that indicates the presence of Lacroix’s descriptive geometry text in the Military Academy, despite the fact that the royal letter that created the Academy originally indicated Monge’s book. Mormêllo (2010, p. 121) transcribed fragments of a written evaluation of books used at the Academy in 1836, which claim that Lacroix’s textbook on descriptive geometry is the most difficult of all, and the least appropriate for elementary education. Figure 1 shows a fragment of a manuscript that indicates this book for use at the Military Academy in 1837. Therefore, we assert that Lacroix's Descriptive Geometry was one of the references for the course at the Military Academy and later for the Polytechnic School.

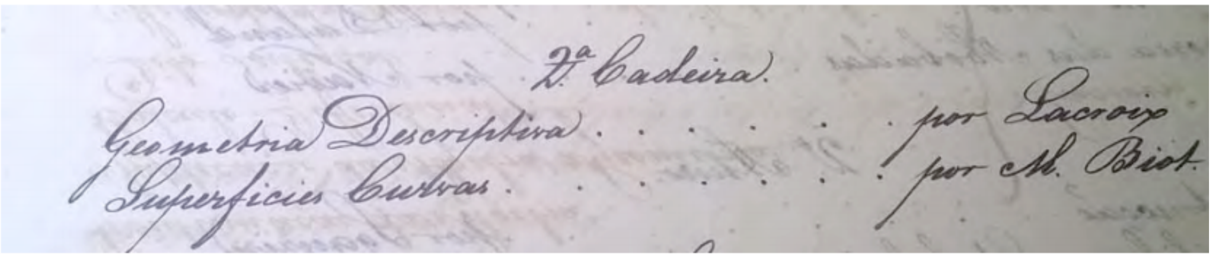

Figure 1: Indicates the use of Lacroix's text in the Descriptive Geometry course at the Military Academy (Source: The National Archive of Rio de Janeiro – Codex IG^{3}5)

The first textbook on descriptive geometry was published in Brazil in 1840 was *Noções de geometria descriptiva para uso da escola de architectos medidores*. It was written by a graduate of the *Academia* Pedro d’Alcântara Niemeyer Bellegarde (1807-1864). It was a 27 page booklet and no copy of it has yet been found. The literature on this subject in Brazil has certain characteristics that must be highlighted. There is a predominance of foreign literature, particularly in the French language, at the institutions that offered the discipline during the 19th century. At the end of this century and at the beginning of the next, it is possible to find texts on descriptive geometry written by Brazilians. The texts were part of the requirements for entering into teaching positions at institutions of higher education.

During the 20th century, there were an even greater number of textbooks written by Brazilian authors on descriptive geometry. We note the predominance of an approach that is similar to the approach described by Barbin (2015). The *élémentations* are included in Brazilian textbooks on descriptive geometry. The objects are studied from a variety of particular positions and the methods of *rabattement*, rotations and changes of planes, are shown as basic skills that should be learned by the students. In addition, surfaces are also studied from several particular cases, according to the classification given by Monge. In the Brazilian texts on descriptive geometry, we find whole chapters dealing with *rabattements* and rotations, which show that these themes are considered to be basic abilities, which should be developed by students in Brazil.

At the end of the 19th century and at the beginning of the 20th century, the book *Éléments de Géométrie Descriptive avec nombreux éxercices par F.I.C.*[2] was one of the texts of reference for the

[2] This book was part of the French collection named *Cours de Mathématiques élémentaires* and included elementary books on arithmetic, algebra, geometry, trigonometry and descriptive geometry. The collection belonged to *the Institut des Fréres des Ecoles chrétiennes*. F.I.C. are the initials of Frere Ignace Chaput.

descriptive geometry course at the Polytechnic School of Rio de Janeiro[3] and other institutions, such as the *Colégio Pedro II*. The books that composed this French collection were translated and adapted to Portuguese by Eugenio de Barros Raja Gabaglia, a former director of the National Gymnasium (the name of the *Colégio Pedro II* at the end of the 19th century) and a professor of the Polytechnic School of Rio de Janeiro (Lorenz & Vechia 2004). The syllabuses of the subjects at the National Gymnasium in 1898 introduced advanced studies in mathematics, included descriptive geometry as a course (Lorenz & Vechia 2004) and indicated the use of F.I.C.'s text.
With the introduction of F.I.C.'s text, we find preliminary concepts and solutions regarding several problems involving points, lines and planes. This study was done by means of a few particular cases, such as the *élémentations* found in Adhémar's text. Later on, several descriptive methods were discussed in F.I.C.'s text, such as the change of projection planes, rotations and *rabattements*. There is also the study of problems involving angles and applications in the representation of plane figures, the representation of polyhedra, the plane sections of polyhedra and the intersection of a line and a polyhedron. In the second part of F.I.C., the surfaces and their classification are studied, as well as the tangential planes, the representation of the cylindrical surface, cones and surfaces of revolution. We also find the representation of tangential planes in relation to cylinders, cones, spheres and a surface of revolution, plane sections of cylinder, cones and surfaces of revolution, intersections between polyhedra, polyhedra and curved surfaces, surfaces generated by straight lines and surfaces of revolution. In the third part, the quoted plans are presented and, in the fourth part, the study of shadows and perspective is introduced.

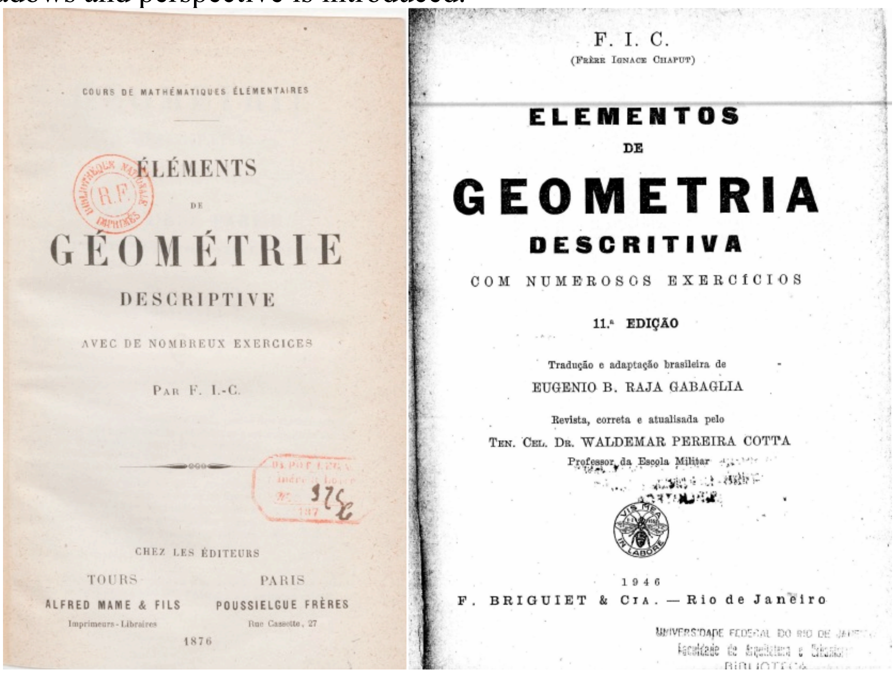

Figure 2: Covers of Éléments de Géométrie Descriptive avec nombreux éxercices par F.I.C. and its Portuguese translation

The Brazilian edition of F.I.C.'s text includes terms that are not found in the original French text. These terms are used nowadays in Brazilian textbooks on descriptive geometry, such as *reta de frente* (straight line parallel to the vertical projection plane), *reta de topo* (straight line perpendicular to the vertical plane), *reta de perfil* (straight line situated in a plane parallel to the projection planes), *plano de topo* (plane perpendicular to the vertical plane), *reta de maior declive de um plano* (straight line whose horizontal projection is perpendicular to the horizontal trace or the plane trace).

[3] Authors also mention that textbooks like C. F. A. Leroy's *Traité de géométrie descriptive* and A. Javary's *Traité de géométrie descriptive* were used at the Polytechnic School of Rio de Janeiro.

This can be seen as an adaptation made for didactic purposes. It aimed to help students understand concepts of the subject by means of an analysis of several particular cases.
After 1920, there were more textbooks on descriptive geometry written by Brazilians. Carlos Süssekind, a professor at the Naval Academy and at the Military School of Rio de Janeiro, wrote the book *Geometria Descritiva*, which was published in 1924. The book was written based on his lectures at the Naval Academy, and the author included the study of perspective, shadows and the design of projections. It addresses problem solving and descriptive methods, without further justifications from a geometric point of view. The next section focuses on a Brazilian professor that published noteworthy work on descriptive geometry.

Alvaro Rodrigues' work on descriptive geometry

Alvaro José Rodrigues[4] (1882 - 1966) was a Brazilian civil engineer who graduated from the Polytechnic School of Rio de Janeiro. He was also a professor at the School of Fine Arts and taught descriptive geometry for 35 years. During 1909 and 1910, he had the opportunity to live in Berlin as a member of a commission created by the government to establish commercial relationships between Brazil and European countries. In his own words[5]

> It was by evaluating the work of the German people that I was able to evaluate the power of technical education and professional improvement, and deem them as major factors for the greatness of that Nation!
>
> Well, it was at these professional schools, the cornerstones of the entire German educational system, that Descriptive Geometry came into my spirit! The role this subject played in teaching drawing abilities in these schools excited me, giving real existence to the dreams, to the cogitations of artists and engineers, inventors and entrepreneurs of works of any kind, for their transformations into drawing projects (transl. by Th. O.).

His work included two volumes about descriptive geometry and a book on parallel perspective. These texts are extremely important in the Brazilian context of publications on descriptive geometry because the author presented not only the constructive methods but also the historical references for its development. In Rodrigues's work, history and theory are presented and articulated in such a way that reveals that the author had profound knowledge on the subject.
The first volume, entitled *Geometria Descritiva - Operações Fundamentais e Poliedros*, was a reproduction of his classes given in the first year of the painting, sculpture and engraving course and in the first year of the training course for drawing teachers at the National School of Fine Arts in Rio de Janeiro. The first edition is from 1941 and presented the subject following a similar structure to books like F.I.C., starting from the representation of points, lines and planes. The volume also presented the study of polyhedra representation, as well as the resolution of the problems of intersection between these solids. Finally, it deals with other methods for representing three-dimensional objects. Rodrigues based himself for this on a broad spectrum of historical authors, from Brook Taylor (1749), B. E. Cousinery (1828), C. L. Bergery (1835) to Wilhelm Fiedler (1871).
Although he had read Monge's original text, Rodrigues was also influenced by authors of different nationalities. In opposition to the spirit of generality present in Monge's lessons, which begins with more complex figures, authors like F.I.C., Vallée, and Adhémar started from the simplest figures

[4] More details on Rodrigues' life and work can be found in Oliveira (2016).
[5] "Foi avaliando o trabalho do povo alemão que avaliei o poder da educação técnica e do aperfeiçoamento profissional, como fatôres principais da grandeza dessa Nação! Pois bem, foi nessas escolas profissionais, pedra angular de todo o sistema educacional alemão, então vigente, que a Geometria Descritiva, surgiu em meu espírito! Empolgou-me daí por diante, pelo papel que nessas escolas desempenhava na educação da faculdade gráfica, dando existência real aos sonhos, às cogitações de artistas e engenheiros, inventores e empreendedores de obras de qualquer natureza, pelas suas transformações em projetos gráficos." (Bello Junior 1966, p.172)

and went to more general cases and complex figures. In Rodrigues' work, it is possible to identify that study should be done by analyzing many particular cases, so that the fundamental elements are presented in various positions in space. *Rabattements* and rotations are descriptive methods that are widely used in the work and are studied through the approach that goes from the particular to the general. However, Rodrigues expands the repertoire of constructive techniques by including methods like central projections, based on Taylor’s work on linear perspective.

The second volume was called *Geometria Descritiva - Curvas e Superfícies*. From the second edition on, the title changed to *Geometria Descritiva – Projetividades, Curvas e Superfícies*, since it contains an introductory chapter dedicated to projective geometry. The author includes concepts from Poncelet’s *Traité de propriétés projective des figures*. The fact that Rodrigues included a chapter on projective geometry in a book dealing with descriptive geometry makes his work very singular and distinguishes him from other authors. Projective geometry has become an autonomous field of mathematics, as it distances itself from the questions connected to projections and sections of figures, from where it originated. In bringing projective geometry to his descriptive geometry text, Alvaro Rodrigues saw these themes intertwined through their history and from the original texts of their founders. Another Brazilian text that adopts such an historical approach to articulate connections between projective geometry and descriptive geometry is unknown. Lietzmann (1924), in his *Methodik des mathematischen Unterrichts*, shows that this connection between projective geometry and descriptive geometry was a characteristic of German secondary education of the early twentieth century. Thus, Alvaro Rodrigues’ experience in Berlin in 1909 and 1910 may have brought him into contact with this concept to include in the teaching of the subject.

Conclusion

The present article presents some characteristics of Brazilian textbooks on Descriptive Geometry. The reception and diffusion of the subject in Brazil started with the translation of Gaspard Monge’s book. Other books in the field influenced the development of the subject in scientific institutions in Brazil. There was a predominance of foreign literature, particularly in the French language at the institutions that offered a descriptive geometry course during the 19th century. A specific kind of text led Brazilian authors to produce literature on the subject, including the previously mentioned texts, which were written as part of the requirements for the assessment process for teaching positions in institutions of higher education. In the 20th century, several Brazilian textbooks appeared and were widely used in institutions of higher education. These texts are structured according to the *élémentations* presented by Barbin (2015).

In the context of Brazilian textbooks, Alvaro Rodrigues’ work is unique because of the historical approach he presents. The evolution of his work in new editions shows that he was dedicated to the study of descriptive geometry and he expanded the content of his work as he increased his cultural repertoire about the subject. The historical notes on Rodrigues’ work do not contain only names and dates, they present fragments taken from original sources and create a text that interweaves content with history.

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A STUDY ABOUT TRANSMISSIONS OF CALCULUS TEXTBOOKS TO BRAZIL

VINICIUS MENDES COUTO PEREIRA

Abstract

The present study related to the research of analysing how the Brazilian mathematical community became established. In this sense, I have studied how the process of transmitting the basic concepts of calculus occurred, analysing which conceptions of analysis, prevailing in one of the metropolis countries were transmitted to Brazil and how the transmitted knowledge was transformed and reworked in order to constitute a proper production. In this intention I have investigated the process of transmission of calculus textbooks to Brazil starting with the "Traité Eléméntaire de Calcul différentiel et de Calcul intégral" (1802), by Sylvestre-François Lacroix already translated into Portuguese as Tratado Elementar de Calculo Differencial e de Cálculo Integral, the first calculus textbook used in Brazil, from 1812, and the version published in 1842 by José Saturnino da Costa Pereira, Elementos de Calculo Differencial e de Calculo Integral, segundo o sistema de Lacroix, being the first work of proper production in Brazil. In the following decades, one continued, however, to transmit foreign textbooks, mainly French ones, so that for a long time the work of Saturnino remained the only calculus textbook being an own production in Brazil.

In this way I have identified the main characteristics of the calculus textbooks transmitted to Brazil, as well as the different generations transmitted, in the sense established by Zerner (1994).

I intend to present a study of various aspects of calculus textbooks transmitted, comprising the period from 1810 – the year of founding the first institution of higher education - to 1934, the year in which the Italian mathematical school arrived at the Universidade de São Paulo, as well as to reveal, which were the relevant proper productions in analysis.

Research Questions

This study is situated in the search for the understanding how the Brazilian mathematical community was constructed. Thus, the process of the emergence of a mathematical community in a country that initially occupied a peripheral position implies a change in the focus from teaching to research – that is from the use of textbooks imported from a metropolis until the moment in which proper productions were achieved.

On the other hand, considering the inexistence of longitudinal studies about this theme and the impossibility of studying all the mathematics produced during an extended period, I have decided to investigate the evolution and establishment of concepts related to Analysis in Brazil as a key theory in the nineteenth century. I will search which conceptions of analysis were transmitted to Brazil and how the transmitted knowledge was transformed and reworked in order to constitute a proper production.

In this way, I have done a specialised research on a well-defined conceptual field in mathematics and a longitudinal study, i.e.: the establishment of a Brazilian mathematical community with its own production. Thus I have a specific question: How the process of transmission of the basic concepts of calculus was implemented in Brazil. Therefore I should to point out the importance of studying the use of textbooks transmitted from a metropolis until the moment in which own productions were achieved.

Finally, I have the main motivation for the establishment of the present study, the investigation of the process of transmission of calculus textbooks to Brazil.

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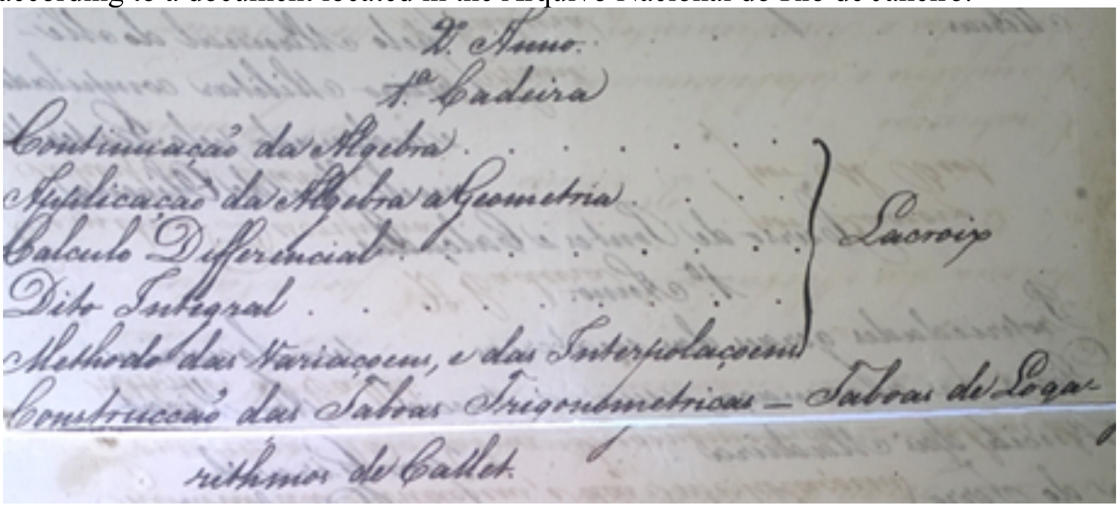
Beginning of the Transmission Process of Calculus Textbooks to Brasil

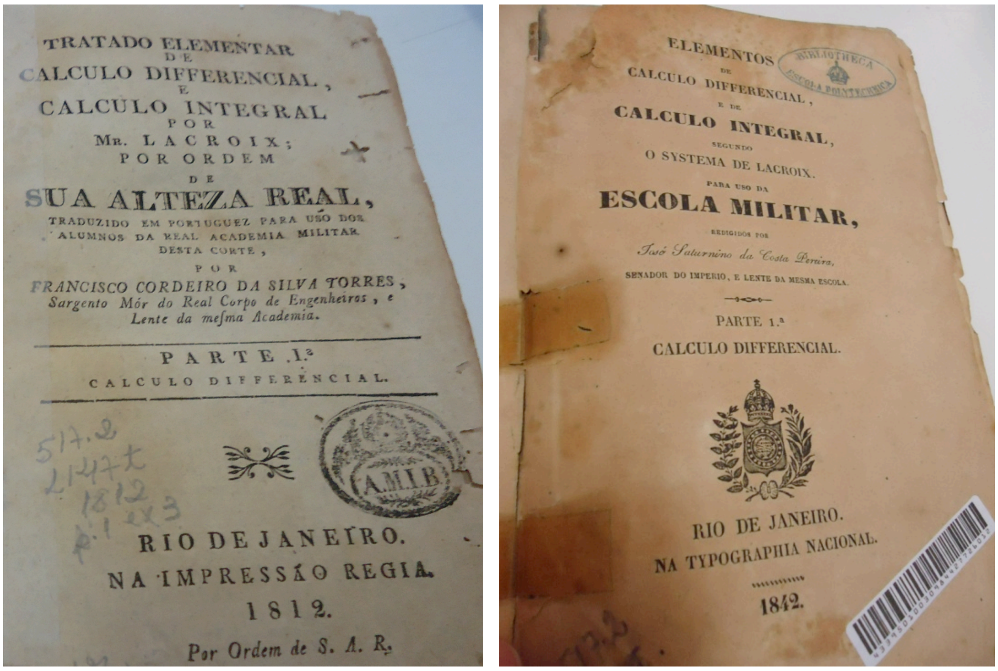
When the Portuguese royal family came to Brazil in 1808, it was necessary to implement minimal infrastructure conditions to house in Rio de Janeiro a contingent of approximately 15,000 people, as well as adjusting it to the new status of capital of the Portuguese Empire. At that time the first institutions of higher education founded by the Portuguese in Colonial Brazil emerged directly linked to the military activities resulting from the Portuguese occupation. Particularly I should point out the creation of the *Academia Real Militar* in 1810.

The textbooks that were to be used for teaching the various disciplines were determined by a royal decree. The textbook *Traité Elementaire du Calcul Differentiel et Integral* de S.F Lacroix was indicated as a compendium for teaching calculus. Although I do not know for sure how long Lacroix's textbook was used at the Academia Real Militar, we know that its use lasted until at least 1837, according to a document located in the Arquivo Nacional do Rio de Janeiro.



2º Anno
1ª Cadeira
Continuação da Algebra
Applicação da Algebra a Geometria
Calculo Differencial
Dito Integral
Methodo das Variaçoens, e das Interpolaçoens
Lacroix
Construcção das Taboas Trigonometricas – Taboas de Logarithmos de Callet.

Figure 1: List of textbooks used in the Academia Real Militar in 1837



TRATADO ELEMENTAR
DE
CALCULO DIFFERENCIAL,
E
CALCULO INTEGRAL
POR
MR. LACROIX;
POR ORDEM
DE
SUA ALTEZA REAL,
TRADUZIDO EM PORTUGUEZ PARA USO DOS ALUMNOS DA REAL ACADEMIA MILITAR DESTA CORTE,
POR
FRANCISCO CORDEIRO DA SILVA TORRES,
Sargento Mór do Real Corpo de Engenheiros, e Lente da mesma Academia.
PARTE 1.ª
CALCULO DIFFERENCIAL.
RIO DE JANEIRO.
NA IMPRESSÃO REGIA.
1812.
Por Ordem de S. A. R.

ELEMENTOS
DE
CALCULO DIFFERENCIAL,
E DE
CALCULO INTEGRAL,
SEGUNDO
O SYSTEMA DE LACROIX.
PARA USO DA
ESCOLA MILITAR,
REDIGIDOS POR
José Saturnino da Costa Pereira,
SENADOR DO IMPERIO, E LENTE DA MESMA ESCOLA.
PARTE 1.ª
CALCULO DIFFERENCIAL.
RIO DE JANEIRO.
NA TYPOGRAPHIA NACIONAL.
1842.

Figure 2: Lacroix and Saturnino's textbooks.

In 1842, the textbook *Traité Elementaire du Calcul Differentiel et Integral* was published by José Saturnino da Costa Pereira, professor at the Academy. Thus I can consider that Saturnino textbook was the first own production of calculus textbooks in Brasil.

On the other hand, besides the textbooks by Lacroix and Saturnino, I know of no other textbooks, which would have been used in the *Academia Real Militar* and its later, transformed institutions. Thus, traditionally the historiography of mathematics in Brazil considers the translation of Lacroix's textbook and the version produced by Saturnino as practically the only two references for the teaching of Calculus for several decades during the nineteenth century.

Since there seems to be only two known Calculus textbooks in most of the nineteenth century there arose the question: How can one study the process of reception and transmission of the concepts of Calculus with only two known textbooks?

Basically, I take two approaches in dealing with this issue. On the one hand, I look for sources that could give us indications of other textbooks used for teaching calculus in Brazil. On the other hand, I have used a methodology to search these works in several libraries and institutions, considering as a basic hypothesis that a listing (and later analysis) of these works with their respective locations, would document part of the process of reception and transmission of the basic concepts of calculus in Brazil.

In this way, I found three textbooks that I consider as "accumulation points", namely Traité Élémentaire de Calcul Différentiel et de Calcul Intégral de S.F. Lacroix, Cours d'Analyse de l'École Polytechnique de Charles Sturm and Élémens de Calcul Différentiel et de Calcul Intégral de Boucharlat. In his study of French treatises, Zerner (1994) seeks to understand how the French treatises relate to the process of establishing rigorous standards in analysis throughout the 19th century, revealing a distinction between three generations.

The first generation is composed of two treatises published in the early nineteenth century, Lacroix (1802) and Boucharlat (1813). It should be noted that these two textbooks experienced a large number of editions, nine each one. The treatises of the first generation do not directly use the infinitely small, with rare exceptions, besides not using continuous functions. Regarding the concept of differentiability, these ones only speak about differential coefficients. According to Zerner, the word "derivative" was proposed by Lagrange, not having reached those works in time.

Sturm's textbook is classified as belonging to the second generation of the treatises characterised by the so-called " principle of substitution of infinitesimals", ruled by the following definition:

> "Two infinitely small quantities *a* and *b* can be substituted for each other and their difference can be neglected either for the limit of a ratio or for a limit of a sum, provided that this difference is infinitely small compared to one of them.[1]" (Zerner 1994, p. 9)

In general, second generation textbooks define the notion of continuous function by restricting the property of the intermediate value.

At this point, I should pint out the reflexions by Schubring (2005) regarding the criticism of the principle of substitution of infinitesimals by the Belgian mathematician Paul Mansion, who has detected inconsistencies in the use of the principle presenting "examples of false application of the principle". Mansion concludes that the condition of the principle is just sufficient, not being a necessary condition. It is thus perceived that the principle remains correct only if all limit processes are conducted simultaneously. Therefore, it is evident that the principle of substitution, widely used in works belonging to the second generation, presents inconsistencies not being valid in general.

Thus, at least until the first half of the nineteenth century one has in Brazil the broad use of the Lacroix textbook and the Saturnino version, with indications of influences of authors like Sturm and Boucharlat.

[1] Deux infiniment petits *a* et *b* peuvent être substitués l'un à autre et l'on peut négliger leur différence soit dans la recherche d'une limite de rapport, soit dans celle d'une limite de somme, pourvu que cette différence soit infiniment petite par rapport à l'un d'eux.

Positivism

Positivism was a philosophical system created by the French philosopher Auguste Comte (1798-1857). The positivists believed that this positive philosophy was the only basis for the development of the specialised sciences and the reorganisation of the society.

The influence of positivism on mathematics at the *Escola Central* and the *Escola Politécnica do Rio de Janeiro* was evidenced not only by various quotations of Auguste Comte in doctoral dissertations presented after 1848, but also by the production of mathematics textbooks.

The textbook *Notas sobre o emprego do infinito* written by Américo Monteiro de Barros, published in 1863, strongly criticised the use of infinity in the teaching of elementary mathematics stating that its use causes serious inconveniences. In this sense, Barros attacked the influence of the French textbooks on teaching mathematics in Brazil, which emphasised the use of infinity. Barros also adopted a polemical position about the use of infinitesimals, stating that for these notions are given a reality that cannot exist, grounding all science on a fragile basis. Thus, Barros' textbook documents a position of strong rejection not only with respect to the use of the concept of infinity, but also of infinitely small quantities.

The textbook *Theoria Elementar das Funcções para servir de introdução ao estudo da álgebra*", published in 1885 by Licinio Athanasio Cardoso was intended to serve as a guide for students about the concept of derivative presenting also a study about functions. In addition to presenting an explicit positivist influence, the work presents a peculiar approach about the concept of derivative based in a notion called *principal state*[2].

The notion of the principal state of an expression is initiated from the indetermination expressed by $\frac{0}{0}$. In this way, the unity is initially defined as the *main value* of $\frac{0}{0}$, i.e., when the symbol results from the cancellation of two identical quantities. In this line of reasoning, unity is seen as the main value among the infinity of values represented by $\frac{0}{0}$.

Then the principal state of the product of any quantity by $\frac{0}{0}$ (from the annulment of two identical quantities) is defined as the quantity itself, since $\frac{0}{0}$ can be replaced by 1. Thus, the main state of $A\times\frac{0}{0}$ is A.

Cardoso argued that the expression given by $y = \frac{a(x-b)}{c(x-b)}$ is constantly equal to $\frac{a}{c}$ for all values $x \neq b$ becoming an indeterminate expression when $x = b$ being its main state $\frac{a}{c}$. In this way the principal state of an expression is defined when the variable is equal to a by means of the notation $\mathcal{P}_a$. It then follows, according to the notation, that:

$$\mathcal{P}_b \frac{a(x-b)}{c(x-b)} = \frac{a}{c}$$

The derivative concept conceived by Cardoso is based entirely on the notion of the principal state of a given expression. Thus considering that in all continuous functions either Δx or Δy tend simultaneously to zero, he defines the derivative of a function as the principal state of $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$ when the increment Δx becomes zero.

$$\mathcal{P}_0 \frac{\Delta y}{\Delta x} = f'(x)$$

I should point out the Cardoso's conception to avoid both the infinitely small approach and the limit method and even Lagrange's algebraising approach. Thus Cardoso adopts a completely alternative approach based on the works of "Mr. Fleury"[3], based on the concept of principal state, alleging that

[2] *Estado principal* in the original.

[3] Mr Fleury was P. Henry Fleury. He wrote the textbook "Le Calcul Infinitésimal fondé sur des principles rationnels et précédée de la Théorie Mathématique de l'infinit (1879)"

this way of considering the derivative makes its notion clearer and its existence more comprehensible. (Cardoso 1885, p. 66)

Calculus textbooks at the Escola Politécnica de São Paulo

The *Escola Politécnica de São Paulo* was the third engineering school founded in Brazil in 1893, continuing a long tradition of mathematics with the function of forming engineers in Brazil. The textbook adopted for teaching calculus at the Polytechnic School of São Paulo was *Premiers éléments de calcul infinitesimal à l'usage dês jeunes gens qui se destinent à la carrière d'ingénieur* (1884) by Hippolyte Sonnet (1800-1879), having had eight editions in France.

In the classic study by Zerner (1994), Sonnet's treatise is classified as belonging to the second generation of French treatises, characterized by the use of the principle of substituting the infinitely small. However, it proved that the textbook in question had characteristics similar to the works of the so-called first generation of treatises, whose authors are Lacroix (1802) and Boucharlat (1813), more specifically I am classifying Sonnet's treatise as a second generation archaism, i.e., a treatise belonging to the second generation but also related to the textbooks of the first generation.

Sonnet makes clear in his preface that he was directly influenced by the reading of the works written by Lacroix, Cournot, Duhamel and Serret. At this point, Zerner stresses the existence of factors in Sonnet's textbooks, which in comparison with other works did not present the same actuality regarding the approach of certain mathematical concepts:

> "I insist about one point: for there to be archaism, it is not enough that the scientific context of the book is outdated, it is also necessary that there are other books much more up to date on the same contents[4]." (Zerner 1994, p.10)

Zerner also emphasizes Sonnet's intention to synthesise in a single work all the calculus knowledge necessary for the future engineer:

> "A brief preface indicates the purpose of this textbook: to bring together in a textbook of modest size (Sonnet uses the word « opuscule ») the parts of the calculus infinitesimal necessary for the future engineer.[5] " (Zerner 1994, p.70-71)

The principle of the substitution of infinitely small quantities is neither enunciated nor effectively used by Sonnet, but the notions concerning infinitely small quantities are made explicit. I should note here that the definition of infinitely small quantity is exactly the one prominent in the treatises of the second generation. "*The textbooks of the second generation include a definition of infinitely small as having zero for limit.*[6]" (Zerner 1994, p.13)

On the other hand, I should remember that, as emphasised by Zerner (1994), continuous functions were only defined, but there were not subsequently applied and no other propositions are established with them. At this point let us remember that, as treatises of the first generation, Lacroix (1802) and Boucharlat (1813) did not introduce continuous functions whereas Sonnet merely enunciates it. "*Collignon and Pauly do not talk about continuous functions, Sonnet and Hagg only define them.*[7]" (Zerner 1994, p.13)

In terms of the notion of limit, it should be pointed out that although the notion is used throughout the work, there is no formal definition of the limit nor are there any proofs using it. Thus, one should remember that Lacroix's textbook also does not present a formal definition for the limit concept. In this way, one remarks another component that brings Sonnet closer to the works of the

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[4] J'insiste sur un point: pour qu'il y ait archaïsme, il ne suffit pas que le contenu scientifique du livre soit dépassé, il faut aussi qu'il existe sur le marché d'autres livres beaucoup plus à jour portant sur le même contenu.

[5] Une brève préface indique le but de l'ouvrage: réunir dans un ouvrage de dimension modeste (Sonnet emploie le mot « opuscule ») les parties du calcul infinitésimal nécessaires au futur ingénieur.

[6] Les ouvrages de la deuxième génération comportent une définition des infiniment petits comme quantités ayant zéro pour limite.

[7] Collignon et Pauly ne parlent pas de fonctions continues, Sonnet et Haag les définissent, sans plus.

first generation. Perhaps it is exactly this point that brings it closer to the works of the first generation, or more specifically, as a second species archaism.

Textbooks at the first universities

The foundation of the Universidade de São Paulo (USP) in 1934 marked not only the establishment of the first university in the country, but also a new moment in the development of mathematics in Brazil. For the first time, students could study mathematics and obtain degrees in mathematics. In this sense, the *Faculdade de Filosofia, Ciências e Letras* da USP became a fertile ground for the transformation of the mathematical environment in Brazil with the mathematicians of the so-called Italian Mission as protagonists.

In this context, the main reference in the mathematical analysis was the textbook "*Lezioni di Analisi*" by Francesco Severi, published in 1933. In addition, Severi's textbook was also a reference for teaching analysis at the *Faculdade Nacional de Filosofia* (FNFi) founded in Rio de Janeiro in 1939, used by the Italian professor Gabriele Mammana.

Severi, in the third chapter of his "*Lezioni di Analisi*", established the concept of number by using Hankel's principle of permanence. In particular, he presented the definition of commensurability between two homogeneous quantities and constructed the concept of irrational number taking into account two classes of rational numbers.

In this way, I am highlighting the influence exerted by the use of *Lezioni di Analisi* in the teaching of analysis in Brazil in the first years of university functioning, influencing the conception of proper textbooks such as the Mathematical Analysis Course (1953) written by José Adelhay, a former student of Luigi Fantappié and professor at the FNFi.

Some Conclusions

In this way, I can note the enormous influence of the French treatises of engineering tradition in the long period of the exclusive function of mathematics in the forming engineers in Brazil. We had a strong impact of the Italian School especially *Lezioni di Analisi* in the first decades from the 1930s from the foundation of the first universities.

Resuming, there was a long period until proper calculus textbooks were composed. The first proper calculus textbooks were produced only in the 1950s, the first textbooks published were strongly influenced by teaching, i.e., they were produced by compilations of lectures notes.

- *Curso de Análise Matemática* written by José Abdelhay, 1953.
- *Curso de Análise Matemática* written by Omar Catunda, 1954.
- *Curso de Cálculo Infinitesimal* written by Altamiro Tibiriça Dias, 1952.

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SECTION USE OF TEXTBOOKS

MULTIPLICATIVE SITUATIONS IN BRAZILIAN MATHEMATICS TEXTBOOK APPROACHES TO DECIMAL NUMBERS

VERÔNICA GITIRANA, PAULA MOREIRA BALTAR BELLEMAIN and ERNANI MARTINS DOS SANTOS

Abstract

Several studies had dedicated many efforts on investigating approaches given to multiplicative structures in diverse contexts, when considering natural numbers. Nonetheless, very few researches addressed multiplicative structure in other themes of school mathematics. This paper discusses some results of a research regarding the types of situations presented in Brazilian Mathematics Textbooks for Final Grades of Elementary Schools when approaching decimal numbers, considering Vergnaud's studies of multiplicative structures. The sections dedicated to decimal numbers of three collections of textbooks of final grades of Brazilian elementary schools were analysed regarding the tasks proposed or solved involving multiplicative structure. The results reveal that the complexity of the tasks is due to the existence of mixed problems, mainly those that articulate additive and multiplicative structures. One to many, Partition, Quotation situations are well distributed among the problems involving simple proportionality, and fourth proportionality is less presented. In this field, we identified situations with flexible classification. It also reveals an barrier due the term used to classify situations called by one to many, when for decimal numbers, the problems require the correspondent to a number less than one.

Introduction

Textbooks, at all school levels, play important roles to teach and to students' understanding. In Brazil, especially at Elementary School, textbooks are also considered to delimitate the experienced curriculum. Differently from the prescribed curriculum (Sacristán 2000), presented in official documents such as the National Curricular Parameters (Brasil 1997), the experienced in classroom is often shaped by textbook guidelines as well as by their teacher's manual.

Therefore, it is important to analyse the situations proposed by the textbooks that are one of the main resource available to teachers and to students. This sort of analyses can lead us to understand the different situations that students solve in school context, trying to understand their nature, complexity and which knowledge their resolutions require. According to Vergnaud (2003), it is through representations and problem situations that a given mathematical concept acquires meaning.

In this context, a research project entitled "A study on the domain of Multiplicative Structures in Elementary Education", carried out in a network between 2013 and 2017 within the "Observatory

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of Education" – OBEDUC/E-mult[1], aimed to investigate and intervene in the practice of Elementary School teachers in Multiplicative Structures, based on the "action-reflection -planning-action[2] training model, with a view to forming a group with collaborative characteristics, in three studies (Santana et al 2016).

In the first stage of this research, among other objectives, a detailed analysis of the collections of textbooks most adopted in Brazilian state schools in 2013 was carried out. The present study is part of this analysis. In this project, other studies (Magina, Santos & Merlini 2014; Gitirana et al 2014; Lautert et al 2015) began to explore a diversity of meanings within multiplicative problems and their approaches to multiplicative structures separate each meaning to study each of them deeply, also in the context of textbooks. For an accomplishment of this research two questions were raised:

1. How can students develop math for a solution?
2. How does the multiplicative framework approach continue in other mathematical topics, such as approaching decimal numbers and the fractions approach, for example?

It has been seen that dealing with natural numbers, the mathematic approaches in the textbook is rich in a diversity of situation. Nonetheless, we question whether this diversity is kept within others field. Thus, we seek to identify the types of situations presented in Brazilian mathematical textbooks for the final grades of Brazilian elementary schools to decimal numbers, considering the classification given by Vergnaud (1982) for multiplicative structures.

The Theoretical Framework: Theory of Conceptual Fields

The Theory of Conceptual Fields proposed by Vergnaud (1982; 1983; 1988; 1996) is characterized as a post-constructive theory, which carries a development perspective. A conceptual field can be understood as a set of situations which domain requires a variety of concepts, procedures, and symbolic representations, firmly connected between them.

Vergnaud (1983) points out that understanding of mathematical concepts requires considering situations, invariants, action schemes and symbolic representations. On the one hand, it is through interactions with the environment that an individual adds knowledge to their concepts. On the other hand, every situation experienced by an individual, simple as it may be, involves several concepts. Given this, it is difficult and meaningless to study concepts separately, since a concept never assumes meaning in a single situation and this situation cannot be analysed through this single concept. In this direction, we assume that a concept needs to be experienced through more situations. In this sense, the diversity and complexity of multiplicative relations can be illustrated by different situations – problems that require the use of multiplication and/or division as solution.

Gitirana et al (2014) brings the work of Vergnaud's theory, discussing his classification a wide variety of mathematical problems that were presented to students in grades 2 to 9 of Elementary School. According to the authors, teachers must understand the nature of the different types of problem, since such an understanding can help them to propose situations that provide a broad conceptual understanding of the mathematical contents.

The problems of the multiplicative field can be classified with different ways. One classification is the relations: quaternary and tertiary. Quaternary relations are characterized by a double relationship between two variables which can be of different nature. About ternary relation, it is characterized by a relation between three elements, of the same nature. This classification is important because of the passage from addictive to multiplicative structure. In the addictive, students deal only with ternary structure; meanwhile in multiplicative one, they essentially deal with quaternary relations.

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[1] Project number 15 727, funded by the Coordenação de Aperfeiçoamento de Pessoal do Nível Superior (CAPES).

[2] Model applied and developed by Magina (2008).

Other type of classification of multiplicative situations, also given by Vergnaud, which were discussed and exemplified by Gitirana et al (2014), adopted in the present study are described according to figure 1 below.

Figure 1 – Multiplicative Situations

Multiplicative Comparison: In this type of situation, two quantities of the same nature are compared in a multiplicative way by a number that can represent a ratio or relation, and it is necessary that we think of terms of a ternary relation.
Simple Proportionality: These situations present a ratio of proportionality between four quantities, two by two of the same nature, and which are related by a rate among the quantities considered as different. Simple proportionality problems are divided into four classes of situations: **one to many correspondence** (the relation between variables is explicit, where one object can relate to several others); **fourth proportional** correspondence (when it is not possible to achieve the one to many relationship, that is the one to many relation is implicit); **partition** (the idea underlying the concept of division refers to the one to one distribution of things) or division by **quotation** (a quantity of things or objects is reorganized into pre-established quotas).
Cartesian Product: These situations involve a combination relationship between elements of two or more distinct sets, where a new quantity is obtained from the product of others.
Multiple Proportionality: These situations are characterized by the approach of problems with quaternary relations containing more than two quantities relater to two in which it is possible to decompose two simple proportions.
Following this classification, in this study, we analyze the nature of the mathematical situations presented in three collections of textbooks from the final grades of elementary school in Brazil, regarding multiplicative structure in the sessions dedicated to the study of decimal numbers.

Methodology

In order to study, the situations while studying decimal numbers, we decided to study textbooks approach to decimal numbers. This topic is one of the first school topic in the expansion of the number fields after approaching natural numbers. We selected three mathematical collections of Brazilian Textbooks adopted by the partner schools of the OBEDUC / E-Mult Project. The selected collections were:

- TB 1: Centurión et al 2011 - Portas Abertas
- TB 2: Dante, 2011 - Coleção Apis
- TB 3: Giovanni Jr and Castrucci 2009 - A Conquista da Matemática

Once selected the three collections of textbooks, the chapters dedicated to the study of decimal numbers were collected. From this, we identified all situations of multiplicative structures and we analysed them considering the following criteria:
1. The identification of multiplicative situations in:
(a) Resolved activities;

(b) Proposed activities.
2. Analysis of each situation identifying:
(a) Whether it has a single structure or was a mixed situation (additive and multiplicative; multiplicative-multiplicative);
(b) What kind of structure of the situation (shared classification of the referential proposed by Vergnaud);
The survey of the situations collected was carried out by three independent judges and an analysis of the multiplicative problems was carried out through a discussion between three judges.

Analysis of the results

After analyzing the textbooks chapters, 87 multiplicative situations were identified. In Table 1, we have the distribution of the situations considering the problems proposed and the problems solved, in relation to the multiplicative field.

TextBook	Proposed Problem	Solved Problem	Total
TB 1	46	01	47
TB 2	18	02	20
TB 3	13	07	20
Total	77	10	87

Table 1 – Number of multiplicative situations per collection considering proposed situations and solved problems

It is observed that, in the three collections, the multiplicative situations are concentrated in proposed problems (88.5%), which are usually taken as prototypical in the teaching of school mathematics, the solved situations serve as examples in general for students. The largest volume of situations is present in TB 1 (54%) in relation to TB 2 and 3 (23% each). This distinguish the approach to decimal numbers of TB1 from the others, which seems to be richer in problem solving.
Regarding the analysis of each specific situation among all the three textbooks, initially the results reveal that all them concentrates the problems into simple structure. In general, 36% of the situations are of Mixed structure. Nonetheless, a great percentage of TB1 and TB2 situations are dedicated to mixed structure problems (Figure 2). Regarding TB3, it dedicates more efforts on simple structure problems.

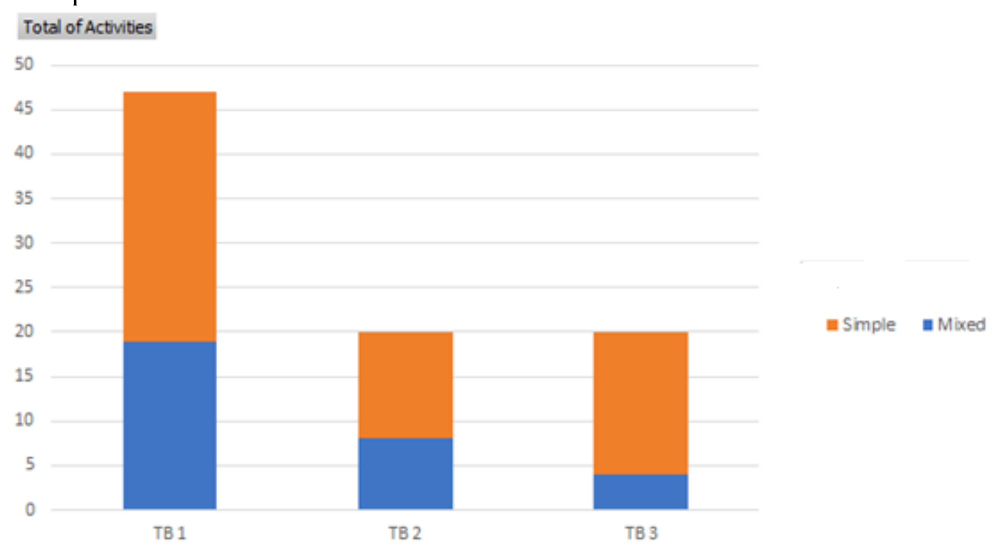

Figure 2 – Distribution of situations of simples and mixed structures

The complexification of the tasks is due to the mixed problems, mainly those that articulate additive and multiplicative structures. The large majority of mixed problems involves an additive situation concatenated with a multiplicative one or vice-versa.

The Figure 3 illustrates the distribution of problem situations involving the multiplicative conceptual field in the three textbooks (situations with simple structure), according to classification presented in Gitirana et al (2014).

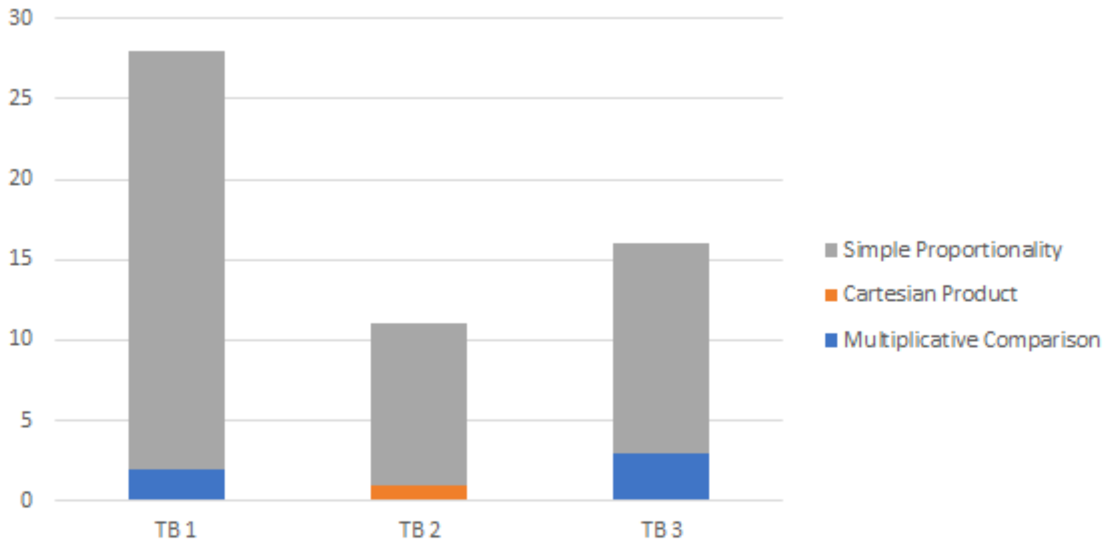

Figure 3 – Types of multiplicative structures among situation

It is observed a concentration of simple proportionality (80%) and a 14,5% of multiplicative comparison. Only one task involves cartesian product (bilinear problems) and no one multiple proportions. This shows a huge concentration of multiplicative problems into one type of structure.

Looking at proportionality situations, one to many, partition, quotation and fourth proportional are well distributed among the problems involving simple proportionality, and fourth proportionality is less presented as shown in Figure 4 below.

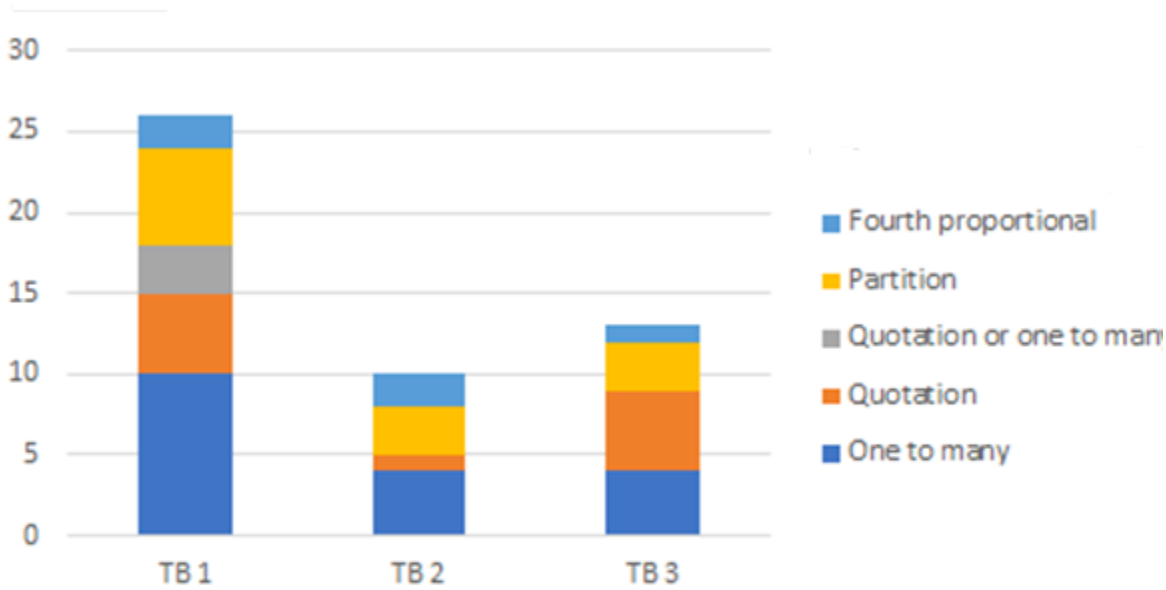

Figure 4 – Simple proportionality situations

All the four types of simple proportionality appear in all TB. Nonetheless, in each TB different structure are focused. Quotation is a structure defended to enable good contexts to make sense of division between two rational numbers less than one. This structure appears in all TB, nonetheless TB2 present smaller dedication to this type of problem. TB3 is the opposite, quotation is the substructure most exploited in it.

In this field, some situations have a missing data, in fact, it requires students to bring it, from his/her previous knowledge. Thus, these problems can be classified as: partition or quotations, depending on the data students choose to add to the problem. Thus, it becomes a problem with flexible classification.

In this field of decimal numbers, it was also revealed a barrier on the use of the term “one to many”, when for decimal numbers, the problems require the correspondent to a number less than one.

Final Remarks

The field of decimal numbers in the textbooks analysed reveals that most situations present a simple structure, although a considerable volume of situations are of mixed structure. In these situations, the increase in complexity is given to problems of mixed structures, especially corporate and multiplicative structures.

Among situations of simple structures, as situations of simple proportion prevail, despite being a good theme to explore problems related to area, for example, allowing a good sense for multiplication of rational numbers.

With regard to the types of problems involving simple proportionality, there is good distribution among different subtypes and there is a good exploration of quotation problems, important to allow the meaning of division of rational numbers.

Considering that textbooks have a fundamental role in Brazilian school curriculum and that the situations proposed in them are important in the formation of mathematical concepts, it is pertinent to promote a greater knowledge about the nature of the problems that are presented to students in the classroom.

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TEACHER FIDELITY DECISIONS AND THE QUALITY OF MATHEMATICS INSTRUCTION

OK-KYEONG KIM

Abstract.
This study examined fidelity decisions (FDs)—teachers' decisions on whether to use, modify, or omit each of the resources provided in the curriculum, or to add a new element to enact lessons—and their impact on lesson enactment within and across tasks and lessons. We particularly examined whether various FDs help teachers steer instruction to meet the mathematical goals of the lessons and whether they promote high cognitive demand. The findings of the study reveal teacher capacities that are needed to make appropriate FDs to transform the written to enacted lessons productively, which include recognizing important mathematical points and addressing them in instruction, and noticing and bridging gaps in the resources provided by the written lessons. Also, it is important for curriculum designers to make the goals and intentions of tasks, activities, and lessons as transparent as possible to teachers. Simply listing goals at the beginning of the lesson does not seem sufficient

Introduction

Teachers make various decisions when they use curriculum to plan and enact a lesson.[1] They decide whether to use the task (lesson or unit) in the curriculum and, if so, how to use it. The curriculum usually includes various kinds of resources regarding how to enact the task (lesson or unit), such as questions to ask, and representations, models, and strategies to use. Teachers decide whether to use, modify, or omit each of these elements provided in the curriculum. I call such decisions *fidelity decisions* (FDs), which indicate various possible adaptations teachers make as they use written lessons to design instruction. One important question to ask is how such FDs impact the quality of enacted lessons, or the quality of the transformation from the written to the enacted.

This study examined the kinds of FDs and their impact on the quality of the enacted lesson, especially those that support or hinder the accomplishment of the goals of the written lesson and those that promote students' engagement at a high or low level of cognitive demand. Mathematical points of the lesson and cognitive demand are two important aspects that can indicate the quality of instructions and opportunities for students to learn in the lesson. Research questions are: What fidelity decisions do teachers make within individual lessons? What impact do such fidelity decisions have on the enacted lesson in terms of mathematical points of the lesson and cognitive demand?

Theoretical Foundation

The term *fidelity* indicates the alignment between the written and the enacted lessons in general (Remillard, 2005). Fidelity of curriculum implementation has been investigated from different perspectives, such as curricular coverage (Tarr, Chávez, Reys, & Reys 2006), textbook integrity (Chval, Chávez, Reys, & Tarr 2009), and fidelity to the authors' intended lesson and fidelity to the literal lesson (Brown, Pitvorec, Ditto & Kelso 2009). Unlike previous studies that focused more on overall implementation of curriculum, Brown et al.'s study examined whether critical elements of

[1] In this paper, the term, *curriculum, or curriculum program*, indicates a set of curriculum resources for a certain grade band (e.g., elementary) developed and published by a group of authors. This includes instructional resources for everyday teaching, such as student texts, teacher's guides for individual lessons, and other supplemental materials, such as workbooks, homework books, and curriculum guides.

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the lesson were implemented, in order to determine the level of fidelity in individual lessons observed.

It is important to analyze fidelity of curriculum implementation in small, meaningful chunks, such as tasks or lessons. I expanded Brown et al.'s approach by investigating FDs teachers make at three levels (within individual tasks, within individual lessons, and across lessons) in order to see their impacts on the enactment of the written curriculum, especially how certain FDs support or hinder accomplishing the goals of the written lessons. In doing so, I examined FDs and their impact on the enacted lessons in both zoom-in and zoom-out ways, and also unpacked the complexity of FDs in terms of meeting the mathematical goals in individual lessons as well as across lessons. Although I see the importance of Brown et al.'s examination of the authors' intended lesson, I focus on the goals presented in the written lesson. That is because teachers in general do not have easy access to the authors directly; rather, they have to rely on the written materials to interpret the goals of the lesson as presented in the curriculum.

Each written lesson has particular mathematical goals and objectives. Usually they are identified at the beginning of each lesson, which helps teachers "articulate the mathematical point" (Sleep 2012) of the lesson. However, not all critical mathematical points are clearly identified or addressed in the lesson. Enacting the lesson toward the mathematical point is a challenging task for teachers. As I analyzed mathematical points of individual lessons and across lessons, I found that mathematical points are multi-faceted and can be described from a dual perspective: conceptual foundation as well as procedural competence. Teachers' FDs can promote or hinder the opportunity for students to engage with conceptual and procedural aspects of mathematical points.

In examining the impact of FDs on the enacted lessons, we also consider the cognitive demand of the enacted task. The level of cognitive demand indicates the kind of opportunity for students to learn (Stein, Grover, & Henningsen 1996; Stein & Smith 1998). Certain FDs increase, maintain, or reduce the cognitive demand of the task, which significantly influences the quality of the enacted lesson.

Teachers' use of curricular resources in individual lessons has been investigated (e.g., Brown & Edelson 2003; Choppin 2009, 2011; Lloyd 2008; Remillard 1999, 2000, 2005). A number of these studies focused on orientations teachers developed in using recommended resources when enacting lessons (e.g., Remillard & Bryans 2004), and identifying types of adaptations teachers make (e.g., Forbes & Davis 2010; Seago 2007; Sherin & Drake 2009). In my view, these different ways of curriculum use are indeed decisions teachers make whether to follow curriculum suggestions or introduce new elements of instructional design. I call all of these teacher decisions FDs, which include use, change, omission, and addition. *Use* occurs when teachers engage with curriculum suggestions almost as recommended; *change* occurs when teachers modify curriculum suggestions that significantly alter the intended meaning; *omission* occurs when teacher does not use critical curriculum suggestions; *addition* occurs when teachers make inputs not specified by the curriculum. I argue that teachers make these decisions because they think it will help them accomplish their goals for students. However, it is not known how such FDs affect teachers' orchestration of instruction to the mathematical point and opportunities for students to learn. Therefore, it is important to investigate the impact of FDs teachers make on the enacted lessons in terms of mathematical goals and cognitive demand of enacted tasks and lessons.

Methods

The data analysed in this study were drawn from a project investigating K-3 teachers' curriculum use in the US: the *Improving Curriculum Use for Better Teaching* (ICUBiT) Project.

Teacher participants and curriculum programs. Data were gathered from 25 teachers in grades 3-5 using five different curriculum programs: (a) *Investigations in Number, Data, and Space* (INV) (TERC 2008); (b) *Everyday Mathematics*(EM) (University of Chicago School Mathematics Project, 2008); (c) *Math Trailblazers* (MTb) (TIMS Project University of Illinois at Chicago, 2008); (d) Scott Forseman–Addison Wesley *Mathematics* (SFAW) (Charles et al. 2008); and (e) *Math in*

Focus (MiF) (Singapore Ministry of Education /Marshall Cavendish International*,* 2008). The first three were programs funded by the National Science Foundation; the fourth was commercially developed; the fifth was originally from Singapore and has gained popularity in the US over recent years. The participant teachers had at least three years of teaching experience and at least two years of using their curriculum program. This study drew on data from five teachers, one teacher per curriculum.

Data sources. The data we used in this study include classroom observations, teacher interviews (introductory and post-observation), and Curriculum Reading Logs (CRLs). Each teacher completed CRLs for each lesson that was observed: on a copy of the written lesson, the teacher indicated which parts they read as they planned instruction, which parts they planned to use, and which parts that influenced their planning. CRLs helped the researchers see teachers' plans for instruction and compare written and enacted lessons. Each teacher was observed for three consecutive lessons in each of two rounds. These enacted lessons were videotaped and transcribed. Also, each teacher was asked questions about his/her teaching experience and overall curriculum use at the beginning of the study, and then asked about specific teacher decisions in the observed lessons after each round of three observations. These interviews were audiotaped and transcribed.

Data analysis. The main part of the data analysis was coding teacher FDs and their impact on the enactment of the lesson. Two researchers chunked Written (W) and Enacted (E) tasks using CRLs and videotaped lessons (transcripts and videos), and created lesson analysis tables. In each pair of W- and E-tasks, we coded teacher FDs on specific resources—whether each of them was used as recommended in the curriculum, changed, or omitted, or whether any new elements were added.

Once FDs were coded, we examined whether each FD positively or negatively influenced the enactment of the task (and lesson), especially whether each FD supported the accomplishment of the mathematical point of the written lesson and promoted cognitive demand (see Table 1).

First, by carefully examining lesson objective(s), key concepts, key ideas, mathematical explanations, and instructional guidance, we identified the mathematical point (MP) of each lesson including both conceptual and procedural aspects. We indicated whether conceptual, procedural, or both aspects of MPs were affected by each FD. When a FD deprived from the MP, we coded it as negative. When a FD promoted a component of the MP but conflicts with another, we took a note of it for further analysis. This enabled us to identify what kind/component of MP was promoted or hindered overall in a given lesson. We also examined FDs in individual and multiple lessons (e.g., merging two tasks, changing the order of tasks, and omitting a lesson) and their impact on the goals of a series of lessons.

FDs' impact on	Code	Description
Lesson goals (mathematical point)	positive	teacher action toward goals
	neutral	teacher action not directly related to lesson goals
	negative	teacher action moving away from the mathematical point/goals of the lesson
Cognitive demand	positive	teacher action increasing the cognitive demand in the written lesson, or maintaining high level as in the written lesson
	neutral	teacher action not related to cognitive demand
	negative	teacher action decreasing cognitive demand in the written lesson, or maintaining low level as in the written lesson

Table 1. Codes for the Impact of FDs on Enacted Lesson

When coding the impact of FDs on cognitive demand, we drew on Stein and her colleagues' elaboration of kinds of tasks and teacher actions in implementing tasks (Stein & Smith, Henningsen

& Silver 2000) and expanded their categories of teacher actions toward low or high cognitive demand. Their intension was to determine the level of cognitive demand when a task was implemented. In our analysis, we thin-sliced teacher decisions moment by moment and coded each action (FD). In doing so, we attempted to do micro-analysis of teacher decisions because individual decisions (FDs) influence the overall task enactment and we hoped to see what kinds of decisions helped to promote high level of cognitive demand.

The code of neutral was given when a particular FD did not impact MP or cognitive demand positively or negatively. These cases were rare but there were instances of neutral impact. Teacher interviews were analysed to see teachers' intentions behind their decisions. After examining individual teachers, we searched for patterns in teacher FDs and their impact on lesson enactment.

Results

In this section, the five teachers' overall and individual patterns of FDs and their impacts on enacted lessons are presented in terms of meeting lesson goals (mathematical points) and promoting cognitive demand. Then, specific examples of FDs and their impacts are described.

Overall patterns of fidelity decisions and their impacts

The five teachers' FDs are summarized in Table 2, along with raw frequency and percentage. They used their curriculum resources in varying degrees; about 20–40% of their FDs were use. In contrast, all of the five teachers often added a new element that was not specified in their resources, ranging from 48.2% to 74.6%. Change and omission were much smaller portions, compared to use and addition in general. EM teacher omitted the most (13%) and MTb teacher had no significant omissions. SFAW teacher had the highest percent of use and lowest percent of addition. MiF teacher had the highest percent of addition; about ¾ of her FDs were addition.

Teacher	Use	Change	Omission	Addition	Total
EM	42 (21.8%)	20 (10.4%)	25 (13.0%)	106 (54.9%)	193 (100%)
INV	48 (19.9%)	16 (6.6%)	19 (7.9%)	158 (65.6%)	241 (100%)
MiF	55 (20.5%)	10 (3.7%)	3 (1.1%)	200 (74.6%)	268 (100%)
MTb	45 (34.6%)	14 (10.8%)	0 (0%)	71 (54.6%)	130 (100%)
SFAW	62 (41.3%)	10 (7.0%)	10 (0.7%)	68 (48.2%)	150 (100%)

Table 2. Summary of five teachers' FDs

Incidences of the teachers' FDs in each category (use, change, omission, and addition) were coded in terms of positive or negative impacts in articulating mathematical points and promoting cognitive demand. Table 3 presents the percent of positive results in promoting mathematical points and cognitive demand in each teacher, respectively. As shown in the second column of Table 3, using the resources provided in the curriculum mostly led to positive results in articulating the mathematical points of the lessons, ranging from 71.2% to 100%. But, the impact of using provided

	Use		Change		Omission		Addition	
	MP	CD	MP	CD	MP	CD	MP	CD
EM	71.2	38.1	40.0	15.0	0	0	65.1	50.9
INV	73.0	63.6	0	0	0	0	24.1	18.4
MiF	98.2	65.5	90.0	90.0	0	0	96.0	85.0
MTb	100.0	97.8	78.6	0	-	-	91.5	90.1
SFAW	79.0	24.2	70.0	10.0	0	0	60.3	17.6

Table 3. FDs and percentage of positive impacts

resources on cognitive demand varied greatly, ranging 24.2% to 97.8%. This is largely due to the types of curriculum they used, whether traditional or standards-based, and the way each teacher used provided resources. In the case of SFAW and EM teachers, for example, mostly the procedural aspect of the MPs was pursued along with low cognitive demand.

The impact of additions on both mathematical points and cognitive demand varied greatly as well. In general, the percent of positive impact on mathematical point is similar to that on cognitive demand in each teacher, except for the teacher using SFAW. It seemed that the kind and quality of additions mattered. For example, in a lesson where students were asked cover a coat with base-ten pieces to find its area, MTb teacher asked students a set of questions regarding how they could do the task in order to guide them with specific things to think about before starting the task. In contrast, in a lesson on mean, SFAW teacher asked students to use blocks to represent individual values and find the mean of them and yet using blocks was limited to procedural aspect of the mathematical point of the lesson along with low cognitive demand. Both teachers added elements not specified in the written lessons, such as questions about symmetry or manipulatives (blocks), to enact the task/lesson, but their impacts turned out to be very different.

Changes led to diverse impacts on mathematical points and cognitive demand. Some changes made to the written lesson significantly improved the quality of instruction. Others reduced the opportunity to articulate mathematical points and promote cognitive demand. In contrast, omissions did not lead to a positive impact on the MP and CD in general. On the contrary, except for SFAW teacher's case, most omissions resulted in negative impacts on MP and CD (88 to 100%). In particular, MiF teacher's omission of models to represent subtracting a fraction from a whole number significantly hindered students' understanding of conceptual foundation and procedural competence and reduced cognitive demand on students' thinking of the task.

The interference between different FDs was also evident in the data. For example, INV teacher's case, the portion of use was much smaller compared to that of addition. While not using the elements suggested in the written lessons, the teacher added new elements that were not productive. In place of use, she also changed or omitted elements from the recourses in a way that significantly affected the enacted lessons. The detailed case of INV teacher below illustrates these interferences among difference FDs that the teacher made.

Table 4 presents percentages of positive and negative impact on both MP and CD in each category of the teacher FDs. The impacts on both MP and CD in three teachers' FDs (INV, MiF, and MTb teachers) showed a high correlation between MP and CD, which means that when the impact on MP was positive, it was the case with CD as well, and when negative on MP, also negative on CD. It was similar in EM teachers' FDs, except for the category of use. In contrast, SFAW teacher's FDs showed a low correlation between MP and CD. This is mostly due to the procedural focus in the lessons.

	Use		Change		Omission		Addition	
	positive	negative	positive	negative	positive	negative	positive	negative
EM	38.1	7.1	25.0	65.0	0	88.0	50.0	27.4
INV	64.6	25.0	0	87.5	0	94.7	17.1	69.0
MiF	83.6	1.5	60.0	10.0	0	100	85.0	3.5
MTb	97.8	0	0	21.4	-	-	90.1	8.5
SFAW	22.6	6.5	10.0	10.0	0	20.0	16.2	19.1

Table 4. FDs and percentage of positive/negative impacts on both MP and CD

Fidelity decisions and their impact: Examples from INV teacher

I now use INV teacher's (Amy) case to illustrate kinds of FDs and their impact on the lesson enactment in more detail. Amy taught third grade using the second edition of *Investigations in Number, Data, and Space* (TERC, 2008). She had taught the curriculum for 6-7 years by the time she was observed. She was confident in using the curriculum and had an established practice of using it. In this section, I provide the overview of the written lessons and enacted lessons to explain FDs. Then, I describe specific FDs in enacting tasks and lessons along with their impacts on the mathematical points and cognitive demand.

The observed lessons were based on a series of six written lessons on an investigation (i.e., a series of lessons on a focused topic) of “Understanding Division” in the unit titled *Equal Groups: Multiplication and Division*. The previous three investigations in the unit are “Things That Come in Groups,” “Skip Counting and 100 Charts,” and “Arrays,” in which students explore equal groups and multiplication. Overall, the written lessons on understanding division emphasize the meaning of multiplication and division and the inverse relationship between the two operations in solving and creating multiplication and division story problems. These mathematical points appear in multiple lessons, which means that they are explored and developed through multiple lessons. It is important to understand why certain mathematical points are repeated across lessons and how the mathematical points are built on previous lessons. Therefore, it is clear that in the six lessons, students are expected to understand division in relation to multiplication.

In fact, there were serious gaps between the lessons observed and the written lessons. Overall, the teacher used activities with significant change in mathematical points of the lessons. Also, she reduced the cognitive demand of the tasks/activities and the lessons. For example, in a lesson segment that requires students to share various solutions for one of the problems they solved and reflect on the attributes of division problems, the teacher focused on solutions to individual problems (one solution per problem), talking about all the six problems students solved as a way of checking students’ solutions to the problems. She did not highlight the mathematical points and meaning across problems. That is, she did not bring up the relationship between multiplication and division, and did not emphasize the meaning of the two operations.

Also, directions she changed and statements she omitted, added, or changed had negative impacts on the enacted lessons overall. In contrast, most of directions and models/strategies she used from the curriculum positively affected the lessons. This implies that in general using the resources in the written lessons helped Amy steer the instruction toward mathematical points and promote a high level of cognitive demand. Overall, change or omission of the guidance in the written lessons and addition of resources outside the curriculum (i.e., the written lessons) influenced the lesson enactment negatively. Below, I describe in detail the kinds of Amy’s FDs and their impacts on her enacted lessons in the cases of omission, addition, change, and use, respectively. Although I explain them individually, the FDs Amy made in these different ways are closely interrelated, affecting the quality of the enacted lessons.

Omission. In Amy’s case, the most fundamental FDs that impacted the enacted lessons negatively were omitting lessons, activities, and curricular guidance. By doing this, she steered the lessons away from their mathematical points and guided students to just solve problems. Through problem solving and discussing their solutions, students should develop the desired understanding of the mathematical points identified by the curriculum designers. However, Amy focused on solving each problem and did not highlight important mathematical points, such as understanding and using the inverse relationship between multiplication and division to solve problems.

The critical omissions she made include one lesson on arrays, which could have helped students relate multiplication and division using the product and their factors. She also omitted a task that provided an introduction to writing multiplication and division story problems. The task especially brings up two related expressions (6×3 and $18 \div 3$) and asks students in pairs to come up with a story problem for each. The focus is on the difference between the two operations and the task gives an opportunity to assess student thinking before assigning the task of creating multiplication and division story problems. The guidance explicitly indicates that teachers need to:

> Listen for student understanding of the difference between multiplication and division. For example, do the problems students make for the expression $18 \div 3$ begin with the quantity 18

> and divide it into 3 equal groups or groups of 3? Do the problems for 6 × 3 involve 6 groups of 3 or 3 groups 6? (TERC 2008, p. 126)

Instead, she spent time on generating key words for multiplication and division. She made comments as students offered some expressions as key words, whether each suggestion would be acceptable in each operation. In doing so, she lost an opportunity to highlight characteristics of multiplication and division in relation to each other. The loss of meaning continued as she enacted the next lesson (creating multiplication and division story problems). See the guidance in the curriculum below on how to intervene when students have difficulty generating their own multiplication and division story problems.

> Help students talk through the elements of a multiplication situation (two known factors and an unknown product) and a division situation (product and one known factor). Write multiplication and division equations with small numbers and ask students to model the action of each with cubes. (TERC 2008, p. 127)

This guidance is followed by the specific script shown below, to use during intervention.

> Look at this equation, 3 × 4=__ (or 12 ÷ 4=__). Can you show me with cubes what this problem would look like? Can you think of a situation to write about in which you might have 3 groups of 4 things (or 12 things divided into groups of 4 or 4 groups)? How can the class poster of "Things That Come in Groups" help you? (TERC 2008, p. 128)

In the guidance above, it is clear that the two operations deal with equal groups and the product and that the two operations have an inverse relationship between them. However, in her intervention, omitting the entire guidance, Amy did not highlight the critical aspect of the operations. Rather, she focused on using key words to determine which operation a given problem required or to create multiplication and division story problems.

Other critical omissions are mathematical statements, directions (teacher questions), and models. In explaining division, the curriculum highlights that "The answer is the number of groups or the number of items in each group" (TERC 2008, p. 119). Also, relating multiplication and division, a chart is used with specific terms such as *number of groups, number in each group, product,* and *equation*. According to the guidance, the teacher is supposed to help students to "recognize that the unknown information for this problem is the product (the number of yogurt cups in all)" (TERC 2008, p. 124). In fact, the teacher rarely used such expressions in explaining multiplication and division, and, as a result, many students were not clear about what makes an operation multiplication or division. For example, when creating a story problem for multiplication, several students did not understand that they had to use equal groups. A story problem like, "I have 10 apples and my friend has 5 apples. How many do we have in all?" indicates that students did not know how multiplication problems are different from addition problems. Relying on key words, to the students "how many in all" could be sufficient to make a multiplication story problem.

While focusing on and creating key words for solving problems, the meanings of the two operations were only implicitly shared. When much confusion was apparent among students while generating multiplication and division story problems, Amy intervened with many struggling students often focusing on key words. She did not explicitly mention that multiplication and division deal with equal groups, using expressions such as *equal groups, number in each group,* or *product*. At best, she said:

> If you have 10 and he has 5 and we want to know how many in all, we're just putting them together. So that's just adding. But if you have a pack of 10 and a pack of 10 and he has a pack of 10 and a pack of 10 then you've got 10, 10, 10, 10. Which is multiplying. Does that make sense, sweetheart? So, multiplication means I have something that has a certain number of something's in it. Like, I have three packs of gum, each pack has 5 pieces. (Second observed lesson)

She also never brought up how a known multiplication combination could be used to solve a division problem. This helps students see and use the inverse relationship between the two operations in solving and generating multiplication and division story problems. Throughout the lessons, she failed to highlight this important mathematical idea. She also failed to recognize this idea even when students brought it up. Solving a multiplication problem of 5 × 7 that used the same combination of numbers in a previous division problem, a student responded that 5 × 7 = 35 by using the related division problem they solved. The teacher began the problem without using this relationship, as if this was a totally different problem. She repeatedly asked, "How do you know that?" Then, they counted by 5s again exactly the same way they did in the previous problem for 35 ÷ 5 = 7. Also, she did not ask critical questions, such as "Describe this problem. What information do you know about this problem? What do you need to find out?" Instead, she asked, "What is our key word? Is this multiplication or division?"

The teacher did not push for multiple strategies at least during the whole group discussion. This is a serious neglect of the curriculum's pedagogical approach. She did not provide students with an opportunity to share multiple strategies and compare them. Her students were not offered a critical strategy of using a multiplication combination to solve a division problem. She talked about all the problems students were asked to solved, one solution per problem based on students' response, rather than focusing on one or two with multiple strategies as suggested in the curriculum.

Addition. Some guidance, questions, and statements Amy added were effective. They promoted students' understanding and required high cognitive demand. For example, every time when solving a story problem, Amy asked students to visualize the problem situation by closing their eyes and imagining what is happening in the problem context.

> To share equally because, here again, get that picture in your head. Kind of close for a second, imagine you and your two best friends standing on either side of you. Mom gives you a deck of 18 playing cards and you're gonna pass them out. … That's exactly what's gonna happen, right? I'm seeing me and my two best friends and Mom's standing in front of me and she's going, "Ok, here's one for you, one for you, one for you." We're gonna share them equally. So we we're taking those cards and dividing them up among our friends. Do you agree? … Do you see it now? (first lesson observed)

The written lessons include specific guidance, such as "encourage students to act out the action of each problem, using cubes or drawings" (TERC 2008, p. 122), to help students understand what the problem is asking them to do. However, imagining the problem situation to figure out what they need to do to solve the problem was her own addition based on her colleague's suggestion at the school district meeting. This visualization helped students see what is happening in the problem, encouraging students to think about the meaning embedded in the problem and relating that to an operation.

She also asked some critical questions that were not included in the written lessons. These questions prompted students to relate the solution process with the problem. For example, students counted by 5s and reached the target number, 35, as a way to solve 35 ÷ 5. Instead of determining the answer right away, the teacher asked students what the answer was to the given problem and how they knew that was the answer. She provided students an opportunity to step back from counting by 5s and relate that to the given problem. She also added a statement, "1, 2, 3, 4, 5, 6, 7. It's how many times you counted by 5. So, our answer is 7." This highlighted the mathematics embedded in the skip counting, i.e., how many groups of 5 are in 35 tells the answer to 35 ÷ 5.

However, as mentioned previously, her addition of using key words in the lessons significantly minimized the positive impact of her added guidance, questions, and statements. Emphasizing key words throughout the lessons, Amy replaced activities and directions with those around key words. For example, she omitted an activity of generating and discussing story problems for 6 × 3 and 18 ÷ 3 and added an activity of generating words and expressions that cue multiplication or division. She asked students to underline key words and determine which operation to use, and find "the numbers" in the problem to execute the operation determined. She also made problematic statements usually around key words, such as, "If it says 'in each,' it's gonna be a division problem." She made students' problem solving mechanical in this way—find key words, determine the operation to use, find the numbers to use, solve the problem, and write the number sentence, reducing cognitive demand greatly, and misguiding students' thinking about multiplication and division.

As a result, after spending two days generating multiplication and division story problems, still more than half of her students were not able to complete the task. On the third day of classroom observation, there was a range of student-generated story problems. Some students had stories but no questions; some students did not have multiplication or division contexts (addition or subtraction instead); some students had numbers that do not work well (34 things divided equally into 3 or 4 groups); students had only one type of story problem (both multiplication or both division).

Change. Her significant change of given resources was mostly around mathematical goals of the lessons and mathematical statements provided in the written lessons. She used the problems and tasks provided in the written lessons, but the way she used them altered MFPs significantly. She also changed goals of discussion and moved away from the MFP that should be highlighted through discussion. She stated that at the beginning: "The reason being the primary objective of us correcting these papers is so that you can talk about what the key words are when you're creating a multiplication or division problem." In fact, the written lesson directs teachers to have students share their solutions to two particular related problems (one multiplication, $4 \times 5 = 20$, and one division, $20 \div 4 = 5$) and highlight what each operation means and how they are related to each other. Instead, Amy went through all the problems students solved, one by one, to identify key words and determine which operation to use.

Also, she omitted most of the important mathematical statements or changed them to promote a different meaning. In particular, Amy significantly altered mathematical statements provided in the written lessons when she highlighted the mathematics students need to learn or use in the lessons. For example, she mentioned several times, "Division sentence always starts with the biggest number." The statements included in the written lessons are: "Each division problem gives a total that must be divided into equal groups. The answer is the number of groups or the number of items in each group" (TERC 2008, p. 119). Certainly Amy altered the meaning of division that students need to learn and did not highlight the important attribute of division—dealing with equal groups.

Overall, her changed directions to guide students to engage in the mathematics of the lessons and altered mathematical statements greatly minimized students' learning opportunity in the lessons of multiplication and division in the way they were designed.

Use. Amy used problems, tasks, and activities provided in the written lessons. She assigned them to students and discussed the meaning of each problem by helping them visualize the problem situation and solutions to the problems. She also had MFPs in her mind as she read them while preparing for the lessons. She also used models, representations, and strategies that were included in the written lessons, such as drawing pictures of equal groups and counting by a certain number to determine the product or the number of groups. However, her use of resources (lessons, tasks, directions, mathematical statements, models) was mainly based on decisions on what to do, not necessarily about how to do. For example, she did not follow the guidance regarding how to use the problems to highlight the meaning of multiplication and division and the inverse relationship between the two operations. Most of directions and guidance were ignored when they addressed mathematical points embedded in the problems and how to help students understand the big ideas and complete their tasks using the big ideas. Although Amy read the guidance, she did not clearly relate directions with identified MFPs. Likewise, she rarely used mathematical statements that highlight the relationships and meaning of multiplication and division. Instead, she omitted or altered those important statements and often added inaccurate statements that are not included in the written lessons (e.g., "division starts with the biggest number").

To summarize, basically Amy altered the written lessons on multiplication and division significantly and did not meet the many of MFPs sufficiently. Although at times she made appropriate adaptions that supported mathematical goals and high cognitive demand, her ignorance or alterations of critical resources, such as directions and mathematical statements, as well as inappropriate additions caused her to fail to create opportunities for students to learn the meaning of and relationship between multiplication and division.

Summary of Findings

The FDs explored in this study varied across the five teachers. One thing common across the teachers was that they had a tendency of adding new elements to enact the lessons from the program they used. In the case of change or addition, its impact on mathematical points and cognitive demand varied greatly. Some teachers tended to make a decision that supported students' learning of mathematical points of the lessons and their engagement with high cognitive demand; others did not. It is assumed that this was in part because of the curriculum program they used. Interestingly, omission was mostly unproductive across the enacted lessons observed. The INV teacher's case illustrates specific impacts of her use/change/omission/addition on students' learning of multiplication and division. These findings facilitate the discussion below.

Discussion

Teachers can make various FDs depending on their classroom situation. However, such decisions need to be made in accordance with the mathematical goals of the lesson. The case of Amy highlights that making appropriate FDs (use, omission, change, and addition) greatly depends on teacher capacity of recognizing important mathematical points and addressing them in instruction. This result is aligned with what Sleep (2012) refers to "mathematical purposing." According to her, mathematical purposing involves articulating the mathematical point and designing instruction to the mathematical point. Essentially, making right FDs is based on understanding the mathematical goals of the lesson and determining which guidance to use and how to use it in order to teach the lesson toward the goals.

The results of the study also reveal that making appropriate additions to enact lessons requires teacher capacity of noticing and bridging gaps in the guidance provided by the written lessons. Amy, as described in this study, made numerous additions to the written lessons while enacting them. Some of them positively influenced the enacted lessons; others hindered meeting the lesson goals and reduced the cognitive demand of the task. For example, she asked students what their answer would be once they skip counted by 5s as a way to solve 35 ÷ 5. The written lessons include skip counting as a strategy and provide examples of using this strategy. However, there is no additional guidance regarding how to talk about this strategy with students. Amy specifically added an important question that made students reflect on the skip counting and how that leads to the solution to the given problem. In contrast, she thought that it was important to add key words that were not specified in the written lessons. She determined to add this element to the lessons to help students know which operation to use. This decision indicates a lack of her understanding of the MFPs, which influenced her to misidentify the gap.

In fact, capacities needed to make good FDs that we described above elaborates Brown's notion of Pedagogical Design Capacity (PDC), which he defines as "a teacher's capacity to perceive and mobilize existing resources in order to craft instructional episodes" (Brown 2009, p. 29). He further describes, "PDC describes the manner and degree to which teachers create deliberate, productive designs that help accomplish their instructional goals" (ibid.). Therefore, examining kinds of FDs and their impacts on instruction and providing appropriate teacher preparation and education will support teachers to develop PDC that is needed for productive curriculum use.

Curriculum designers need to make the mathematical point of the lesson clear in terms of when it is introduced and how it is developed through a series of lessons. Although MFP 2 is greatly emphasized in the written lessons that Amy used to teach multiplication and division, it is not clear how MFP 2 is met in Lesson 4.1. In this lesson, a strategy is included of using a known multiplication combination (4 × 5 = 20) to solve a division problem (28 ÷ 4), that is, creating 5 groups of 4 and then adding some more groups of 4 to find the answer. Other than including this strategy, this lesson does not clearly indicate why MFP 2 becomes an important mathematical goal to accomplish and how this goal can be met. Subsequent lessons do not provide clear explanations either, although instructional activities in the subsequent lessons more explicitly target this MFP. Without sufficient knowledge, teachers may miss this important mathematical point in instruction, as Amy did, and teachers will greatly benefit from explicit explanations about mathematical points and how lesson activities accomplish them in the written lessons.

The findings of the study suggest that professional development on curriculum use is needed even for teachers who have used the given curriculum for a long time. Amy recognized the usefulness of visualization when she heard that at the district's meeting and used it in her teaching. She was open to suggestions and tried to learn new things, but she still had past habits of and beliefs on accustomed teaching moves and decisions. She thought that key words really helped students understand what multiplication and division are, although she noticed her students struggled a lot. Teachers like Amy need revisited professional development highlighting the essence of the teaching approaches and rationale behind each activity and connections across activities and lessons. For the first couple of years using a new curriculum, teachers read carefully; later they tend to rely on their past experience and colleagues, rather than using curriculum carefully. Missing important elements in the lessons, Amy confessed that she just skimmed through the lessons, as she had already taught them for 6-7 years. This justifies that professional development on using a curriculum is necessary for experienced users as well.

This study explored what kinds of FDs teachers make and how they impact lesson enactment within and across tasks and lessons. We particularly examined whether various FDs help teachers articulate the mathematical points of the lessons and whether they promote high cognitive demand. The findings of the study reveal particular teacher capacities that are needed to make appropriate FDs to transform the written to enacted lessons productively. Also, it is important for curriculum designers to make the goals and intentions of tasks, activities, and lessons as transparent as possible to teachers. Simply listing goals at the beginning of the lesson does not seem sufficient.

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A COURSE ON MATHEMATICS TEXTBOOK ANALYSIS IN THE TEACHER TRAINING CURRICULUM: THE EXPERIENCE OF UNICAMP

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Abstract

We present the experience of the undergraduate course on Mathematics textbook analysis in the teacher training programme at Unicamp (Brazil) from 2014 to 2016, based on qualitative analysis. Our findings corroborate a number of novel practices both in the lecturing process and in theoretical methodology. Our contribution spans over both intrinsic and comparative analyses of textbooks, the acquisition of good practices from textbooks of various countries, the production of textbook material to a high standard of content and layout and the possibilities of reflexive feedback from students into the course structure itself.

Introduction

The undergraduate teachers training programme (*Licenciatura*) at the University of Campinas (Unicamp) has recently incorporated the one-semester (60h) course on *Analysis of Mathematics textbooks and teaching materials* as a mandatory curriculum requirement. The course had existed before as an optional subject and it has attracted substantial interest from students since its total restructuring, by the first-named author, in 2014. Its objective is to prepare the students, as future teacher, to systematically evaluate textbooks from the perspectives of individual quality, relative quality and consonance with international standards. To that end, a number of novel practices have been introduced, and the aim of this paper is to explain its innovative methodology, based on qualitative analysis.

For our purposes, the term *textbook* designates books as such but it also encompasses similar printed teaching support materials, such as hand-out sets, examples sheets and so on. The critical examination of the textbooks addresses content, structure, language, layout, examples and exercises, both in variety and in quality.

1- The official system of textbook evaluation in Brazil

The use of standardised textbooks has been ubiquitous in Brazilian classrooms for decades. Following the international norm, the textbook is the teacher's most adopted resource (Valverd 2002). The primary role of the textbook is to support the teacher, who is free to use it in their own way, integrating it, for example, with other media such as the computer as a tool, as well as online content, video, concrete materials and further bibliography. In Brazil, the assessment and distribution of textbooks are performed by a federal programme under the Ministry of Education.

1.1- The National Textbook Programme (PNLD)

Mazzi and Amaral (2017) produced a historical account of Mathematics textbooks in Brazil, tracing it back to the first known examples and studying the evolution of relevant public policy. While a

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number of official initiatives at local and national level have been set up over time to ensure minimal quality standards, year 1985 marks the consolidation of the current assessment framework, with the establishment of the Brazilian Federal Government's National Textbook Program (PNLD)[1]. The Programme schedule is organised in three consecutive 3-year cycles, each dedicated to one of the three segments of schooling (Elementary, Middle or High school). Every year, independent authors may submit their collections for the current segment to be certified by the PNLD committee at the Brazilian Ministry of Education (MEC). To do so they must pair up with a publisher, who applies on their behalf for evaluation.

Approved collections are announced by MEC in the *PNLD Textbook Guide,* which is sent to every state school (though not to private schools) in the country and also made publicly available on the Ministry's webpage. Each school adopts independently the collection best suited to their pedagogical project at each level of schooling (Brazil 2012). Finally, collections are purchased from publishers and sent to every public school pupil. It should be noted that the analysis takes place every three years for each cycle. "Thus, each year, MEC acquires and distributes books for all students in a segment, among Elementary school, Middle school or High school" (Brazil 2012, p.2, translated by the authors). Distributed books are expected to be retained and returned for use by other students in subsequent years.

It stands to reason that the teacher has an important role in this process, and yet most teacher training programs in Brazil offer little systematic criteria, if any at all, for that crucial choice. The object of this paper can be seen in this context as a relevant set of skills found severely lacking in the current Brazilian undergraduate teacher training degrees.

1.2- PNLD criteria and procedures

Every PNLD public call establishes a set of criteria for textbook evaluation, with rather explicit sufficient requirements for approval. This includes a common core, applicable to all school subjects, addressing for example universal ethical principles, no reference to religion or merchandising, the presence of some interdisciplinary perspective, actuality of concepts and compliance with a number of laws, regulations and manuals. On the other hand, subject-specific criteria for Mathematics textbooks consist of certain curricular guidelines and a list of cognitive skills to be developed, such as logical thinking, rational argumentation, mathematical modelling and so on. Finally, some few criteria for direct rejection are provided, such as conceptual errors, methodological inconsistencies, and failure to cover all broad subareas of Mathematics, as prescribed by the curricular guidelines (Brazil 2014; Borba & Selva 2013).

For most of its existence, the actual PNLD assessment work has been performed at an accredited University, by teams comprising academic specialists in Mathematical Education (from different Brazilian institutions), as well as teachers of the three levels of schooling and other Education professionals. Submitted collections are analysed in a double-blind review process. The referee reports are then examined by the Programme's steering committee, which issues a pass-fail verdict.

2- Textbook analysis in Brazilian teacher training (*Licenciatura* in Mathematics)

In Brazil, teacher training is an independent subject-specific undergraduate degree called *Licenciatura*, which typically lasts 8 to 10 semesters. For the purposes of this article, that term will refer specifically to the *Licenciatura* course in Mathematics.

2.1- Current literature

Rosa, Ribas & Barazzutti (2012) report on the lecturing of a *Licenciatura* course dedicated in part to the evaluation of high school textbooks in Mathematics. Their points of analysis included the tasks proposed in a given book, the methodological approach of the content and its adequacy to the proposed age group. They follow an analysis script somewhat adapted from the PNLD criteria, in which they examine in broad terms:

[1] Programa Nacional do Livro Didático, in Portuguese.

- The physical characteristics of the textbook, eg. cataloguing data, number of pages etc.
- The diversity in subareas of mathematics, such as numbers and operations, functions, algebraic equations, Euclidean geometry, analytic geometry, statistics and probability etc. Are all fields addressed? Which ones get more and less emphasis? Is this emphasis consistent with the grade for which the book is intended?
- Syllabus and connections between topics. Is content selection appropriate? Are there explicit interconnections between the contents of different chapters?

In addition, a sample of two specific contents is selected by the course students for detailed analysis, according to the previous three axes as well as:

- The quality of content, how it is introduced and developed. Are there links to previous knowledge by the students, and in what form? What types of exercises are offered, which proportion address repetition and memorisation, as opposed to more elaborate problem situations? Do they allow students to test different strategies? Do either contents or exercises have inconsistencies? Is there incentive for student-student or student-teacher interaction in proposed activities? Is there any employment of other teaching resources? Is there any kind of contextualization with social practices and/or other fields of knowledge?

Finally, the authors stress the importance of analysing a textbook before using it in class, since good material can assist in the construction of lesson plans and their time-management, as well as suggest alternative classroom activities. Conversely, they warn that the teaching-learning process may be hindered when the textbook is in disaccord with the objectives set by the teacher, in which case it can drive the teaching proposal astray or limit the exploration breadth around a given concept.

Salla (2012) argues that some undoubtedly relevant aspects such as conceptual errors, biases, methodological inconsistencies, problems in layout and overall lack of standards are already evaluated from the outset by the PNLD process. She suggests that teachers should add a further layer of analysis in terms of:

- how the textbook relates to knowledge that students bring from outside school;
- whether possibilities of discussing alternative resolution strategies are presented;
- if the contents are motivated from a contextual problem, as opposed to an emphasis on algorithmic procedures;
- whether activities encourage experimentation, valuing pupils' individual strategies and enabling logical and autonomous thinking;
- finally, whether there is a balanced presence of the various subareas of Mathematics, while observing that the PNLD Textbook Guide entry for each collection already contains a graph representing their respective proportions.

Lima et al. (2001) is arguably the first systematic effort by a team comprised essentially of mathematicians to engage with school textbook quality. In some sense, it is the only Brazilian reference to offer a category-based grid with definite *a priori* standards stated in objective terms. It structures the intrinsic analysis of a textbook around six main axes. The first one, Concepts, is by far the most present in their actual analyses and it unfolds into painstaking detail, as per the following scheme:

<table>
<tr><td rowspan="5">* Concepts</td><td rowspan="4">Mistakes</td><td>Typos and misprints</td></tr>
<tr><td>Faulty reasoning</td></tr>
<tr><td>Incorrect or incomplete definition</td></tr>
<tr><td>Poorly stated or vague concepts</td></tr>
<tr><td colspan="2">Excessive formalism</td></tr>
</table>

	Inadequate language
	Imprecision or omission
	Obscurity (ambiguity or self-contradiction)
	Confusion of concepts
	Objectivity (balance between topics)
	Connections between topics
* Manipulation	
* Application	
* Pedagogical qualities	
* Adequacy to contemporary social reality	
* Constructive role and fairness of the assessment process itself	

Table 1: Six axes of intrinsic analysis of a textbook. Source: Lima et al. (2001)

It should be noted that Lima et al. (2001) then set out and colossally apply this grid to *the entire contents of each of the (twelve) textbooks* from the preceding year's PNLD Guide. Their conclusions are rather damning for every single one of them, and we believe that this had a lasting impact in the textbook authors' establishment, and indirectly contributed to raise the national standards of the PNLD itself. To the best of knowledge, this experiment has not been henceforth replicated.

2.2- The Unicamp context

Since its relatively recent foundation in 1966, the University of Campinas (Unicamp) has risen to the position of Latin America's top University[2]. It is public, tuition-free and maintained by São Paulo State; its budget is protected by law at a fixed proportion of the State's yearly tax revenue, currently around R$ 3 billion or approximately U$ 1 billion. The community encompasses about 2k faculty and 35k students, of which 19k undergraduates and 16k graduate students. Unicamp is strongly research-oriented, offering 153 postgraduate degrees, relative to 66 at undergraduate level. It responds for about 15% of Brazil's total scientific output, and its patent production is second only to oil giant Petrobrás, among both the public and private sectors.

In Unicamp, the Mathematics *Licenciatura* programme is based at the Institute of Mathematics, Statistics and Scientific Computing (IMECC)[3], where, for good or worse, it is taught almost exclusively by research mathematicians, and the remaining credits are offered at the School of Education. The degree lasts 8-9 semesters (3120 hours) and it consists mostly of evening classes, seeing as many of the students work on day jobs – quite a few of them actually already teach in some informal capacity. Admissions are subject to the University's dedicated *Vestibular* exam, which covers the whole high school curriculum, weighted by some minor emphasis on the Mathematics score. The junior class typically consists of 70 places, of which some 25-30 eventually graduate every year.

Following changes in Federal and State level pedagogical directives, the degree curriculum underwent a recent reformulation (2013) to incorporate more independent written production and elements of practical experience and cultural diversity. In this context, the formerly optional course on *Analysis of Mathematics textbooks and teaching materials* was also reformed and became mandatory. It is currently offered every other semester on Friday evenings (19:00-23:00) and it accommodates up to 25 participants.

3- Methodology

This article addresses some qualitative conclusions based on three iterations of the course on *Analysis of Mathematics textbooks and teaching materials*, lectured to Unicamp's *Licenciatura* classes by the first-named author. It should be mentioned from the outset that, not unlike most other

[2] cf. Times Higher Education assessment, 2017.
[3] Instituto de Matemática, Estatística e Computação Científica, in Portuguese.

faculty working with Mathematics *Licenciatura* classes, the lecturer is a research mathematician with virtually no authoritative training in Education. The course evolved therefore as a two-way dialectical learning process, in which students' pedagogical background often took prominence and helped shape the course dynamics, as well as its materials. Indeed, the focus of our investigation lies in the various methodological resources developed during these experiences and it is largely derived from students' own perceptions and independent intellectual production.

In particular, on the issue of comparative analysis, there are very few references describing the process of choice by Mathematics teachers (e.g. Rosa, Ribas & Barazzutti 2012) and one of the aims of this paper is to contribute to this relevant research subject.

Our source of data consists essentially of coursework by students, produced over the years 2014 to 2016 supplemented by oral accounts from students themselves. Since the course itself has been recently created, its approach to pedagogical and methodological questions follows a contemporary perspective, quite often derived from current theoretical trends which help shape the students' pre-existing views of Mathematical Education.

4- Outline of the *Maths textbook analysis* course at Unicamp

4.1- The five tasks, abilities and skills

The course is organised in fifteen weekly workshops of 4h each, divided in five four-week modules (up to overlapping, see below) in which teams of 4-5 students must complete a concrete task, assessed by a written report and an oral presentation of findings discussed with the whole class. The tasks address the following abilities:

Task 1: Vertical analysis - to criticise and improve on a textbook.

Task 2: Horizontal analysis - to compare two textbooks on the same content.

Task 3: Foreign textbook analysis - to extract useful resources from foreign materials.

Task 4: Production of teaching material - to write original content to a high standard of text and layout.

Task 5: Original contribution - to take initiative and suggest an independent contribution.

As to secondary skill developments, Tasks 1 to 3 require the students to formulate and apply methodological protocols and assessment grids and Task 4 requires advanced layout management. Having in mind a gradual build-up of template creation skills towards Task 4, all coursework from the outset must be submitted in LaTeX. Students are also encouraged to coordinate asynchronous teamwork combining chat apps, cloud folders and shared editing alternatives.

Each task module iterates a 4-week cycle, organised as follows:

Week 1: Description and motivation.

Week 2: Formatting and discussion of assessment.

Week 3: Critical pre-submission review.

Week 4: Submission of report and oral presentation.

Notice that subsequent modules overlap: the Week 4 activity of Task n and the Week 1 activity of Task n+1 use up respectively the first and second halves of the same 4h session.

4.2- Teamwork dynamics

In the course experience at hand, students are *not* allowed to form the groups themselves, and are rather reshuffled after each task in order to maximise, in order of priority, (i) the total number of colleagues with which each one interacts during the whole term, and (ii) the heterogeneity of each group, in terms of skills and motivation, as assessed from results of previous tasks and participation in class. This is particularly justified in a workshop-based course which involves substantial extra-classroom commitment, since one most certainly wants to avoid the *good students* clustering up while everyone else is left dependent on direct intervention by the lecturer. Conversely, students confidentially evaluate each team member after a task is handed in, by attributing one of the following three grades:

Grade 1: team member did their share of the work as agreed / default grade, if left blank.
Grade 1/2: team member underperformed the agreed workload, to the point that other team members had to do some of their work in their stead.
Grade 0: team member did not contribute to the task, to the point that all their agreed workload had to be fulfilled by others.
Individual grades are then computed according to the *mode* of the grades assigned by their team members, and these results in a *multiplier* of the original task grade for the group attributed after marking, so e.g. if a group task is given the mark 8.0 (over 10), but a majority of team members decide colleague X contributed only 1/2, then their effective grade for the task will be 4.0. The adoption of mode as a criterion, compared for instance with average, dilutes the effect of personal idiosyncrasies and occasional rivalries. Finally, other fractions could of course be allowed, but this might encourage students to overthink a reaction, which after all should be a last resort. In practice, reductions in effective grade have occurred very seldom, about once or twice in a semester, but students have reported that the *possibility* of evaluating their peers in fairness to their output is recomforting.

5- Detailed description and findings from Tasks 1 to 5

5.1- Task 1: Vertical analysis

Groups examine a connected segment of about 40 pages from a given PNLD textbook, typically comprising two related sections. The analysis focuses on the intrinsic characteristics of the material, especially mathematical correctness, language, pedagogical approach, examples and nature of tasks proposed.
There is very little technical bibliography on school textbook analysis by Brazilian authors. For this first task, students follow essentially the approach proposed by Lima et al. (2001), a former research mathematician and prolific author of *higher education* maths textbooks. The manipulation of the textbooks and the reading of chosen extracts from this theoretical reference support a classroom discussion about the main conceptual issues involved in the task. The key points raised typically include:

- what ought to be the main axes of analysis: eg. mathematical rigour, language, layout, exercises etc.
- which methodological criteria to adopt, i.e., what is a "good" or "bad" instance of each of the previously outlined axes; these are usually based on students' own past experiences both as pupils and teachers.
- the issue of assessing exercise sections is more involved, and some specific discussion develops about what is an appropriate balance between direct manipulation and applications, with a recurrent emphasis on how to detect false contextualisation (problems whose narrative pretends to involve modelling whilst in practice disguising mere, and often implausible, manipulation).
- the inevitability of reductionism: the dangers and responsibilities which any such analysis entails.

Ensuing the discussion, groups gather and must formulate independently their own methodology, the outcome of which tends to be a mild variation on Lima et al.'s own assessment scheme, as seen above. Each team member will then uniformly apply those methodological axes and criteria during homework, leading typically to the mapping and sorting of "faults" in the studied excerpt and some mild statistical exploration of the findings towards justifiable qualitative conclusions. Usually, this comes in the form of verdicts, for instance "definitions tend to be sloppy and exercises lack contextualisation" or "language tends to be excessively formal for a 5th grade text". Ideally such conclusions should be thought of as hypothetical constructive feedback for the textbook authors.
In parallel, seeing as many students have no prior contact with LaTeX, the class takes a crash introduction, with the help of a teaching assistant. Their objective in this regard is to be able to

contribute autonomously to the group effort by providing their share of the analysis directly in LaTeX code, usually via some online platform like Overleaf or Sharelatex.

5.2- Task 2: Horizontal analysis

Groups compare segments of about 30 pages on the same topic, from two distinct PNLD textbooks. The goal now is to be able to justifiably *choose a winner* among the samples offered, regardless of their intrinsic merit. This task reflects upon a common school teacher experience, since each time a new PNLD list is published, every school *must choose* one collection to adopt based on samples sent by the approved printing houses, and our premise is that if any two options can be systematically compared, then upon finitely many comparisons teachers should be able to rank the whole list of samples and make a rational informed choice.

Again, there is no ready-to-use bibliography by Brazilian authors to support this important decision. In some sense the only available source are the PNLD guidelines themselves, but then again the decision only takes place among nationally approved collections, hence, under the working assumption that the national assessment is minimally coherent, these guidelines are insufficient by definition. In practice, overworked teachers must make these choices at very short notice and with virtually no procedural guidance, so the process is exposed to anti-pedagogical criteria such as authority, tradition and personal idiosyncrasy.

At Unicamp, we took therefore upon ourselves to establish the basic framework for a content-based comparison grid, which is perfected each year from previous iterations of the task. The main elements are:

- An assessment grid is a point-based system which rewards relative good performance of a textbook over the other about each subtopic or content unit, corresponding roughly to one entry in the official syllabus (eg. sketch of the graph of a hyperbola, or distributivity of integer multiplication).
- Subtopics are surveyed and divided into *even* and *odd*, according to whether they occur in both textbooks or in only one of them, respectively.
- Even subtopics are compared in merit, based on some form of simplified "locally vertical" analysis, and at every instance the winner is awarded some points.
- Odd subtopics are examined per se, and usually assessed as positive, irrelevant or detrimental, being accordingly rewarded or even sometimes punished with negative points.
- A typical methodological pitfall lies in so-called *false odds*: topics which do occur on both textbooks, and therefore are even, but under such different pedagogical guises as to elicit a first impression of being different altogether. For instance, a subtopic may occur as opening motivation for the chapter on one book and as a side remark next to a theorem on the other, and they should be treated as even because in a way or another the teacher can trust that the *syllabus will be covered* in that respect. The resolution and assessment of false odds tend to require quite a bit of interaction between the lecturer and individual groups, as well as some wider classroom sharing of controversial cases.
- Another common pitfall is the temptation to produce *two vertical analyses*, thereby misplacing time and energy towards discussing the individual merits and shortcomings of each book, as opposed to sometimes frustrating actual task of choosing the *least bad* one. This impulse is understandable, since the students feel naturally inclined to apply the skills they just acquired in Task 1, and it must be constantly monitored by the lecturer during progress discussions.

The standard satisfactory report plays out somewhat as a sports match, in which textbooks score or lose points until a winner emerges. The central point of this is the fact that the grid must be established *before* actual analysis occurs, so that assessment derives from universal values and consensual pedagogical decisions, rather than case-by-case subjective preference. As a counterweight, however, students are encouraged to make minor updates to their methodology once, after the second week of the task, to accommodate totally unexpected phenomena and to exclude methodological expectations which did not materialise at all - hence have no purpose for comparison.

5.3- Task 3: Foreign textbook analysis

Students are made aware, in some cases apparently for the first time, that Mathematics is arguably the most universal of school subjects and that its teaching has been addressed from very different world views for centuries. This task consists in examining a 30-page section of a foreign textbook from our library's collection, with the sole and explicit goal of extracting good practices, both in content and in layout, which in their opinion constitute relative innovations and *can be effectively applied* to improve Brazilian textbooks. To state it clearly, it is an exercise in constructive plagiarism and technological catch up.

Currently available materials are from Cuba, France, Japan (in Japanese), Russia (in English) and Spain, mostly obtained by donation from visiting faculty, postdocs etc over time.[4] Clearly a preference is given by students to textbooks in Latin languages, which can be efficiently read by most, but some have ventured for instance into A. Givental's translation of the Russian classic[5] *Kiselev's Geometry* (Kiselev 2008), yielding interesting results (see below).

Groups are asked first of all to do some minor geographic research and establish a comparison between the education systems of the relevant country and Brazil, as well as broader socio-economic factors such as population and per capita income, Gini index, PISA scores and literacy rates. An interesting finding, for example, is that Spain is soundly comparable to São Paulo State in most of these variables, albeit teachers are much better paid, whereas Cuba has much fewer resources but relatively superior results.

Here are some highlighted elements which emerged from this exercise over recent years:

- French and Spanish textbooks tend to have a much more solid chapter and sectioning structure: sections begin with a recap and a statement of goals or skills to be developed, which is then matched by some form of progress assessment at the end. This may seem trivial to non-Brazilians but it is hardly ever correctly implemented, if it is to be found at all, in the Brazilian textbooks surveyed in this course.
- Some Spanish textbooks tend to be explicit as to the difficulty of exercises, or to the specific skill being practiced, relative to the chapter's goals. This is usually done by some visual code involving colours or "difficulty bars"; students have found this to be a time-saver both for the teacher's lesson-planning and for the pupil's independent study.
- Cuban books have relatively poor printing quality, but they tend to be very carefully adjusted to local reality: for instance, the first few sections in a 1st grade primary school textbook are strictly pictorial, since children are meanwhile learning how to read, and written text is gradually introduced along the book with increasingly complex grammar, so as to follow their progress in Spanish along the year. Students assert that this sort of interdisciplinary pedagogical integration is very different from standard practice in Brazilian primary schools.

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[4] The reader is very much invited to send us donations from their own country, with our utmost gratitude.

[5] A. P. Kiselev (1852-1940) was the Russian and then Soviet writer of a hugely influential set of school textbooks, which were widely adopted in Russia and the USSR from 1892 to the mid 1970s, and also gave rise to adaptations in Eastern Europe and in China (Kiselev 2008, Translator's foreword). His books on Euclidean geometry have a distinctively systematic yet practically-minded approach, with an emphasis on structured proof and ruler-and-compass construction. Indeed, "(d)uring the 1930s, these became accessible if not to all, then to almost all schoolchildren, and this is what brought Russian school mathematics its deserved fame. (…) Millions studied using Kiselev's textbooks, and their proofs and arguments, their relatively complex transformations and algorithms, became accessible to millions. " (Karp 2012, p. 408).

- Kiselev's 19th century pedagogy in geometry is mostly seen as a monstrosity by students' sensitive contemporary eyes. They argue that the emphasis on rigorous definition-theorem-proof structures and elaborate ruler-and-compass constructions (aimed at 6th graders…) and the scarcity of contexualised examples are simply too distant from current canons in education to be applicable in the modern day classroom. However, one group has made a very interesting use of those features, by a clever and rather straightfoward adaptation of a sample section into a set of activity sheets for Geogebra
- Although the assessment criteria in this task are necessarily more flexible, to account for the diversity of materials and outcomes, groups are already well-aware that the next task involves the production of original material, so they tend to take very seriously every hint that might give them a headstart in that near future.

5.4- Task 4: Production of teaching material

Students must apply the skills acquired in Tasks 1-3 into the original production of two sections of a hypothetical textbook, spanning over 20-40 pages, to a high standard of content and layout. While of course this is an opportunity to consolidate their conceptual achievements so far, the task is first and foremost an experience of the passive pole in criticism, a proficiency which tends to quickly hypertrophy in younger generations' analytical culture. Students report that wearing the author's shoes is a humbling experience, and that feeling just how hard it is to put their best into paper tends to inspire a certain benevolence toward textbooks they may have viciously denounced in Tasks 1 and 2.

Each group must choose two related topics, establish a pedagogical approach and create a LaTeX template accommodating all the structural elements required. In particular, they must create dedicated visual language to designate recaps, definitions, examples, curiosities, reminders etc. The A-standard for this material is both (i) to be visually indistinguishable from a commercially available textbook and (ii) to stand on its feet upon subsequent vertical analysis by the lecturer. Here are some of our systematic findings:

- Seeing as the creation of a LaTeX template from scratch (or the major adaptation of something available online), as well as intense decision-making on the selection, phrasing and ordering of content, must happen within just over three weeks, it might seem that this measure is altogether unachievable. However, students tend to surpass all expectations and produce some really outstanding material.
- Successful groups systematically apply lessons learned from foreign books in Task 3 and show a permanent concern not to incur in faults detected in Task 1.
- Successful groups maximise their work capacity by assigning to the most LaTeX-savvy member an exclusively editorial role, essentially creating environments and visual resources upon request from the "contents team" and compiling together their contributions.
- The least able or least motivated students tend to be assigned to online searching for useful images or to browsing existing textbooks for good exercises to borrow; in virtually no observed instances did any team member stand idle.
- Only just about one group each year tends to decisively fail Task 4.

We conclude with some highlighted Task 4 works from each of the course iterations; non-Portuguese speakers may still appreciate the layout and structure choices, as well as some elements of the pedagogical approach:

- 2014: http://www2.ime.unicamp.br/~ma225/2014Tarefa4-GrupoA.pdf
This is a 5th grade chapter on Fractions, divided in the two sections *Revision of fractions* and *Operations with fractions*. It demonstrates careful planning of chapter structure, beginning with motivation and statement of goals and following, for each unit, the consistent expository pattern of content - example - practice. The text reflects considerable attention to the adequacy of written and visual language to that specific age group.
- 2015: http://www2.ime.unicamp.br/~ma225/2015Tarefa4-GrupoB.pdf
This is an 8th grade chapter on Proportions, divided in the two sections *Proportionality* and *Thales' theorem*. In this beautifully designed template, a particular pedagogical priority is placed on exercises, both as motivation and *a posteriori* practice, and their difficulty is highlighted to allow both teacher and student to gauge their planning and expectations. At the end of each section, a Summary page is included, just before the exercise sheets, and a Geogebra construction activity is proposed. At the end of the Chapter there is a selection of real questions appearing in Vestibular admissions exams across the country, and a card game is proposed.
- 2016: http://www2.ime.unicamp.br/~ma225/2016Tarefa4-GrupoC.pdf
This is a High School 1st year chapter on quadratic polynomials, divided in the two sections *Quadratic functions* and *Inequalities in degree 2*. Although the template is less impressive than the previous two examples, this document reflects appropriately the level of mathematical formalism for this age group, and it reveals a deliberate pedagogical choice to motivate quadratic phenomena from the mechanics of sport, as opposed to say ballistics or stale "free-fall" physics situations. It also offers some graph-plotting activities on Geogebra.

5.5- Task 5: Original contribution

The final task of the course pushes for an independent initiative proposed by the groups themselves, according to two guiding approaches: (i) to extrapolate the immediate scope of the course, or (ii) produce a lasting contribution to further iterations of the course itself. Approach (i) has resulted for instance in the vertical analysis of a *Physics* textbook or the development of software modelling activities to complement an otherwise good textbook, while several interesting outcomes of (ii) are discussed below. Once the task is presented, groups must form a consensus around a proposal, which is then validated into a concrete project with the lecturer's input. The expectations and assessment criteria are then directly negotiated between each group and the lecturer.

Task 5 seeks to stimulate initiative, independence, creativity and generosity to future generations of students and to wider society. These values are well-reflected in many contributions so far, of which we highlight the following:

- The course website: http://www2.ime.unicamp.br/~ma225/
- The official webpage for the course was itself an outcome of Task 5 from the 2014 class, entirely designed and developed by the students themselves.
- Course manual: aimed at facilitating future generations' assimilation of the course's rationale and goals, it includes an instructive description of each task, a specific section on LaTeX for textbook templates and a detailed report of the 2014 class experience from the students' perspective.
- Slideshow: the actual big screen supporting material currently by the lecture, containing hyperlinks to the course's webpage, examples from previous years, standards for efficient

workshop dynamics during the class and standards for group communication and submission of tasks, among others.

- LaTeX templates: a number of ready-to-use templates for the use of future groups in each of Tasks 1-4, accounting for subtleties such as age group-specific suggestions of visual resources etc.
- Online games inventory: http://www2.ime.unicamp.br/~ma225/jogos/
- Conceived as an online support platform for teachers, this compendium of online games is organised according to the national curriculum, so that one can easily find links to online resources sorted by specific curriculum entry. Quality was a big concern, and a methodological decision was made to offer *no link at all* for a given topic if no pedagogically sound resource could be found for it. The idea of course is to be gradually expanded by future classes.

Afterword

We believe the findings of our qualitative analysis can contribute both to the theoretical field of Mathematics Textbook Analysis and to the practice of school teachers in a number of ways, based on our Brazilian experience. First, by establishing some systematic patterns to anchor the criticism and improvement of a given textbook. Second, by formulating methodological cues to support the decision-making process of teachers among a sample of alternative collections. Third, by highlighting the potential input of textbooks from different cultures and epochs, both in content and in layout, relative to the Brazilian standard. Fourth though not least, to corroborate the perception that future teachers, while still in-training, already display huge creative energy and, albeit not necessarily the strongest students in Higher Mathematics as such, can vastly outperform expectations when provided a collaborative and stimulating classroom environment.

Finally, we hope this account may motivate similar experiences in other teacher-training institutions, which can then be compared and evolve together into a core set of good practices to be adopted, in various guises, on an international scale.

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NORWEGIAN TEACHERS' USE OF RESOURCES FOR PLANNING INSTRUCTION IN MATHEMATICS

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Abstract

This paper presents findings from two master theses about seven Norwegian teachers' use of resources before, during and after instruction in mathematics in Norway. Although the teachers had a range of resources available, it was found in both thesis that all teachers almost exclusively used the curriculum material available at their school, the teacher's guide in particular. However, it was also found that the teachers who had participated in professional development were less dependent of the teacher guide. Teachers who attended professional development tended to use the curriculum objectives in the national curriculum as a starting point rather than the teacher guide and included other resources in their planning.

Key words: teacher's guides, curriculum material, curriculum material use, curriculum material affordances, professional development.

Introduction

Design and distribution of curriculum material is one of the oldest strategies to influence what takes place in the classroom (Ball & Cohen 1996; Davis & Krajcik 2005). In Norway, teachers can freely decide if, which and how much, they use curriculum materials and other resources. There is a strong tradition of using curriculum materials published by commercial publishers. The curriculum material is often a package that typically consists of a student textbook including exercises, a teacher guide, and web-resources of various kinds including, but not limited to, exercises, games, tests and films. Thus, this material potentially has a great impact on instruction in Norwegian classrooms.

Currently there is no national quality assessment of published curriculum materials in Norway. After 100 years with a national quality assurance of published curriculum materials, the government decided to abolish the system with quality assurance in the year 2000. There were two main arguments for this decision. First, parents and students should have stronger influence on choice and use of curriculum material in schools. Second the national curriculum should be the main political management tool to decide the content in instruction (Bratholm 2001). This puts a large personal responsibility on the teacher to develop their lessons in order to reach the competence objectives in the national curriculum. However, many teachers still continue to lean too much on curriculum materials to achieve the competence objectives in appropriate ways (Grave & Pepin 2017). The lack of national quality assurance of published curriculum material was one of the reasons that led to this investigation about what kind of resources teachers use and how they use them. This paper reports from two case-studies including seven teachers in upper primary school in Norway.

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Literature review

Teachers working with resources

Resources in mathematics plays an important role in pupils' and teachers' environment. Over the past three decades there have been a considerable amount of research about curriculum materials and resources in mathematics teaching (Adler 2000; Remillard 2005). It was looked at resources in a wide perspective: curriculum materials including other text resources, ICT and human resources (Adler 2000; Gueudet, Pepin & Trouche 2012). In this paper the focus is on the teacher's use of curriculum material.

Teachers notice and use curriculum material differently, depending on their experience, intention and competence. Teachers and resources influence one another (Brown 2009; Remillard 2011). Looking at how teachers use curriculum material to achieve teaching objectives, *The Design Capacity for Enactment* framework (DCE) is relevant. In this relationship, teaching is design, where the teacher designs lessons in the classroom by modifying existing material or integrating new material. The teachers' ability to perceive and mobilize the pedagogical ideas that are embedded, Brown (2009) calls pedagogical design capacity. This approach identifies three different ways of teachers work with the curriculum resource: the teachers *adapts*, *offloads* and/or *improvises* with the material to adjust it to their teaching (Brown 2009). When teachers offload, they follow the curriculum material slavishly. When adapting, they make some changes related to their own experiences. Finally, if they improvise, they use their own strategies and the connection to the curriculum material becomes vague. Who the teacher is as a reader, influences what and how they read. Remillard (2011) describes different ways we can look at the teacher as a reader. We can look at why, what and when the teachers read and who they are as readers. Teachers participating in PD, gain new experiences and competence and can potentially change the ways they interact with their resources. Thus, resources can play an important role in teachers' professional development.

Teachers professional development, PD

In Norway there is an increasing interest in developing mathematic teachers' knowledge for teaching. This can be seen in both: the big national strategy called *"Competence creates quality - Strategy for continuing education for teachers and school leaders until 2025"*; by the Ministry of education and research (Kunnskapsdepartementet 2015), and in smaller and local PD-projects and innovations on municipality level and at school level.

The ultimate goal of professional development is improving students' learning through the mechanism of improving instruction (Doerr, Goldsmith & Lewis 2010). The design of PD can be influenced by research from both empirical studies and small scale qualitative studies.

The research on professional development suggests that mathematics professional development is effective when it promotes mathematics teachers' growth in four major areas.

- 1. Builds teachers' mathematical knowledge and their capacity to use it in practice,
- 2. Builds teachers' capacity to notice, analyze, and respond to students' thinking
- 3. Builds teachers' productive habits of mind, and
- 4. Builds collegial relationships and structures that support continued learning.

In addition, research suggests that three broad features of professional development support these goals.

- 1. Substantial time investment,
- 2. Systemic support, and
- 3. Opportunities for active learning (Doerr, Goldsmith & Lewis 2010)

Our research

The research presented in this paper tries to answer the following questions:

- 1. How do teachers use curriculum materials available at their school?

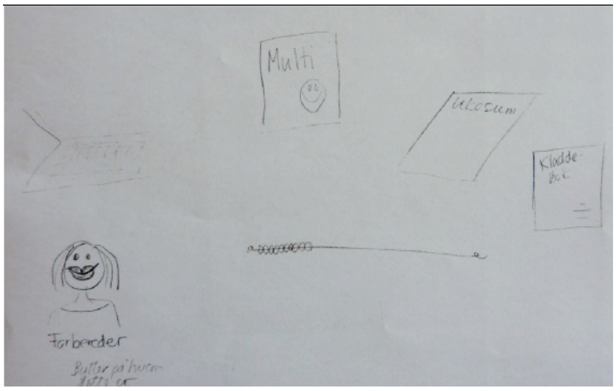
- 2. In which way are the teachers' use of these curriculum material influenced by professional development?

In answer to these research questions; the findings from two master theses during 2012 and 2013 (Justnes 2013; Svingen 2014) were looked at. The data was collected from seven teachers from three different upper primary schools in Norway. All seven teachers used one of the two most dominant curriculum materials available in Norway, at the time. All teachers were interviewed about which resources they used, when and how they used them, and about their rationale for their choice of resources. The teacher's private notes for planning and observation of their classroom teaching was also collected.
Since Justnes's study aimed to explore how the teachers' use of resources developed, she conducted her data collection twice, before the teachers attended PD and four months into the PD-program. The data collection was a combination of interviews where the participants made a *schematic representation of resource system (SRRS),* observation of both planning and teaching, and collection of the teachers notes. These notes included private notes; they made while planning their teaching; notes to communicate their plan to each other, and notes to communicate plans to the parents and pupils (extended schedule/work plan/timetable).
Svingen's study went on to analyse the teacher's guides involved and conducted a document analysis in addition. The categories from the document analysis was used as a framework on how the curriculum material contributed to the teacher's practice. Svingen further set up a case study to answer how the teachers used the curriculum material before, during and after instruction. The data collection was a combination of semi-structured interviews and structured observation of three teachers.
The first findings from the two case studies will be presented separately.

Short presentation of the main findings in the two case studies in question

Case study 1 – Mathematics teachers work with resources, Justnes (2013)

The research question for this study was: How will four teachers develop, in terms of resources, while attending professional development?

SRRS Teacher 1 before PD:	SRRS Teacher 1 during PD:
	

The main findings from Justnes:

1) **A wider definition of resources.**

The analysis showed that all the teachers perceived the curriculum material and the teachers-guide as their main resource before they started their PD. The teachers defined the resources they used more broadly after four months in their PD. They included their own experiences and the other members of the staff/team, as a resource. The teachers had started to plan more of the content together as a teacher-team, the teachers then perceived themselves, as more confident in their role, as mathematics teachers. Hence their work with their resources was perceived as less individual.

2) Strengthened capacity to noticing pupils thinking.

Before the PD-program started, none of the teachers talked about the pupils as resources. Four months into the PD-program, the teachers also included pupils as a part of their resource system. They both perceived pupils as resources for each other, and for planning further instruction. The teachers claimed stronger focus on noticing pupils' strategies and understanding, and this influenced their planning of instruction more than before or instead of the "pace" proposed in the teacher guide. A growth indicator in noticing is according to Jacobs, Lamb, and Philipp (2010) a shift from being ruled by the progression in the textbook to students' current understanding. Teaching based on the analysis of students' understanding is in other research related to the improvement of pupils learning opportunities (Jacobs, Lamb & Philipp 2010).

3) Development of community of practice

Justnes found that they ways the teachers started to work together can be described as the development of a community of practice. Communities of practice are groups of people who share a concern or a passion for something they do, and learn how to do it better as they interact regularly (Wenger 2008). This includes mutual engagement, a joint enterprise, and the development of a shared repertoire, which Justnes found evidence for in the material.

4) Support for development

According to NCTM support from the system is important for professional development. Both *time* and *opportunity to meet,* are important factors that enable PD (Doerr, Goldsmith & Lewis 2010). Based on the previously mentioned findings from this study; we argue that facilities must be offered so that teachers can develop communities of practice to support teachers' development of a productive mindset, since this is a mechanism that contributes to improved teaching in mathematics.

Case study 2 – Analysis of two teacher's guides – characteristics and teachers use of them - Svingen (2014)

The research questions for this study were: When studying the teacher's guide in two different curriculum materials, what characteristics are there? How do the teachers use the teacher's guide when they are; planning, carrying out and evaluating instruction?

The main findings from Svingen's study were:

1) Characteristic of the teacher's guides and how they are used

The two teacher's guides differed in three ways. The first was related to user-friendliness. That was dependent of how easy it was to bring the teacher's guide in the classroom and in which way it was easy to get an overview of both; the content in the pupil's textbook and of the support the teacher could use in instruction. The second characteristic was about what view of mathematics the teacher's guide promoted. One supported an instrumental understanding of mathematics, while the second emphasized more relational understanding of mathematics. The third characteristic was related to differentiation. One supported differentiation where the students move forward, where the other one gave examples of how the students could work in depth with the same topic as their classmates. User-friendliness was an important factor in which way the teachers used the teacher guide. The one teacher guide which was huge and gave little support, was used only to copy an overview of what tasks the student should work with on different levels. The teachers used the textbook instead to plan instruction.

2) The importance of curriculum material

The three teachers in this study, used either the teacher guide or the textbook as their main resource before and during instruction. They used a few other resources. Two of the teachers used continuing education and training, they had participated in as a resource in their planning

3) The potential in the teacher's guide

The main purpose for the teacher's guide, is to support teachers in planning instruction and during instruction. The characteristics of the teacher's guide; can tell us something about what kind of support the teachers get and what kind of instruction the teacher guide promotes. A characteristic which was in common for the two-teacher's guide, was the transparency. If the teacher guide is transparent, rationale for decisions are made visible and it's easier for teachers to interact with the content in an appropriate way. In what way are the goals for the lesson explained? How are the mathematical concepts explained? Is the rationale for why activities are important explained? Will the teachers get some support in which way different topics are related to each other? The findings in the analysis of the two teacher's guides, show that both of them were not transparent. They gave little support to the teachers in the decision-making process.

4) Use of the analytic scheme for the teacher's guide

The development of the analytic scheme for the analysis of the teacher's guide; gave insight in both qualities in the teacher's guide, but also what happened in the classroom. The instruction is a complex situation; and to look at instruction through the analytic approach to the teacher's guide, gave new insight in where it is important that the teacher's guide supports the teachers.

Results and discussion

All though the original research, questions in these two master theses were different, several of the findings are related to one another, and this paper will report on these findings.

Result 1: Teachers depend heavily on the curriculum material available at their schools

Both studies found that the curriculum material is the main source for the teachers in planning instruction, despite that other resources are available. The first SRRS and interviews from Justnes's study showed that the teacher's guide is the starting point for further planning for the four teachers in this study. Hence, the teacher's guide determines both topic, pace, classroom activity and tasks, which Brown (2009) calls *offloading*. Svingen's study also showed that the curriculum material was the most dominant resource for their planning. This is in line with other research, which also finds that curriculum material is the primary source for mathematics teachers; and also legitimate and decide content; and how the content is sequenced (Freeman & Porter 1989; Robitaille & Travers 1992; Pepin & Haggarty 2001).

However, the teachers in question used the curriculum material in different ways. In curriculum material where the teacher's guide gave little support for the teachers, they used the pupil's textbook instead. In planning, the teachers decided which examples from the textbook they should use and what tasks the pupils should work with. They were *offloading* the textbook. Teachers who used curriculum material that gave suggestions on what they could do in instruction, followed the suggestions in the teacher's guide and offloaded the teacher's guide.

Result 2: PD influenced teachers' perception and use of resources

One of the findings in Justnes's study was that the teachers mainly perceived the resources available at their school, as their resources before they started PD. This included curriculum material, but also their own experience as teachers and for some of them, their colleagues. The decisions for teaching was mainly based on the suggestions from the teacher's guide in the curriculum material, however influenced in various degrees by their own experience and sometimes by input from other teachers. Their planning practice was individual.

After four months the teachers also included the PD-course, or elements from it, and input from pupils as a part of their resource system. The teachers reported that they perceived themselves as less dependent on the curriculum material in their planning. They planned for more open tasks and group work for their pupils, which had been a part of the focus in the PD-course. This led them to search for tasks and content from other resources than the curriculum material. Because of the increase in group work, the pupils were able to participate in plenary discussions and oral communication with each other in greater extent. This gave the teachers opportunity to listen more carefully to pupils’ mathematical thinking and take this in account in their further planning. Hence, they included pupils in their resource system.

The PD-course led the teachers to meet regularly and discuss their mutual goal of increasing inquiry-based learning in their mathematics teaching. The teachers reported, and observations showed, more cooperation and joint work with their resources, and development of a joint repertoire of resources and teaching methods.

Similar findings come from Svingen’s study; where we found that teachers with continuing education and training added knowledge from their education into the planning of instruction. One of the teachers participated in continuing education in assessment for learning. She added some activities that made the pupils’ learning and thinking more visible than the curriculum material prepared them for. She referred directly to the education as a resource in her planning.

Discussion and concluding remarks

Both studies found that the curriculum material is important for the teachers planning, despite that other resources are available. However, the teachers in question used them in different ways. The first SRRS and interviews from Justnes’s study showed that the teacher’s guide is the starting point for further planning for the four teachers in this study. Svingen’s study looked at the use of teacher’s guides closer and found that the participating teachers were looking for different aspects to use for their planning. They were, in particular, looking for three aspects; how to deal with the competence objectives in the national curriculum, differentiation and to expand their repertoire of teaching strategies. Looking back into the data in Justnes’s study, this is connected to the changes in teachers use when they joined the PD-course. After they started their PD-course, they had another starting point than the teacher’s guide when planning instruction. The new starting point for their planning was the national curriculum objectives and the pupils’ responses from last lesson. The PD-course contributed to more cooperation between the teachers where they discussed both curriculum objectives, teaching strategies and how to meet different pupils reasoning.

Even though we found that the professional development course under consideration had an impact on the teachers´ flexibility in choosing and using recourses; it should be noted that participation in PD is not systematic in Norway and varies between schools and districts. Hence, many teachers are left with the curriculum material as their main resource for teaching. This puts the curriculum material in position to provide professional development for teachers in Norway. But that makes some demands on the curriculum material and, in particular, the teacher’s guide. The two teacher guides under examination in this study were not transparent. They gave the teachers few possibilities to take part in the decision-making process. As long as the teacher’s guide does not make the rationale for decisions transparent, it will be hard for the teachers to interact with the content in a productive way. As shown in Justnes’s study, the PD-course made the teacher more flexible in how they interacted with the teacher’s guide. It might be explained by the teachers’ growth; in mathematical knowledge; capacity to notice, analyse and to respond to pupils thinking; build productive habits of mind and build collegial relationship (Doerr, Goldsmith, and Lewis 2010). They had increased their ability to take part in the decision-making process together. The PD-course compensated for the lack of transparency in the teacher’s guide.

There is a potential for improving instruction through curriculum material, since it is accessible for teachers in an easy way. A keyword is transparency. The rationale for choices, needs to be expressed explicitly. But there is one main limitation, the teacher’s guide cannot compensate for a collegial relationship, where teachers discuss and develop their understanding for instruction, the power of working with colleagues.

There are several limitations in small studies, which have a small number of teachers and are carried out in a short time span. Nevertheless, there are findings in the two studies which can contribute to a deeper understanding of how professional development and curriculum material can improve instruction and further improve students’ learning.

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SECTION TEXTBOOKS AND STUDENT ACHIEVEMENT

ENEM AND MATHEMATICS TEXTBOOKS FOR HIGH SCHOOL: AN ANALYSIS OF THE VOLUME OF GEOMETRIC SOLIDS

KATY WELLEN MENESES LEÃO, ROSILÂNGELA LUCENA and VERÔNICA GITIRANA

Abstract

Each year, Brazilian students undertake the National High School Exam (ENEM), aiming to enter university. Textbooks play a significant role for students' development. So, we consider that there should exist compatibility between mathematics textbook proposals and the mathematical knowledge required in ENEM. An analysis of this compatibility was undertaken regarding the content of volume. The Anthropological Theory of Didactics was used to analyze tasks and techniques involved to correctly solve each task in the last six editions of ENEM and in a Brazilian Mathematics textbook collection for high school. The results showed that most contents and abilities required to solve ENEM questions of volume were found in the textbook's approach. However, differently from ENEM questions, the textbook approach rarely correlates them to another field of knowledge, as well as to contents from other blocks of school mathematics, such as to proportion and to percentage.

Keywords: Textbook, Volume, Anthropological Theory of Didactics, ENEM

Introduction

ENEM, Exame Nacional do Ensino Médio, is a Brazilian evaluation test for high school students created in 1998 by the Ministry of Education. Many state and private universities use their results as a way of selecting students to their courses. Since 2009, when ENEM results began to be adopted by state universities in their admission process, there has been a collective mobilization to think about the compatibility between school teaching and the content required in ENEM. The National Exam contains contextualized questions, related to several areas of knowledge. We start from the point of view that School approach to a content and any evaluation test should have compatibility. Textbooks, are one of the most used resources in classes, thus its approach and the ENEM should be compatible.

Our focus is on the block of school mathematics - magnitudes and measures -, more specifically on the magnitude volume, aiming to answer the following questions: (A) which tasks involving the magnitude volume are proposed in ENEM questions? (B) Which techniques are required to solve ENEM questions? and (C) Are these tasks and techniques addressed in textbooks?

This paper discusses part of a research that aims to investigate the compatibility between the knowledge required by ENEM tests and the textbook approach. For this, we analyzed six editions of the exam, applied in the years 2011 to 2015, concerning the questions involving the concept of volume, and a collection of Brazilian textbooks that were the most distributed in state schools by

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PNLD 2015 (Brasil 2014) - a national distribution program of textbooks, which also evaluates the textbooks to be chosen from a guide by the teachers.

Theoretical Framework

In this section, we will discuss the Anthropological Theory of Didactics (Chevallard 1999) which composes our theoretical framework. We also found the concept of volume and its perspective as magnitude, according to the didactic hypothesis of Douady and Perrin-Glorian (1989), seeking to reveal the importance of its treatment within the magnitudes and measures cadres.

Anthropological Theory of Didactics

In this work, we seek to understand the Chevallard (1999) proposition of the Anthropological Theory of Didactics - ATD. In theory, the Praxis of didactic organizations and mathematical organizations in relation to the knowledge enable us to analyze concepts, procedures, and algorithms used in the accomplishment of a certain task. For Chevallard (1999), "mathematical organization is the study of the mathematical activities that are proposed by the institution and the didactic organization refers to the way the study is made around the mathematical organization" (Freitas 2014, 4).

According to Chevallard (1998 apud Santos & Menezes 2015, 649), "an ATD must be faced as a development and an articulation of the notions, the elaboration of which aims to allow thinking in a unified way a great number of didactic phenomena, that arise at the end of multiple analysis". ATD offers us a methodological apparatus for the study of several facets of Didactics process that takes place in the classroom, such as didactic contract, time management and didactic transposition.

The ATD defends that practical and theoretical elements that conduct human action can be described according to a praxeological organization, involving a task, T, which is performed by means of a technique, τ, thus justified by means of a technology, θ, and then, such technology is justified by a theory, Θ. This block is what we call the praxeological organization $[T, \tau, \theta, \Theta]$.

Chevallard (1998, p 2) states about the task, T: "Specifically, a kind of task exists only in the form of different types of tasks, the content of which is narrowly specified. Calculating ... is a kind of task". According to Chevallard (1998, 2, our translation), "A praxeology relative to T specifies (in principle) a way to accomplish, to perform the tasks $t \in T$: to such a way of doing, τ, we give here the name of technique". A set of tasks and techniques is called the practical-technical block $[T, \tau]$, the know-how block. This set of elements needs a theoretical fundament that bases it, a set of elements that theoretically justifies such techniques. One of these elements is called technology, θ, where it presents itself as a rational discourse on the technique τ, a theoretical justification of the techniques used, usually by a demonstration. The other element that underlies the practical technical block is called theory, Θ, which is the most elementary mathematical concept that conceptually supports technology.

> In turn, technological discourse contains assertions, more or less explicit, in which reason can be applied. We then move on to a higher level of justification-explanation-production, than of theory, Θ, which takes up the role of technology regarding technique. (Chevallard 1998, 4, our translation).

The set of technology and theory forms a block called technological-theoretical block, $[\theta, \Theta]$, the block of knowledge, representing the knowledge that was implicitly used to solve a problem.

In this research, we are interested in the practical-theoretical block of praxeological organizations. We will use the practical-technical block $[T, \tau]$ that gives us support to identify and analyze tasks, T, and techniques, τ, in ENEM and textbook questions, besides the practical block, that justifies the techniques, τ, applied in the resolution of the tasks, T, related to the volume concept of geometric solids. Our perspective is to verify the compatibility between the volume of solid approach in the textbook and the skills required to solve ENEM questions.

Volume of Geometric Solids

By volume of solid, we have a definition by Moise (1990, 352):

Given V a class of measurable sets in space, there exists v: V → R - a function in real positives such as if M ∈ V, v(M) is called volume of M, where measurable sets are the sets in which we can associate a measure with its elements. We will call the elements V of solids. Following the definition of volume above, every solid that has volume needs to satisfy the following properties:

- *Being a convex solid every solid the line of which that connects any two points of its interior is entirely contained in it, all convex solid has volume;*
- *The union, intercession, and difference of solids have volume;*
- *If a solid M is bigger than a solid N, the volume of M will be bigger than the volume of N*
- *If the volume of the intersection of two solids is empty, the volume of their union is the volume of each separately;*
- *If M is a parallelepiped of base b and height h, the volume of M is b×h;*
- *If M and N are solids in space and α0 and α1 are parallel planes, being any measurable cross-section, where transversal cuts are the intercession of M and α0 and the intercession of N and α1, and the transverse cuts have equal two by two sizes, then M and N have the same volume.*

The definition presented for volume of a solid reveals aspects of a concept studied in geometry. Today, the concept is considered one of the geometric magnitudes, within the magnitudes and measures block of school mathematics. As Lima and Carvalho (2010, 136, our translation) affirm, one of the reasons for this change comes from the "need of a greater attention when teaching the concept of magnitude in general, not limited to geometric magnitudes, being relevant to its study the comprehension, measurement, and representation".

The concept of volume as magnitude comes from the adaptation of didactic hypothesis derived from research developed by Regine Douady and Marie-Jeanne Perrin-Glorian (Douady & Perrin-Glorian 1989), which distinguishes three frames for comprehension of area as a magnitude: geometric frame, magnitude frame, and numeric frame, as affirmed by Figueiredo, Bellemain and Teles, (2012, 2).

In these frames (Figure 1) are inserted mathematical objects, their relations, formulations and mental images that the subject associates to the objects at that moment.

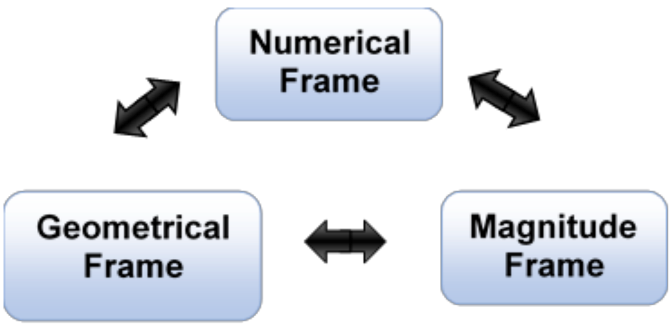


Figure 1: Didactic model of frames representation. Adapted from Douady and Perrin-Glorian (1989)

In the geometric frame are presented plane surfaces of mathematical objects. By adapting the concept of volume, we can describe how objects have volume in the physical world. The numeric frame refers to the actual values that the volume can have, being a non-negative real value. The magnitude frame is the proper notion of volume and can be classified as being the equivalent class of figures with the same volume. One of the reasons to differentiate magnitudes and geometric objects is that each object can be observed in several ways, including several magnitudes. A cube can be analyzed according to its volume, the length of its edge, its main diagonal, the diagonal of one of its faces or the area of their faces.

Review of the Literature

In this section, we will discuss the importance of the study of volume in the magnitude frame and its approach in the textbook, discussing some relevant aspects, according to research carried out on the theme in focus.

The way the magnitude volume is presented in Brazilian textbooks has changed over the years, as affirmed by Bellemain and Lima (2010, 167): “We noticed, for example, that concepts related to this field are more articulated than other mathematical contents, and, when treated separately, that is done in chapters distributed along the textbook, and not left to an isolated section of each book, usually at its end.”

Morais (2012) affirms that in ENEM tests, the use of formulas relating to other aspects of the volume concept is overmuch prioritized, when the use of formulas alone does not support the transition between numerical and magnitudes frames. “Therefore, they are important tools for resolving volume problems. but it is worth noting that their use doesn't affect the comprehension of the concept, since it differs from magnitude.” (Morais 2012, 43)

This author, in his analysis of the volume concept in textbooks, observed that in most of the 7 collections approved by the 2012 PNLD for high school, approximately 93% of activities were about measurement. The author considers that the questions in which it is necessary to assign a numerical value to volume, by either using formulas, composition and decomposition or Cavalieri Principle, can influence the student to understand only the numerical aspect of the volume concept.

On the frame of geometric magnitude, his study becomes of fundamental importance because:

> [...] geometric magnitudes reveal themselves as complex conceptual fields, where deeper analysis is required in order to understand the learning difficulties of the students, intervene in a relevant manner and favor the establishment of articulations between multiple possible conceptions of concepts related to magnitude. (Bellemain & Lima 2002, apud Brito 2003, 31, our translation)

It is also important to emphasize that geometric magnitudes were previously studied in the field of geometry, where only numerical value and volume properties were treated. One of the biggest problems in the study of this magnitude is the non-differentiation between measure, magnitude and geometric object, as "students have little knowledge of volume as magnitude, they change it either to solid, or to number" (Morais 2010, 2, our translation). Morais emphasizes another confusion, already discussed in Bellemain and Lima (2002) that refers to the necessary differentiation between the geometric object and the physical object. A dice, for example, is a physical object that geometrically can be modeled by the geometric object called "Cube".

Another point to be considered, as Figueiredo, Bellemain and Teles (2012, 2) affirm, is that "the content has social meaning and, because of that, it becomes the school’s responsibility to promote students’ development of skills and competencies regarding volume".

Methodological Path

Initially, to map the task types that appear in mathematics questions of ENEM tests related to the volume concept of a geometric solid, we delimited the data collection to ENEM tests applied from 2011 to 2015. In a first mapping of the questions, we identified questions related to the concept, properties and geometric solids of volume.

After mapping the questions that involve the concept of volume, to analyze the tasks presented in these questions and possible techniques to obtain correct solution, the questions were solved. In this process, we seeked to use different strategies to solve all the mapped questions, in a process of theoretical analysis. On the basis of the Anthropological Theory of Didactics - ATD (Chevallard, 1999), we sought to classify the questions looking at the tasks involved in them. Many of these questions had an interdisciplinary feature, articulating the concepts of volume with others, such as proportion and percentage. Our expectation about this mapping consisted in having a quantitative and qualitative panorama regarding the questions exploring geometric solids volume and to the task types and techniques classified from the mapped questions.

Analysis Results

The analysis structure reveals a mapping of the ENEM questions, regarding magnitude and measurements. Among these, only those that dealt with geometric volume magnitudes were selected and grouped according to the techniques of resolution required. Such classification allowed us to group the questions identified in the textbooks related to volume, allowing a comparative analysis of the data.

Classification of ENEM questions

The six ENEM tests contained 77 questions, out of a 270 total, that dealt with magnitudes and measurements, 16 dealing with the volume of solids.

All the questions about volume from ENEM tests had, as the correct answer, to find a measurement, what shows the privilege of the numeric frame. In a first glance, we could think of a simple categorization of all the question involving the calculation of volume. However, a deeper analysis shows differences. Some started from the volume to obtain one length of the solid, as a problem of inversion, while others required different tasks in their resolutions. Thus, some criteria were taken to assist with the mapping of the multiplicity of tasks involved in each question. Some of the criteria also involves techniques and other questions involve contents of other mathematics blocks of school mathematics:

- (FV) Use of a volume formula: questions that involve calculating the volume using the lengths of a solid by the *use a volume formula* – in which we observe the passage from the geometric frame to the numeric frame.
- (CV) Composition and decomposition of volume: questions that involve operations with volumes, disjoint or not, such as adding or subtracting. Such operation deal from geometric to magnitude frames. It can be observed that the union of solids belongs in the geometric frame and the operations of volume, in the magnitude frame.
- (TU) Transformation of units: questions that require students to transform units of measure, volume, capacity, or length.
- (PC) Use of the Principle of Cavalieri: questions in which the used formula is not easy to see. It is necessary to compare the area of sections of distinct solids in order to calculate their volume.
- (PO) Use of percentage: questions that require the use of percentage.
- (P) Use of proportionality: questions that involve the concept of proportionality and/or ratio of proportionality.
- (MC) Use of multiplicative comparison: questions in which comparison between volumes is required, by analyzing the increasing or reduction of the volume through the multiplication of its edges by a number different from zero.
- (AC) Use of additive comparison: question which involve the use of an additive comparison of measures.
- (PS) Produce a new solid by visualization corresponding to a new measurement given.
- (FA) Formula of area - questions which involve calculating the area of a figure.
- (PG) Identification between physical objects and geometric models - questions which involve the identification of a geometric model given a physical object.

Group	Criteria	Sub-tasks involved	Total

1	FV - TU - P	(A) Calculate volume given radius and diameter; (B) Transform the unit of measure from cm^3 to mL; (C) Calculate volume given ratio of proportionality and total volume.	1
2	FV - CV	(D) Decompose/compose the volume of a solid; (A) Calculate the volume of each solid decomposed given its lengths; (E) Calculate one length of a solid given its volume.	3
3	FV - AC - PS	(F) Produce (by visualization) a new solid corresponding to a new given measurement; (E) Calculate one length of a solid given its volume; (G) Determine the additive relation between lengths.	1
4	PG - FV - FA – TU - PC - P	(H) Identify correspondence between the measure of a physical object and the geometric models; (I) Calculate the value of comparative relation between the measurement of the trapeze bases in a situation of simple proportionality; (J) Calculate a measurement, given the comparative relation and the compared measure; (A) Calculate the area of a figure; (K) Calculate the volume of a solid; (B) Transform the unit of volume.	1
5	FV - PO	(L) Calculate the length of a solid by increasing percentage; (A) Calculate the volume of a solid given its lengths; (E) Calculate the lengths given the volume of a solid; (M) Calculate increasing percentage of measurements.	4
6	FV - CM	(A) Calculate the volume of a solid given its lengths; (E) Calculate lengths given the volume of a solid; (N) Compare volumes or measurements.	3
7	P - CM	(O) Compare volumes given the ratio of proportionality between lengths; (C) Calculate the length given the ratio of proportionality.	3

Table 1: Classification of the questions from ENEM (adapted from Leão 2017, 25)

Each question was classified considering if it involved or not one of the seven aspects above. We also analyzed in which questions it was necessary to visualize the solid, with the drawing of the solid or not, to reveal the importance of geometric visualization as essential element to solve the question.

After mapping the questions related to the geometric magnitude volume (Leão 2017) and creating the criteria for categorization, a classification was sought for the questions. We organized them into seven groups. In general, if we look at the content of volume, the majority of questions relates to calculating the volume of a solid; nonetheless, they involve different tasks. Instead of grouping by task, we looked at techniques identified to solve the questions, seven in total, as can be seen in Table 1.

An important result, that can be seen in table 1, is that formulas (FV and FA) are very valued to solve these questions. Most of questions requires students to calculate the volume or area by giving the lengths. Meanwhile, only three out of 16 questions require students to deals with composition and decomposition of volume, which also shows a great valorization of the numeric frame even in ENEM test.

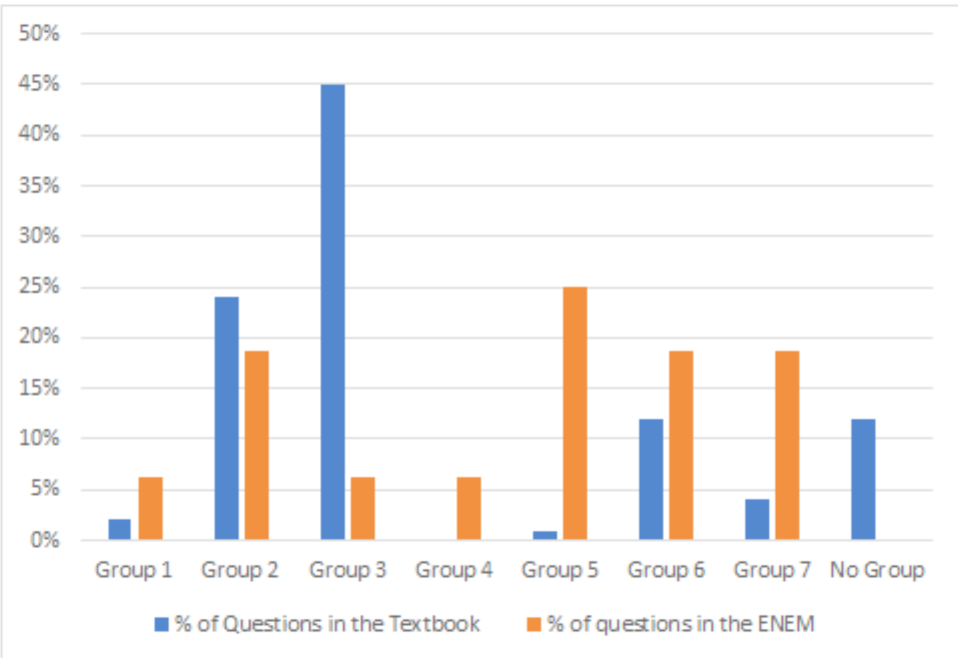
Despite the tasks to solve the questions being the same, the techniques required in the resolution of each question were not the same, nor disposed in the same order. The analysis allowed us to conclude that the challenge level of the questions is given by the chaining of tasks to solve the questions. We discovered some other aspects required in ENEM, such as connection with different areas of knowledge, proportion and percentage.

Classification of the questions of Textbook in relation to ENEM questions

After classifying the ENEM questions, we started the analyze the ones from textbooks, looking for questions that involved the volume magnitude. We looked at 91 questions that dealt with the volume magnitude, distributed in the 3 volumes of the textbook collection analyzed. Only volume 2 of the collection had the volume concept approached satisfactorily, within the whole chapter of spatial geometry.

Among the 91 questions from the textbook, only 2 are classified into group 1, which uses the formula of volume and unit transformation only. On the other hand, within the questions that use the formula and the composition and decomposition of volume, we found 22 questions, classified into group 2. In group 3, in which most questions are found, a total of 41 questions make use of the formula, comparisons additives and some properties of the solids. We did not find questions that fit into group 4, where the questions use the Cavalieri principle, volume formula, area, and proportionality, and only one question was classified into group 5, which uses, in addition to the volume formula, percentage. We found 11 questions that fit into group 6, which involve the use of formula and volume comparison. Only 3 questions used proportionality and multiplicative comparison in their resolution, without the use of the formula. We have 11 questions that did not fit into any of the groups initially formulated. These questions use other combinations of techniques than the described, that were not seen in the ENEM test.

Figure 2: Percentage of questions from Textbooks and ENEM in each group that deal with volume.



We can see with this classification that a great part of the questions presented in the textbooks analyzed the use of formula, in counterpart with the little use of different knowledge inherent to other areas of mathematics, such as percentage and proportionality, reiterating previous research that indicates the predominance of the numerical aspect.

The graph (Figure 2) brings a comparison of the percentage of questions (among those which explore volume) in each of the resources.

About the ENEM questions analyzed in this paper, we could see that volume treated as a number is present in most of the questions, as well as in the textbooks analyzed. The graph shows that the textbook concentrates its approach of volume into question from groups 2, 3 and 6 while ENEM does it from groups 2, 5, 6 and 7.

We also found that some aspects that are required in ENEM questions did not receive great emphasis, such as connection with other areas of knowledge, proportion, and percentage. In this case, we saw many questions that involve proportionality and percentage in their resolution. 5 out of 16 questions, about 6%, involve percentage. In the textbook, we found only one question that uses percentage, an ENEM question that is present in our analysis and at the end of the textbook. Only 11 of the analyzed questions used proportion in their resolution. Regarding the Cavalieri principle, we did not find any question in the textbook that explicitly use such concept, but in the explanations of the sections, the concept was widely used. The concept of composition and decomposition was seen in 3 questions and multiplicative comparison in 3 questions as well. We also verified that the textbook brings many questions from ENEM, some seen here in this work, using, beyond the formula, composition and decomposition, transformation of unit and proportion.

Observing the percentage of questions classified in each group, from ENEM and the TB, textbook, some TB questions did not fit into the groups created for the analysis of the questions of the tests. We then listed these questions, that make up a total of 12% of the questions from the textbooks, of which resolution techniques are among the techniques described in different configurations of the groups already listed.

Conclusion

In the analysis of ENEM questions we observed the predominance of the numerical aspect of volume, being present in all the questions analyzed. However, many of them also used other aspects for its resolution, such as and decomposition composition and transformation of units, as well as the relation with other internal areas of mathematics, as numbers and operations, which are used in more than half of the questions.

In the analysis of textbooks, we could note that only the volume 2 of the collection had the concept of volume approached satisfactorily, within the spatial geometry chapter. In that chapter, the use of the Cavalieri principle was requested several times in the introduction to the solids' volume, but without any mention to questions directly involving the concept. The questions involved, in general, the numerical aspect of the volume magnitude, with questions that also used composition and decomposition, a transformation of units and proportion. We did not find questions that dealt with percentage in their resolution, remembering that the analyzed questions from ENEM that involved this concept were 1/4 of the total, and may prejudice the student in the construction of knowledge to correlate different concepts in solving a problem.

In the analysis made from the groups that were listed by similar resolution strategies, we observed that the number of questions that only use formula for its resolution in the textbook is about 45%, as opposed to only 6% of ENEM questions and the questions that use percentage and formulas in its resolution in the exam is about 25% when only 1% of these questions are represented in textbooks, thus stating that the relation with other knowledge, besides the use of formula, is still being undervalued by books.

In general, the textbook satisfactorily addresses the skills and competencies required by ENEM, as far as volume is concerned, when dealing with related questions, but when the knowledge of other areas of mathematics is required, the approach is short or nonexistent. While the textbook helps the student in his preparation for the ENEM test, in relation to the magnitude volume addressed in this study, it falls short in its correlation between distinct areas, internal and/or external to mathematics.

We believe that it is necessary and relevant to advance this research regarding the expansion of the analyses, based on the increase in the number of collections of high school mathematics textbooks evaluated by PNLD. We intended to analyze the three most chosen collections by mathematics teachers in Brazil, but the time needed to complete the study made this goal impossible. Analyzing more collections would generate a panorama about how the study of volume has been approached in these works, as well as how much these have contributed to the education of high school students and to their preparation to undertake the mathematical test of ENEM.

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SECTION DEVELOPMENT OF TEXTBOOKS

REDESIGNING OPEN TASKS IN MATHEMATICS TEXTBOOKS

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Abstract

In this paper, we analyze the role of improving designing tasks processes from the perspective that tasks do not exist separately from the pedagogies and didactics associated with their proposals. We used a Brazilian textbook project called "Matemática do cotidiano". We focus on how the results of classroom implementation can help to redesign the task increasing its power for potential creative reasoning. After the analysis, we found a need for adapting the task concentrating on a degree of openness, and the need to help the teacher to manage the debates in the classroom to encourage creative thinking.

Introduction

The activities regularly presented in mathematics textbooks for elementary education are usually closed, simple and without challenges. Task design is part of recent interest in analyzing the role of the task and the teacher when using open-ended tasks in textbooks. Previous research results showed that participants who practiced with ill-structured tasks performed worse than those practicing with well-defined tasks (Boaler 2015). Many authors state that the role of teachers is to broaden the textbooks proposals by opening and challenging them. In fact, we assume that tasks do not exist separately from the pedagogies and didactics associated with their proposals.

Research into the design and use of mathematical tasks in instructional settings should accommodate student intentions, actions and interpretations to at least the same extent as those of the teacher. In fact, Lithner (2008) suggested that a key variable in learning mathematics through task solving is to analyze the reasoning that students activate in relation to specific tasks. The use of inquiry, discussion and reflection of ideas is critical to student learning. Moreover, the tasks that teachers select for their classes are fundamental and characterize their work (Stein & Smith 2009). In such a framework, how can teachers be helped to improve the mathematical creativity potential of an interesting open task by increasing cognitive issues? How can it be done in a textbook?

In this paper, our focus is on describing elements of creative mathematics thinking that appear when observing classroom experiences, thus improving the presentation of rich contextualized tasks used in mathematics textbooks. Our goal is that with these results, teachers will gain practice improving their students' mathematical creative thinking. We specify the need for including explanations to the teacher about his/her role in classroom discourse when managing such rich mathematical tasks.

Theoretical issues

To improve the epistemic mathematical quality of task design in a student centered perspective it is necessary (Barzel, Leuders, Prediger & Hußmann 2015): to use mathematical examples, connections and diversity of representations in classroom discourse (Adler & Ronda 2017); to give opportunities for reasoning and students' legitimation for promoting mathematisation and retention. It is also important that the teacher have tools for applying suitability criteria when analyzing

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mathematical activities (Giménez, Font & Vanegas 2013). On the other hand, in order to promote creativity for all students it is important to design tasks that improve the ability to recognize and define problems, generate multiple solutions and paths toward solutions, reason, justify conclusions, and communicate results (Leikin & Pitta-Pantazzi 2013).

To analyze the potential of creative mathematical thinking (CMT), we use a set of different dimensions of creativity (according Sala, Barquero, Font & Giménez 2017).

(a) *Openness, generalization and versatility*: giving opportunities for diversity and surprising solutions, which can promote generalized procedures and flexible strategies, adaptable to different uses of mathematics and groups of students.

(b) *Problematization and inquiry*: the incorporation of questioning which helps to problematize students' knowledge, as well as giving the opportunity for students to propose new questions.

(c) *Combining representations:* the need for giving opportunities to explore, use and combine different representations of mathematical objects.

(d) *Exploration and conjecturing:* ways of facing mathematical arguments and reasoning in order to increase fluidity of mathematical objects and processes.

(e) *Connectedness*: the need to establish intra- and extra- mathematical connections when possible to establish mathematical structures. Theoretically speaking, we assume that intra- and extra-mathematical connections should appear when two epistemic configurations are connected by one of their elements such as definitions, representations or arguments (Giménez, Vanegas & Font 2013). Extra-mathematical connections relate mathematical ideas to real world experiences in order to understand and improve modelling processes. Intra-mathematical connections could be using common definitions, using similar examples, same arguments or similar representations (Vanegas, Gimenez & Font 2016).

(f) *Validation*: offering tools to students for self-validation and control of their mathematical proposals, as we did for problem solving and modeling processes.

(g) *Emotion*: Giving opportunities for the activation of emotions, promoting mathematical communication and debate with emerging mathematical consensus.

Methodology

We use some tasks from a Brazilian textbook called "Matemática do cotidiano" (Lopes & Giménez 2015). They are contextualised tasks in which students should be active participants in the educational process, rather than receivers of ready-made mathematics, thus developing mathematical tools and insights by themselves. According to such a perspective, the teaching-learning process should be closer to the creativity proposals cited above.

For our study, we observe two regular classes where two geometry tasks are performed (tasks 1 and 2). Task one, was implemented with 30 students of 10-11 years of age and task two was implemented with 30 students of 11-12 years of age. In both cases, the teachers have 10 years of experience and their practice is characterized by listening and involving the students. The teachers do not have a strong mathematical background. During the implementation, we assume that the interaction process should elicit rich threads of student reasoning, facilitating an integration of the critical role of classroom milieu and related socio-mathematical norms in the conduct of the instruction (Prediger, Gravemeijer & Confrey 2015). Such a process gives opportunities for careful attention to argumentation, explanation and giving attention to generalizability of findings (according Kieran, Doorman. & Ohtani 2016). The analysis of these practices will enhance task-design, evaluation, analysis and revision of the learning arrangements.

The data

We consider the tasks and the journal writings as data of the research process. Let us explain the tasks implemented and analyzed as examples to observe the role of the task and the teacher.

Task 1. "Design the blueprint of a building according to the following measurements. Use a sheet of paper to do each place. Cut out each of the building's spaces, and paste to the building having the following: 3 rooms (3m x 3m each); 1 dining room (4mx6m); 1 dining room (3m x 6m); 1 corridor (5m x 2m); 1 kitchen (7m x 3m); 2 bathrooms (2m x 2m each) and 1 garden (3m x 12m)" (Lopes & Giménez 2015).

Task 2, "Find the maximum area defined by the pieces of pentominoes (after discussing about this shapes). This open statement promotes an immediate discussion about introducing conditions. The first possibility is to find the maximum sized rectangle having the pieces as borders. The second problem is to consider the maximum area of a "tunnel shape" as it is shown in figure 4, and the pieces being a curve creating a possible irregular figure.

We believe that both tasks are open enough for students to find intra–mathematical connections, promoting other creativity dimensions such as different representations and exploration-conjecturing dimensions, giving opportunities for creating a diversity of solutions. During the observations, we think a large diversity of individual ideas, strategies, solutions, findings, pre-concepts, etc. will also appear. It allow students, to actively and collaboratively reinvent geometrical ideas about the role of measurement, reflect about area measurement, and use multiplication procedures and spatial relations as mathematical objects.

Results and discussion

The text of **Task 1** is open and related enough to the real world to promote changes in a teacher's mathematical mind. The teacher suggested that she was surprised, because she thought that the problem had a single answer using proportional reasoning. She "*never imagined the power of children's discussion using different scales when doing their drawings due to the openness of the question*". In fact, the teacher thought that children would use a square paper, in which a square means one square meter. Thus, the teacher did not take into account the ***openness and versatility*** of the task to open not only the arrangements of the pieces but also the construction of the spaces themselves. It was unique for the teacher in that new unexpected extra-mathematical connections appeared because the children generated a discussion about the possibilities of circulating space, relating the task to everyday life, instead of simple measurement observation and scale framework. In fact, when the teacher put two possible designs (as we see in figure 1) on the blackboard, the

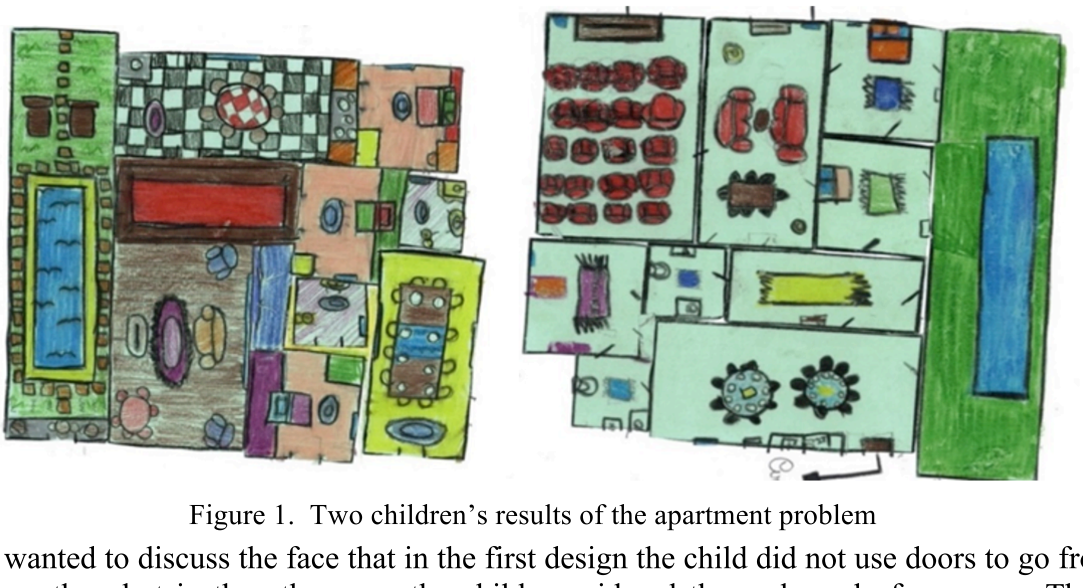

Figure 1. Two children's results of the apartment problem

children wanted to discuss the face that in the first design the child did not use doors to go from one stage to another, but in the other case the child considered the real need of passages. They also talked about the need of a possible corridor, and other inventions not present in the proposal. Some students designed a house without doors, only looking to the mathematical conditions, forgetting the real contextual conditions (as we see in the left example in figure 1). Some of these ideas were used by the teacher as ***problematization*** or inquiry, but some others not, as in the following example. In fact, many students tried to create a "compact apartment" and when possible, close to a rectangle, which is not a constraint in the initial proposal. It is a creative potential of the task that was unexpected for the teacher.

The book for the teacher had a simple comment about the task "Look at the personal answers and discuss them". The researchers understood that the experienced teacher solved and managed part of the discussion spontaneously to clarify some intentional mathematical meaning (the object scale). However, the teacher did not manage the difference of scales to see them as different viable strategies, in order to promote fluent and versatile strategies, and possible ***generalization*** of the task, therefore gaining creativity potential. It seems that in a new version of the task, some suggestions will help to invent new problems based on the constraints.

During the experience, the students clarify by themselves some of the contextual constraints, and the situation itself doing good ***connections*** without the expectation of the teacher. Some students use three bathrooms, which is not common for most families, to see a possible generalization suggesting the use of more than one stage of each room. Therefore, the teacher did not use all the power of transforming the task to promote a set of new questions as new problems suggested by the children. Such transformations give opportunities for ***conjecturing***.

Such a rich open task also provides opportunities for the teacher to teach additional new skills and for students to practice unexpected mathematic skills such as the use of the same area in different positions, giving for instance the possibility to find the unexpected relation between perimeter and area. It would help to elicit new ***intra-mathematical connections***.

We assumed that the task promoted positive ***emotions*** related to different possible right answers, present in the task itself. As a research team, we also observe that children are proposing ideas, and perhaps using technology; not only being engaged in the problem but also being involved in a creative mathematical classroom discourse. In fact, after the classroom, the children say that, "We did an interesting problem yesterday. Julia for instance told us: '*I never imagined that an area problem is important when we buy an apartment*'. Carla explains, '*I liked building my own house. I observed that Mario also made a good apartment that I like a lot with a terrace in front of the swimming pool*!'"

We know that it is not an easy task, because it provokes a compromise to see some "wrong arguments" in terms of ***validity dimension*** of creativity. For instance, some children overlap two rectangles, to see a regular shape of an apartment. We observe that it is difficult for the teacher to accept. During the school experience, the teacher in task 1 noticed that "*the task could be more challenging, by observing children's drawings: 'Where is the corridor in your answers? Did you manage to have a rectangle?"* With such proposals, we will redesign the task for the next textbook publication on the potential of creativity registered during the school experiences from a "student centred" perspective. The textbook's authors immediately state that "next edition, we will put a new sentence: 'Discuss your answers in pairs, with your colleagues'". Moreover, to add a comment for the teachers such as, "Observe not only the different possible apartments, but also the possibilities for children to create new mathematical problems and relations. Focus on the possible relations between area and perimeter".

Observing **Task 2**, after many trials, we see the students proposing different representations helping them with further visualization issues such as problematization and the power of ***openness*** to present different strategies and solutions such as the ***creative and versatile generalization*** process. During the task discussion, children use different parameters and analyze their influence on a real situation. In figure 2 on the left, a student tries to find a large hole and an internal rectangle. In the middle, a child tries to connect the pieces as an external rectangle (as we see in figure 2) however she found the measurement of a hole. In the right figure, another student discussed if it is a maximum internal area.

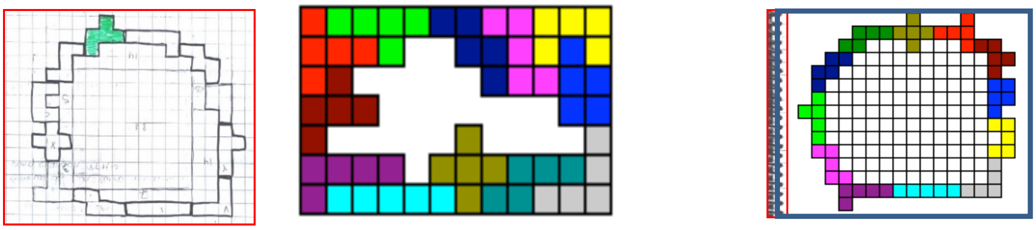

Figure 2. Three different answers for creating a large hole with the pentominoes pieces.

Thus, ***problematization*** not only appears because of the open task, but it broadens the inquiry perspective by promoting ***conjectures*** and trials for proving.

The only convincing argument for having one bigger hole than the other is the counting process. When the children discussed the problem, a new set of problems proposed by the children appears. During the task discussion, children use different parameters and analyze their influence on a real situation. In fact, rich open activities are not enough by themselves to promote it, but teacher engagement helps to increase ***openness*** when the teacher hears and legitimates the students' answers. It is difficult to include this issue in a textbook, but we tried to consider it in a section called "didactic orientations". For instance, we will propose including a comment for the teacher in a redesign, "Use the development of the task to see what happens with a rectangle as a hole? Is there a figure having the maximum size? Is it possible to see a rectangle outside and a rectangle as an internal hole? ". In figure 3, we see two trials for a maximum and minimum internal rectangle made by Joana.

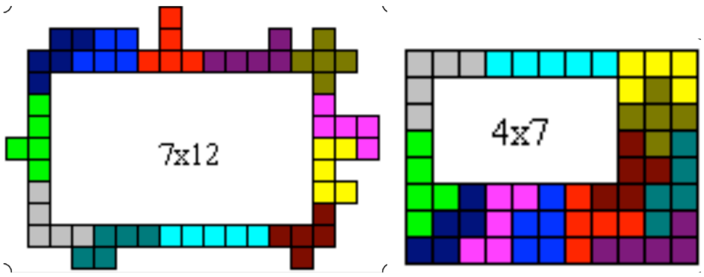

Figure 3. On the left hand, Joana found a rectangle bigger than the right one.

Joana found several rectangles as possible answers as we see in figure 3. She could not prove that the left one is the maximum area. She could conjecture that the right image corresponds to the minimum. Task 2 is also open enough to promote new questions, after observing different trials. In fact, such a task was introduced because some children proposed in a previous version. It is a good example of redesign showing a ***problematization*** process. We can see an example of a children's solution in figure 4.

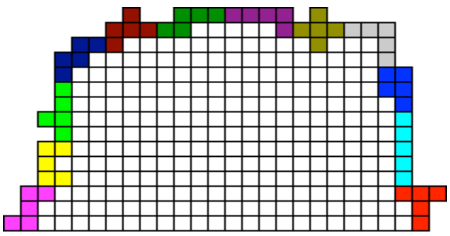

Figure 4. Conjecture about the maximum area of a tunnel shape limited by pentominoes

In such implementation, ***emotional issues*** appear when we observe some answers of the students. Let us see in figure 5. The children told the teacher "I did not solve it" (left side) or "I'm the best", "Wonderful, wonderful…" "Super good" "I created the minimum rectangle". These are just some examples of exciting answers given by the children.

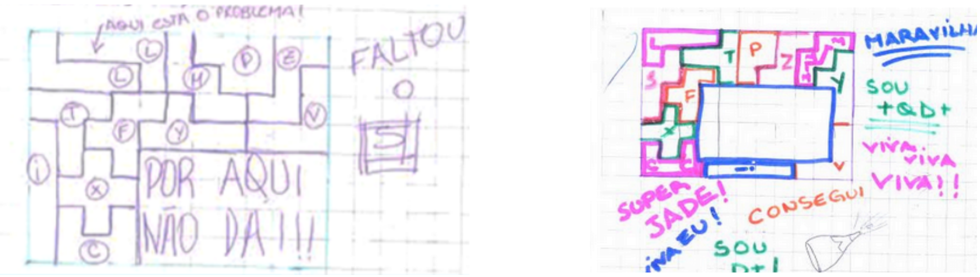

Figure 5. On the left, Gabriela's trial without a solution. On the right, Joana discovering the minimum rectangle as a hole.

We mention that the task promoted the observation of conceptual differences in 11-12 year-old students' knowledge when they reflect on the classroom process using this activity. For this purpose, the teacher in task 2 promotes the use of writings about implementation as a self-regulation process. The activity also promotes ***intra-mathematical connections*** as a creative dimension. In fact, analyzing students' work (as presented in Gabriela's work), we observe some creative dimensions when constructing mathematical processes. Joana speaks about perimeter and area colloquially, as "contour", not yet associated with the measurement. The idea of area used is counting, a consequence of the fact that the polyminos can be treated in the domain of discrete numbers. She uses her own terminology (assumed and / or created by the group) as the idea of the "track" (as a race circuit) associated with the idea of contour. Joana also explains conceptual and procedural relationships. Joana implicitly assumes that the flat figures have a perimeter and knows how to calculate the perimeter. She relates the pieces and their drawings with the quantification of possibilities, and evokes a relation between parts and motions and part invariance / rotation. Joana reflects positively on the "educational" value of working with pentominoes in the area problem and assumes that she has developed geometric problem solving skills. Finally, Joana formulates new problems used in class involving construction with the pieces and the measurement of area (minimum and maximum).

In her writing, Gabriela shows appreciation for as detailed communication as possible in her first experience of writing about her processes and her knowledge, and she links the contents of conceptual (object, relations, properties) and procedural nature. We perceived that Gabriela values recording in the form of drawings (with legends) and an appreciation for the domain of geometric terminology when it names the isometric transformations (rotation, translation and reflection), and a beginning of use of notations. She recognizes the provisional results obtained by the group of colleagues against the information of other results outside their personal context, posing a reference to the fact that there are other better results in relation to the problem of the gap limited by pentominoes, which is an open problem not yet demonstrated.

In such activities, all the students negotiated some mathematical constraints involved in the task. In fact, there are not contextual conditions, but each context implies that the students understand that many different contexts could give possible answers. We also observed that when a problem involves opportunities for using a computer tool there is a new opportunity to frame new negotiation by introducing new kinds of representations and interactions, because students interrupt the dialogues of the colleagues, showing their interest in the debate itself. In such cases, the gesture appears to be important among students. It is difficult to include such management tools in a textbook, but we must tell the teacher that it is not enough to talk about dialogue without explaining something about the type of dialogue, which increases creative potential.

Conclusions

According our results, we found that when redesigning mathematical tasks, we assume that textbooks should include some explanations for the teacher about creative mathematical potential about the tasks and about how to manage the school interactions In fact, many creative potential dimensions observed are present in the task itself as we observed in both tasks presented:

(1) ***Openness, generalisation and versatility*** appear if we express a certain degree of openness of a task and we assume the importance of the role of the teacher using textbooks. We increase openness when we present questions as general and as open as possible to provide opportunities for focusing on extra-mathematical connections, such as architectural design or gaming. In our examples, there is enough openness to promote flexible understanding. (2) As for ***problematization,*** we found that rich open tasks promote ***conjecturing inquiry*** processes, a research attitude for creative thinking as was found by other studies (Leikin & Pitta-Pantazi 2013). (3) To conserve the demand of using ***different representations,*** increasing diversity of possibilities until the moment of mathematics consensus, when the task is open enough to promote autonomy. (4) ***Exploration and conjecturing attitude*** appear in both tasks. But to give instructions for the teacher to manage the architectural debate serves to discuss what is needed to have a sustainable house, the human needs of room size according the number of members in a family, and so on. The changes of the questions reveal a *creative* conjecturing position, more than *imitative* (Lithner 2009). (5) ***Connectedness*** in some particular tasks appears because measurements relate to everyday life and the use of scale as a way to represent real world or gambling situations (6) ***Validation***. The role of the teacher is essential to encourage children to validate solutions, and even to accept that many solutions can be acceptable as a conjectured framework given from the dialogue. The mathematical consensus in natural debates appears not only when accepting the validity of the colleague's solutions, but with the student's legitimation (Adler & Ronda 2017), understood as an acceptation of students' authorship. In our implementation, we see that children's writing journals gives deep opportunities for self-regulation and validation increasing creative mathematical potential. (7) ***Emotional dimension.*** Contextualized tasks seem to be enough to engage children to grow mathematically, but it is necessary for children to be legitimate in their negotiations of mathematical meanings. The framework of a playing task activates strong emotions, because each student can reflect by using a trial and error strategy available to all.

According to our results, it is necessary to include a set of comments in the textbooks to drive an appropriate questioning dialogue in order that the teacher should redesign the management process. The debate of students' answers gives new opportunities to create new, different problems. Sometimes (as in task 1) in order to enlarge the potential of different intra-mathematical connections, it is not enough to have an open problem, but rather to manage a discussion that suggests the children invent new problems. To improve problematization, it is necessary to discuss mathematical objects, such as the role of possible different scales, or the relation of area-perimeter. There is a need for expanding generalization and inquiry exploration. Time for having a natural debate serves as self-validating mathematically their proposals. This dialogue should articulate not only the variety and originality of answers, but also discussion distinguishing the contextual and mathematical aspects. During a redesign process, it is also important that the didactic comments help the teacher to intertwine and manage the interactions in order to focus on mathematical meanings. The results suggest that the teacher who reads a textbook should know elements of the impact of the classroom experience, thus resulting in the need for increasing group validation processes and self-regulation.

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TEACHING STATISTICS IN TEXTBOOKS: THE PNLD AND THE TEACHER'S HANDBOOK

GILDA GUIMARÃES and NATÁLIA AMORIM

Abstract

This study aims to investigate whether there is influence of the prescribed curriculum (the PNLD guidelines) on the curriculum presented to teachers (teacher's manual presented in mathematics didactic collections) specifically on teaching and learning of statistics. For that, we analysed the last five editions of the school textbook guide published by the PNLD and four (4) collections of textbooks approved in the 2007, 2010, 2013 and 2016 editions. We observed that the PNLD's textbook guides indicate less competences than the didactic collections presented in all the years in focus. However, it is noteworthy that 2016 Guide shows a great expansion of the competences to be developed in the teaching of statistics, formalising some competences that already appeared in the books, and going beyond all the competences hitherto related in the didactic collections.
Key words: statistical teaching; textbook; teacher's handbook; PNLD.

1. Introduction

This study aims to investigate whether there is influence of the prescribed curriculum (the PNLD guidelines) on the curriculum presented to teachers (teacher's manual presented in mathematics didactic collections) specifically on teaching and learning of statistics. This is so important taking into account the important role played by the teacher's handbook when presenting the structure and didactic organisation of the collection, as well as theories and methodologies that can contribute to better use of the proposed activities.

The creation of the PNLD, in 1996, had as purpose the evaluation, purchase and distribution of textbooks, and, currently, has been carried out in triennial cycles, meeting all levels of basic education in Brazil. The PNLD Guidelines presents a review of the approved collections, resulting from an evaluation process carried out by teachers from educational institutions in various regions of Brazil, with theoretical comments and reflections on the collections, helping with the choice of the textbooks that teachers will use. In addition, it brings criteria used and competences to be met.

Reflecting on issues that construct a rationale about curriculum development, Sacristán (1998) states that when we define the curriculum, we are describing the functions of the school itself in a given historical and social moment, in its content and in the ways in which it organises and presents itself to teachers and students. Sacristán (1998) proposes a model to interpret curriculum organised in six levels of development.

2. About Curriculum

From a set of disciplines of a course to the organization of learning and teaching processes and their management, there are several different definitions for ‘curriculum’. As Santos (2012) argues, it is necessary to speak of curricula, since these are social practices that are established in the complexity of diverse and plural educational networks, creating tensions between the instituted and instituting processes.

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When reflecting on the curriculum or making some reference about some element that involves curricular practices, it is necessary to have clear its scope, all aspects it embraces, its influence in the school routine and in all school spaces from its construction until its materialisation into a classroom. It is necessary to understand that this construction does not take place in a neutral way or without a purpose, that it always involves political and social interests, which are constituted with the purpose of meeting certain educational goals, and that it is always in constant transformation.
Sacristán (1998) states that when we define the curriculum, we are describing the functions of the school itself in a given historical and social moment, in its content and in the ways in which it organises and presents itself to teachers and students. It is an option historically configured within a culture, politics, society and school, carrying, therefore, values and assumptions. Several agents interact on the curriculum, creating different areas of action, from its implementation with what has to be taught, through the prescriptions, to its evaluation as a way of verifying its quality, which materialises through the pedagogical practices that are constructed on those several influences. The curriculum is organised around a distribution and specialisation of content through courses, levels and modalities, and that it differ in the different levels of the school system and in the various competencies that are established by age.
In this sense, Sacristán (1998) proposes a model of interpretation of the curriculum organised in six levels, or moments, of development (curriculum prescribed, curriculum presented, curriculum modelled by teachers, curriculum in action, curriculum executed and curriculum evaluated) with different mutual degree and strength of influence, but which are always interrelated, reciprocal and circular.

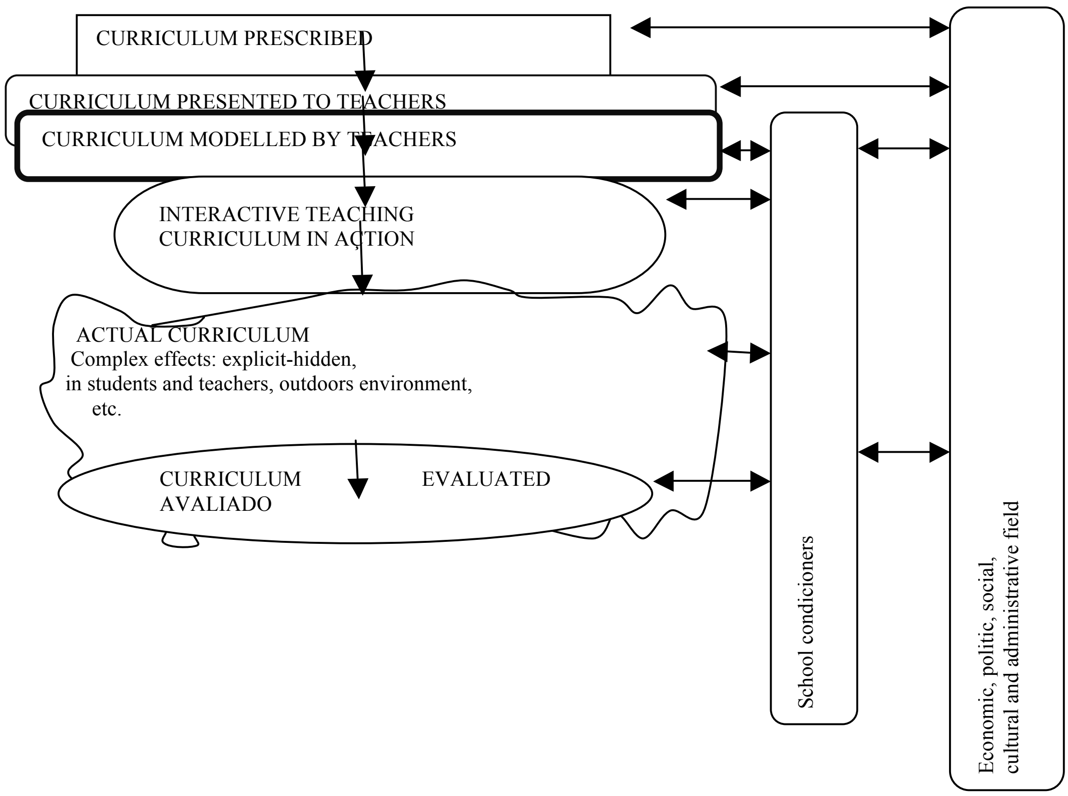

Figure 1: The objectivation of the curriculum in the process of its development Source – Sacristán 1998, p. 105.

According to Sacristán (1998), curriculum prescribed is the one that

> "in any educational system, as a consequence of the inexorable regulations to which it is subjected, taking into account its social significance, there is some kind of prescription or

orientation of what should be its content, especially in relation to compulsory schooling. They are the aspects that act as reference in the ordering of the curricular system, they serve as starting point for the preparation of materials, control of the system, etc." (p.105)

Curriculum presented is the one that has

> "a series of means, elaborated by different instances that usually translate for teachers the meaning and contents of the curriculum prescribed, performing an interpretation of it. The prescriptions are usually very generic and, to the same extent, are not enough to guide the educational activity in class. The very level of teacher training and working conditions make it very difficult to set the practice from the curriculum prescribed. The most decisive role in this regard is played, for example, by the books". (p.105)

The textbook is a very strong didactic resource in schools, and assumes the role described by Sacristán (1998) as "presenters of the curriculum pre-elaborated for teachers" (p.150). Its use is considered almost inherent to the practice of the profession, denoting the dependence of teachers on some material that structures the curriculum, develops its contents and exposes teachers to teaching strategies and methodologies.

When the didactic collections are being analysed, it is fundamental that the Teacher's Guidance Handbook is evaluated also, both in the general part and in the parts that are specific to the activities proposed in the books.

3. About the Teacher's Handbook

According to the PNLD (2016) edict, the Teacher's Handbook is a mandatory part of the textbook and plays a very specific role in the teaching process, besides presenting the student's own book to the teacher. The Teacher's Handbook should explain the theoretical and methodological assumptions underlying its didactic-pedagogical proposal. The book should be a source of reliable references, presenting a formative role, as it engages in a direct dialogue with the teacher. It should present the textbook unit by unit, activity by activity, clarifying goals, anticipating possible paths of students' development and their difficulties, helping the teacher to systematise the contents worked, discussing the relevant didactic choices, among others.

In this study, we are interested in reflecting on the influence of the *curriculum prescribed* (the PNLD Guidelines) and the *curriculum presented to teachers* (teacher's handbook presented in the didactic collections), in relation to the teaching of statistics.

4. Statistics in the research cycle

Regarding the teaching of statistics, we understand that it is essential for the formation of a conscious citizen, capable of making autonomous decisions in the face of statistical data or information presented to him/her at any moment. The need to communicate statistically is increasingly frequent in our society.

The school plays a fundamental role, because some knowledge is not characteristic to human development. School intervention is paramount for the acquisition of specific knowledge.

Amorim e Guimarães (2016) had analysed the objectives proposed in the PNLD Guidelines for the initial years of elementary education for the statistics axis in the 5 (five) last editions of the Mathematics Textbook Guidelines, years 2004, 2007, 2010, 2013 and 2016. Figure 2 shows the competence per Guidelines.

PNLD Guidelines	Teaching goals/ Competences related to the Teaching of Statistics
2004	- Know how to represent and interpret data in non-Cartesian graphs.

2007	- Work with issues relating to physical or social reality data that need to be collected, selected, organized, presented and interpreted critically. - Make inferences based on qualitative information or numerical data. - Articulate the axes of mathematics
2010	- idem to 2007
2013	- idem to 2007 and 2010
2016	- Work with issues relating to physical or social reality data that need to be collected, selected, organized, presented and interpreted critically. - Make inferences based on qualitative information or numerical data. - Articulate the axes of mathematics - Reflect on: the research question, variable types, different types of graphs and tables, the relationship between numerical line and graphs, the mean and measure of variability.

Figure 2 – Goals/competence per Guidelines. Source – Amorim & Guimarães, 2016, p. 6

The authors affirm that there are two defining moments. The first moment concerns the changes presented between guidelines 2004 and 2007, since the goals for the teaching of statistics are to provide students with a reflection on the function of statistics and not just on some types of graphic representations, like in 2004. From the 2007 Guidelines onwards we observe a concern about the actuality of the data, linked to the physical or social reality and under a research perspective, involving collection, selection, organization, representation, analysis and inferences based on qualitative information or numerical data of the data collected.

The second modification of the curriculum prescribed can be observed in 2016 Guidelines, which places research as the structuring axis of the teaching of statistics, considering the importance of experiencing research and all its stages. It stresses the importance of working with more than one quantitative or qualitative variable, creating criteria to classify, distinguishing between tables and charts, understanding scales, estimating data analysis. It also states that we must consider measures of central tendency related to the amplitude, and its meaning as a descriptive measure of a set of numerical data.

This study aims to reflect specifically on the influence of the curriculum prescribed (the PNLD Guidelines) and the curriculum presented to the teachers (teacher's handbook presented in the didactic collections), taking into account the important role played by the teacher's handbook when presenting the structure and didactic organisation of the collection, as well as theories and methodologies that can contribute to better use of the proposed activities.

5. Method

We analysed, for this research, 4 (four) collections of mathematics textbooks, including books from 1st to 3rd year, approved in the 2007, 2010, 2013 and 2016 editions, making a total of 76 books. The criterion to choose the collections was that they were the best selling, therefore, probably the best used, textbook collections in Brazil.

Thereafter, we compared the proposals presented in the PNLD Guidelines with the 5 (five) last editions of the Mathematics Textbook Guidelines, years 2004, 2007, 2010, 2013 and 2016 and the guidelines for teachers presented in the Guidance Handbooks.

To carry out the analyses with the teacher's textbook, we tried to identify the concepts or objectives presented in the general part and in the specific part addressing the teaching of statistics[1] (information processing). In Figure 3, we give an example of a unit of analysis presented in the general part. From this paragraph, we can affirm that the collection aims to propose the learning of

[1] The mathematical axis that addresses the teaching of statistics was called information processing.

different types of graphs, construction and interpretation of a table, construction and interpretation of graphs, classification, data collection and research.

Figure 3: Sample from the general section of the teacher's manual

> As tabelas e os diferentes tipos de gráficos devem ser construídos e interpretados pelo aluno como um recurso capaz de resumir, apresentar e classificar dados coletados numa pesquisa. Em especial, os gráficos permitem uma rápida impressão visual; apresentam de forma imediata, mais rápida e simples esses dados coletados.

Source – collection *Ápis*, Volume 2, 2010, p. 49.

In Figure 4 we give an example of a unit of analysis presented in the specific part, in which we can affirm that the collection aims to articulate axes of mathematics, different types of graphs, interpret a table, fill in a graph or a table, collect data and perform research.

Figure 4: Example of text of the specific part of the teacher's handbook

> A atividade 7 da página 205 também faz uma integração de números (*números e operações*) com tabelas e gráficos (*tratamento da informação*). Nesse caso, trabalhamos com um novo tipo de gráfico – o **gráfico de setores** ou **gráfico de pizza**. Dê tempo aos alunos para interpretarem a tabela, pintarem os setores já delimitados do gráfico e depois responderem às perguntas. Estimule-os a fazer pesquisas ou enquetes e registrar em tabelas e gráficos as informações coletadas.

Source – collection *Plural*, Volume 2 , 2016, p. 419.

In this way, all texts referring to statistics were identified. Firstly, it was analysed which concepts were cited and then analysed the conception of education. Furthermore, we observed if the teaching conception addressed had as a presupposition the learning of statistics should consider all phases of the cycle of research, stimulating the students to reflect on the function of the research.

6. Results

We started by establishing relationships with the goals related to the teaching of statistics presented in 2004 Guidelines and the Teacher's Handbook of the 2007 collections (Figure 5). The 2004 Guidelines had only two goals: to represent and interpret data in graphs. All collections referred to these goals, and three of these collections refer to interpretation of the data, among the data and beyond the data. The conclusion is that the collections meet the goals indicated in 2004 Guidelines, and most go further still, specifying different types of data interpretation. In addition, the handbooks of the 2007 collections still refer to a large number of goals that must be developed in teaching statistics.

Goals in PNLD Guide	**Collections 2007**				**Collections 2010**				**Collections 2013**				**Collections 2016**			
	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
real data	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
articulation between axes		x	x	x			x	x			x	x			x	x

articulation between areas	x	x	x		x	x	x	x	x		x	x	x	x	x	x
research	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
aim	x		x	x	x	x		x	x	x	x	x	x	x	x	
sample			x												x	
method of collecting	x	x	x			x		x	x	x	x	x	x	x	x	x
instrument of collection							x									
collect data	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
classify	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
variable Type	x	x	x								x			x	x	
graphic representation	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
graphic interpreting	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
tables representation	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
tables interpreting	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
graph types			x				x				x	x	x	x	x	x
scale		x		x												
conclusion			x				x				x		x	x	x	
inferences			x				x				x				x	

Figure 5 - Comparison between the PNLD Guides and the teacher manuals

Comparing the 2007 Guidelines with the handbooks of the 2010 collections, we observe that all of them refer to work with actual data, articulation between the axes of mathematics, data collection, data organization/classification, representation and interpretation of tables and graphs, as proposed in the Guidelines. The handbooks for the 2010 collections refer to most of the competences related to teaching of statistics presented in the 2007 Guidelines. Two collections do not refer to the articulation between the axes. Again, the handbooks for the 2010 collections refer to other competences related to "information processing" learning that are not explicitly stated in the 2007 Guidelines. There is also an absence of reference to the sample and scale occurred in the 2007 handbooks.

Comparing the 2010 Guidelines with the 2013 collection handbooks, we can see the need to work on issues related to data from the physical or social reality that need to be collected, selected, organized, presented and interpreted critically. There must be inferences based on qualitative information or numerical data, articulation of mathematics axes, and work with different tables and graphs. We can show that half of the collections meet the guidelines' indications.

The handbooks for the 2013 collections refer to most of the competences related to teaching statistics presented in the 2010 Guidelines. Two of the collections refer neither to the articulation between the axes nor to different types of graphs. No collection refers to different types of tables, and Collection 3 is the only one that refers to the need to make inferences. Again, the handbooks for

the 2013 collections refer to other competences related to the "information processing" learning. Reference to the type of instrument is absent from the handbooks of 2010.
In observing the 2013 Guidelines related to the handbooks of the collections of 2016, we note the permanence of the same goals, such as the need to work with questions concerning physical or social reality data that need to be collected, selected, organized, presented and interpreted critically. Inferences must be made based on qualitative information or numerical data, articulation of axes of mathematics and different tables and graphs.The handbooks for the 2016 collections refer to most of the competences related to teaching statistics presented in the 2013 Guidelines. Two collections do not refer to the articulation between the axes. Note that all collections refer to work with different types of graphs. No collection refers to different types of tables.
Again, the handbooks of the 2016 collections refer to other competences related to "information processing" learning as articulation between areas, work with research, delimiting objectives, methods, types of variables and conclusions.
In a differentiated, 2016 Guidelines presents in-depth proposals on the teaching of statistics. In addition to the competences presented in previous years, 2016 Guidelines list, in item "Content Approaches": developing a research, formulating questions, defining (qualitative and quantitative) variables, describing measures (by arithmetic mean), integrating statistics, probability and combinatorial, establishing categories of variables (creating criteria for a classification), discerning between tables and charts, changing from chart to table and vice versa, understanding scales, mean and amplitude, conclusions.
We observed that the PNLD's textbook guidelines indicate less competences than the didactic collections presented in all the years in focus. However, it is noteworthy that 2016 Guidelines show a great expansion of the competences to be developed in the teaching of statistics, formalising some competences that already appeared in the books, and going beyond all the competences hitherto related in the didactic collections.
With regard to the analyses of the teacher's handbook and the teaching objectives presented in the guidelines on information processing, we found that the general and specific part details better activities and suggestions of activities and readings present in the collections in all the years, thus meeting criteria established by the PNLD edict.
Comparing the propositions of the PNLD guidelines with the textbook guidelines for teachers, we observe that the guidelines and the textbooks are consistent regarding the goals presented for the teaching of statistics. The handbooks refer to what is explained in the guidelines, either in the general part or the specific part, or in detailing and explaining an activity further, in the 4 collections analysed.
We can identify that although the handbooks refer to more concepts than the guidelines, when analysing the activities we also noticed a great focus on activities related to graphs and tables, compared to the ones of construction of the research. All collections mention research, but the student is rarely asked to conduct his or her own research. Some phases of the research are not explored, such as research question, scale or sample. Few collections are able to present activities that require that students draw their conclusions. We hope that the publication of the 2016 Guidelines incentivises the next editions of the collections to incorporate and discuss all phases of a research.
However, it is important to note that this study is only a small sample of the collections approved. Nevertheless, we could observe differences between the collections, evidencing that the exclusion criteria are very broad. Those criteria can accept different methodologies and approaches, disapprove collections with conceptual errors, error induction, outdatedness, prejudice or discrimination of any kind; comply with legislation, guidelines and official standards relating to elementary education; accept coherence and appropriateness of the theoretical-methodological approach assumed by the work, with regard to the explicit didactic-pedagogical proposal and the goals; respect to the interdisciplinary perspective in the presentation and approach of the contents.

We believe that it is necessary to find ways to encourage managers, educators and authors to alter their proposals, since it is necessary to change the activities related to statistical education (information processing) for a quality education. Researchers such as Lopes (2012), Kinnear and Clark (2014), Barreto and Guimarães (2016), Evangelista and Guimarães (2015), Leavy and Sloane (2015), Cabral (2016), among others, highlight that children are able to understand much more than what has been proposed for this level of education.

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LINGUISTIC, CULTURAL AND PEDAGOGIC DIMENSIONS OF GEOMETRY: NAVIGATING TEXTBOOK DEVELOPMENT IN A CROSS-NATIONAL PROJECT

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Abstract

In this paper we explore the complexities of cross-cultural collaboration in a textbook development project involving Kazakh teacher-authors and UK consultants. Communicating through a translator brings with it inevitable problems. However, working to resolve communication difficulties revealed that these problems arose not only from linguistic differences but also from fundamental cultural differences in our understandings of mathematics and of pedagogy. We reflect on some examples from our work with Grade 7 Geometry.

Introduction

The Secondary Education Textbook (SET) project began in November 2014 as a collaboration between University College London Institute of Education (UCL IOE) and the Nazarbayev Intellectual Schools (NIS), a non-governmental organisation in Kazakhstan with responsibility for developing curriculum, assessment and pedagogy in its own group of schools for high attaining students, in support of general national educational aims. NIS has worked and continues to work with a number of international partners to support its objectives, drawing on best international practice, with a major aim not only to develop high quality education within NIS schools and more widely in Kazakhstan but also to develop capacity within the country to sustain and grow high quality teaching, teacher training, curriculum and assessment development and production of teaching resources (Bridges 2014). In the case of the SET project, the collaboration aimed not only to write and publish new textbooks for 10 secondary school subject areas but also, importantly, to support the development of local textbook authors and of textbook publishing in Kazakhstan.

This paper focuses on the work of the SET mathematics team, comprising a group of six authors (one of whom is third author of this paper), two UCL IOE consultants (first and second authors) and a member of the NIS translation service (fourth author) who served as interpreter during team working sessions and translated written materials throughout the project. As we developed our collaborative work to produce a mathematics textbook for Grade 7 NIS students, we encountered a number of difficulties. In this paper, we describe and discuss some of the major sources of difficulty, focusing in on Geometry as a particularly problematic – but interesting – area of the curriculum. Our reflective analysis of the issues arising during our work identifies both cultural and

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linguistic factors that affected communication within the team. This raises issues that need to be taken into account in such cross-cultural projects.

Curriculum, Pedagogy and Assessment

A key factor in the success of educational reform is the extent to which curriculum, assessment and pedagogy are aligned (Barnes, Clarke & Stephens 2000). The ambitious work undertaken by NIS has sought to develop these three elements simultaneously, drawing on international research and expertise, while valuing the existing knowledge and expertise within Kazakhstan and the cultural heritage of the nation. In the case of mathematics education, there is justifiable national pride in the success achieved by many students in a mathematics curriculum that has drawn on Russian traditions together with a consequent desire to retain the scope and rigour of this curriculum. There is, however, an ambition to reform the curriculum as a whole to develop students' '21st century skills' as well as traditional academic content. An important part of the brief originally given to the UCL IOE consultants was to ensure that the new textbooks would support forms of pedagogy consistent with this ambition, including inquiry-based learning and development of student independence and creativity. In mathematics a central aim was to bring out the 'big ideas' of mathematics and to develop students' thinking skills while engaging with these big ideas.

In advance of the writing of a new set of text books, the NIS had developed a new curriculum for all its schools. In mathematics, the new curriculum added problem solving to an extensive set of facts and skills, also specifying the order in which topics were to be studied. A new programme of assessment had also been developed, with tests set at the end of each of the four terms that make up the school year, based on the curriculum objectives mandated in the course programme for that term. The mathematics team of authors and consultants were necessarily constrained by this packed curriculum. The teaching and the textbooks had to cover the curriculum objectives and cover them in a defined order; this restricted our freedom as textbook designers and challenged our ability to develop the desired pedagogy within the textbooks

Negotiation and compromise

A critical feature of our growing collaboration was the development of mutual respect for the experience and expertise of each side in the partnership. On the one hand, the UCL IOE consultants brought considerable experience of a range of educational contexts (schools, further and higher education) in the UK and elsewhere in the world (Mozambique, Ghana, South Africa and Brazil) and expertise in teacher education, curriculum development and the design and production of teaching resources. We also brought knowledge of a body of international research in mathematics education – though we have come to realise the domination of this research by Anglophone and Western European traditions, often silencing knowledge arising from other traditions, including in particular that of the ex-Soviet republics. The group of authors on the other hand, brought their insider knowledge of the education system and traditions of Kazakhstan, the cultural norms of Kazakh society in general and of educational contexts in particular, as well as many years of successful classroom experience as teachers in a variety of schools across the country.

The work of designing a textbook had to draw on the experience of the team but, as we came to understand better the differences between our experiences, we realised that many compromises were necessary on both sides. While the authors aligned themselves with the NIS aims and appreciated the value of pedagogic approaches suggested by the consultants, we all realised that there was no point in attempting to impose a pedagogy that would challenge core cultural values or deskill teachers. For example, an important area of difference lay in our expectations about the development of mathematical concepts. This issue arose early in the project during the writing of a chapter about indices as the chapter author sought to introduce a^0. The authors as a group, drawing on their teaching experience, set a high value on students learning the definition and practising its application; they contended that, once learned, Kazakh students will always remember and apply the definition of a^0 correctly. On the other hand, the consultants saw the introduction of a^0 as an opportunity to model an inquiry-based approach as well as to build conceptual and procedural knowledge, suggesting: Let the student try to answer the questions ‘What is the value of a^0?’ ‘How can you multiply a by itself zero times?’ However, the authors knew that many teachers in their schools would not be ready for such open-ended discussions in the classroom. After a long discussion we reached a middle ground acceptable to both, involving definition and practice together with guided opportunity for discussion.

A lesson that the consultants had to take on board was that the Kazakh team own the project and it had to be their textbook. As advisers and mentors, consultants can use persuasion, argument, research knowledge and experience to seek a middle ground but not to take over the writing of a unit. We had to search for compromises that would be both ‘effective’ and ‘permitted’. These compromises demanded extensive discussion and negotiation – made additionally challenging by language differences. All the authors had Russian as their first language and, while they had some facility with English, this was not usually sufficient to support effective negotiation. The role of the interpreter was thus critical but was not straightforward.

For example, creating common understanding in the topic area of ratio and proportion challenged us all, including our interpreter. In England, the National Strategy for teaching mathematics, a government initiative to strengthen mathematics learning (DfES 2001), had placed great emphasis on students knowing, understanding and being able to apply definitions of *ratio* and *proportion* and being able to distinguish between the two concepts. According to these definitions, ratio is used to compare two parts (the ratio of girls to boys in a classroom) whereas proportion is used to compare one part to the whole (the proportion of girls in the class). When the consultants attempted to introduce this distinction into the relevant textbook chapter, the author team said that they did not use two words but only *отношение (ratio)*. This prompted us to ask if this difference was linguistic or mathematical: was the same word being applied to two distinct concepts or were we conceptualising the domain itself in different ways? For an interpreter who is not himself a mathematician, catching the nuances of meaning in both English and Russian was undoubtedly difficult. After discussion, recourse to the internet and to a range of Russian textbooks, we agreed that the Russian dictionary translation for *proportion (пропорция)* appeared to be used in contexts where UK English speakers would use the adjectival phrase *in proportion*. In Russian text books, *пропорция* is defined as “an equality of ratios”. In English text books, two figures are defined to be *in proportion* if the ratio of their equivalent measurements is constant. The Russian noun *пропорция* and the English noun *proportion* refer to related but distinct objects, while the distinction made by the English National Strategy between ratio and proportion was not found to be conceptually significant within the Kazakh mathematical tradition.

Finding a common mathematical language

“Kazakhstan is the only post-Soviet country that is still poly-lingual …. Other Central Asian countries speak their native tongues.” (President Nursultan Nazarbayev, Astana Times, 21 October 2013). Within Kazakhstan, ethnic Kazakhs make up 66% of the population. While 85% of these can speak and write Russian and over 94% understand it, only 6% of the ethnic Russians, who make up 25% of the population, can read and write the Kazakh language (Lillis 2010). This means that public and government life, including the education system, is still dominated by the Russian language. In 2011 the Government of Kazakhstan put forward a programme to create a trilingual country with Kazakh designated as the national language, Russian as an official language used alongside Kazakh in state and local government affairs and English as an international language that would enable the people of Kazakhstan to benefit fully from international economic, educational and political opportunities. The NIS schools took up the challenge to create a trilingual education system with the aim to have teaching and textbooks available in the three languages. The vision of the SET project is that the student text books will be published in the three languages of Russian, Kazakh and English. This vision was originally based on the naïve view that the books would be written in one language and then translated into the other two. Our experience shows that this process will not produce equivalent meanings. The interpreter on the project has found that in moving from Russian to English or vice versa he has to negotiate the specialised mathematical terms needed to convey the desired meaning in each language. The challenges of translation into Kazakh have not yet been addressed. We are now building a glossary of mathematical vocabulary that give correct and equivalent meanings for a concept in Russian and English. But this is not straightforward – as exemplified above in the case of *proportion*.

Taking another example, in English textbooks the word *line* is used loosely. On the one hand it is used to mean a line in its strict Euclidean sense, having no thickness and extending infinitely, but it is also used to refer to a finite line with end points drawn on a sheet of paper. The word may also appear in a range of everyday contexts, referring to objects that may have very different properties (e.g. railway line, washing line, etc.). A teacher can say "Draw a line with length 5 cm", with the understanding that the line should be straight unless stated otherwise. In Russian, however, these different types of 'line' are named and defined separately. A straight line is written as *прямая линия* (literal translation *straight line*) but is more usually known as *прямая* (literally *straight*); this term is only ever used to represent a Euclidean straight line that extends infinitely. A line that has an endpoint is known as a *ray* and two end points as a *segment*. All Kazakh teachers know this and teach their students to always use the correct term. While these terms are part of the mathematics register in English, they are not commonly used in school mathematics in England.

The challenge of geometry

The Kazakhstan curriculum until recently had a strong Euclidean orientation to geometry, accompanied by a pedagogy based on establishing definitions and theorems, followed by application of definitions and theorems to solving problems. Before the introduction of the new curriculum, the mathematics curriculum in Kazakhstani schools had been compartmentalised into the separate subject areas of number, algebra and geometry. In secondary schools, students had 3 hours of number and algebra and 2 hours of geometry each week. In contrast, the new curriculum provides an integrated curriculum consisting of units of number, algebra and geometry distributed through the year. In the first two terms of Grade 7 there are only 13 hours of geometry in total and in the last two terms there are 45 out of a total of 90 hours of mathematics teaching. The consultants saw the integrated curriculum as a positive step, enabling more connections to be drawn between different areas of mathematics. They did not appreciate how difficult it would be for Kazakh teachers to adapt to this new curriculum model – a model that appeared to downgrade geometry in particular. When the integrated curriculum for mathematics was adapted and extended to national government schools the mathematics teachers in these schools argued strongly against it – and were listened to. In February 2017 the ministry of education responded to this discontent, agreeing to once more separate the teaching of geometry from that of number and algebra, although the NIS schools will continue to introduce an integrated curriculum. When a similar change from separate to integrated curriculum was made in England (during the late 1960s and early 1970s) the process was extended; schools were able to choose between separate and integrated syllabuses for a period of at least a decade, allowing schools and teachers time to adapt their teaching. In contrast teachers in NIS schools are being required to make the change in a single step from one year to the next.

Extensive work during the early stages of the SET project enabled us to agree common understandings of the curriculum objectives and strategies for incorporating aspects of the desired forms of pedagogy into the first chapters of the Grade 7 textbook. As we started to work on geometry chapters, however, the process of negotiation became much more difficult. Initially, in analysing the curriculum objectives for geometry in order to plan the relevant chapters, the consultants found the English translation of the objectives to be unclear and the objectives themselves to be unfamiliar. There was a need for the team to analyse the curriculum together in order to achieve a common understanding. It became clear that, while language differences were an issue, they were intertwined with differences in our cultural expectations and knowledge of geometry. Such differences had not arisen to the same extent in other areas of the curriculum. As our discussions progressed, we found that we moved from debates about pedagogy to debates about mathematics. In this paper, we will illustrate the complexity of relationships between language, culture and pedagogy with three examples.

Example 1: Angle

According to the NIS curriculum, students are first introduced to angle in primary school. The concept of angle that they encounter at this stage is the space between two sides meeting at a common vertex. They work with lots of real life examples, particularly looking at the hands of the clock, finding the angle between the minute hand and hour hand. In Grade 7, students are introduced to a formal definition of *angle* as *the part of the plane delimited by two rays meeting at a common point*. Kazakh learners recognise that a ray extends infinitely from its start point and hence an angle is part of an infinite plane. From the perspective of the consultants, this definition provides just one way of thinking about angle. The cultural expectations of the consultants, based in the UK education system, as well as our familiarity with a body of research in mathematics education (e.g. Magina & Hoyles 1997; Mitchelmore 1998), led us (CM and TS) to see this definition as an insufficient basis for developing an understanding of angle that would support progression.

In the first place we saw difficulty in applying this definition of angle as a static space to the kinds of practical applications of angle that Grade 7 students are expected to deal with, such as the angle a door turns through as it opens or the angle at which an aeroplane takes off from the ground. Indeed, our whole approach to thinking about angle measurement was closely tied to a dynamic concept of angle as a rotation – a concept never explicitly addressed within the geometry curriculum in Kazakhstan. This difference in conceptualisation of angle is reflected in the language: learners in England talk about 180 degrees as a 'half turn' while a Kazakh learner will refer to an angle of 180 degrees as a 'flat angle'.

Further, the consultants also anticipated problems as students progressed to later topics in geometry. For example, in the NIS curriculum a *triangle* is defined as *the part of the plane delimited by three lines that meet at three points* (each pair of lines meeting at a point). As in the case of the definition of *angle*, the focus of this definition is on the space delimited by lines rather than on the lines themselves; when a primary school child in Kazakhstan is asked to draw a triangle, they will always colour it in. So, while a triangle is a finite part of a plane, an angle is a part of a plane that is infinite. With these definitions, a triangle *marks out* three angles but does not *contain* angles. Although the Russian word for triangle is *Треугольник* (literally *three angles*), its curricular definition does not mention angles. Yet when students meet the topic of congruent or similar triangles they have to work with the idea that pairs of triangles 'have' angles that are equal. The textbook authors, drawing on their own educational experience and years of successful teaching, were confident that their students would be able to cope with these apparent inconsistencies. Moreover, a very experienced headteacher of a Kazakh primary school assured us that, over many years of teaching, she is confident that students, once they have learned a formal definition of angle, have no difficulty adapting their concept of angle to work in practical situations. The consultants still find this hard to accept. Our scepticism draws not only on our personal experience but also on European and North American research on how students move from practical to abstract understanding of angle (e.g. Mitchelmore & White 2000). However, we have not found any comparable research in Kazakhstan or other post-Soviet contexts and have to respect the experience of our Kazakh colleagues. As a team we agreed that the text book for Kazakh schools should continue the current approach of giving students a formal definition but would also provide guidance for teachers to support students in moving from abstract to practical applications.

Example 2: The linear function *y=kx+b*

Our second example, while drawn from the part of the curriculum labelled as algebra, also involves the concept of angle. It relates to the curriculum objective "Know the definition of a linear function $y = kx + b$, plot its graph and predict its position depending on *k* and *b*". The coefficient of *x* is named in Russian as *угловой коэффициент*, translated literally into English as *angle coefficient.* In Grade 7, students are shown how to draw the graph of a straight line given its equation $y = kx + b$.

They are told that k, the coefficient of x, is called the *angle coefficient*. They are expected to learn that, when the angle coefficient is positive, the angle between the Cartesian line and the x-axis is an acute angle and, when the angle coefficient is negative, the line makes an obtuse angle (see Figure 1). However, the more general issue of how the value of k relates to the size of the angle between the line and the x-axis is not addressed and it appears that the value of k is not related geometrically to any measure of the steepness of the line or to the fact that the line has constant steepness.

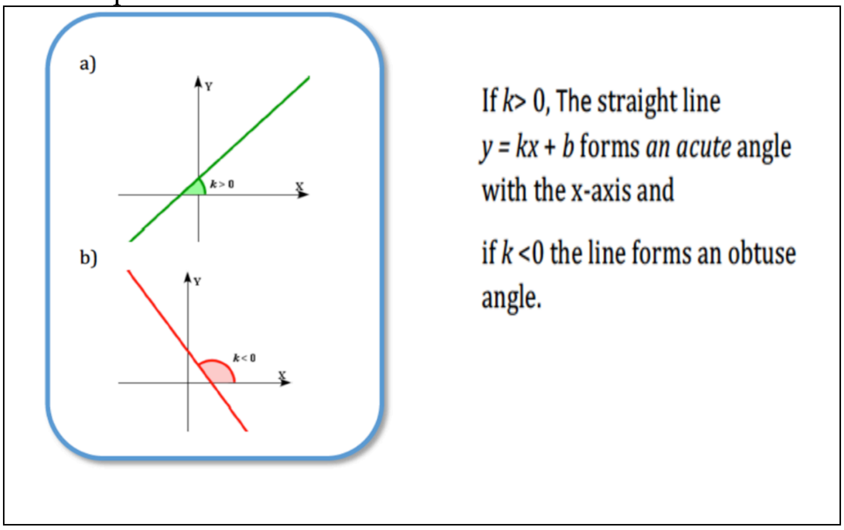

Figure 1: Textbook extract showing the Kazakh approach to gradient in Grade 7

In the Russian mathematics register there is a specialised term for *gradient* (*градиент*) but this term is only introduced to students in upper secondary school when they begin to study calculus, to be used exclusively in the context of gradient function and differentiation. This specialised term also does not appear to be used in geography or topography, contexts in which the English word *gradient* would become familiar to students in England.

In contrast, in England the word *gradient* is introduced in lower secondary school to describe the steepness of a line; the value of the gradient is equal to the coefficient of x. Learners are expected to understand that, as $|k|$ increases, the line becomes steeper. Rather than considering the angle made by the line with the x-axis, they are likely to learn that when k is positive the line slopes 'upwards', while it slopes 'downwards' when k is negative. They may also to be asked to calculate the gradient of a given line by drawing a right-angled triangle with its hypotenuse on the line and finding the ratio between its vertical and horizontal sides. Connections are likely to be made with use of the term *gradient* in concrete contexts such as the steepness of hills.

Again, the consultants experienced conflict with their assumptions about this area of the curriculum, including the importance given in English mathematics education and in the research field to making links between different topic areas and to helping students to move between different representations of mathematical constructs (e.g. Skemp 1976; Acevedo Nistal et al. 2009). For example, it was difficult to accept that students in Kazakhstan were expected to calculate the coefficient by purely algebraic means without any reference to the geometry of the line. Finding a way to resolve our differences is still in progress.

Example 3: Adjacent angles

Our third example also involves angles, or, more specifically, the term *adjacent* used in the context of the study of angles and triangles. In England, the most common use of *adjacent* occurs when working with right angled triangles, distinguishing the side *adjacent* to a given angle from the *opposite* side. Teachers in English schools will often support students to understand the word *adjacent* by reference to its everyday equivalent *next to*. The term *adjacent* itself is not commonly used in everyday speech but is found in a range of formal contexts in English so is likely to be familiar to secondary school students. In Russian, a similar use for distinguishing the shorter sides of right angled triangles is found, translating *adjacent* as *прилежащий*. Etymologically, the

Russian term can be interpreted as 'lying next to' but *прилежащий* is an archaic word, now used only in mathematics. In both English and Russian, *adjacent* (*прилежащий*) *angles* are defined as non-overlapping angles with a common side and vertex, although this usage is not common in schools in either England or Kazakhstan.

Our problem arose when we began to address the curriculum objective given in the English version of the curriculum as "know the definition of adjacent and vertical angles, recognise and draw them". We all initially believed that we understood this objective but soon found that we were at cross-purposes. The consultants and the author team needed to find a common understanding of this objective.

It eventually emerged that the origin of our difficulty lay in the fact that there are two mathematical terms in Russian, both of which are translated into English as *adjacent*. In the Russian version of this particular curriculum objective, the word *смежный* was used, not *прилежащий*. This word is found in everyday usage; *смежный* is equivalent to *bordering* or *close* (e.g. *смежные участки* – neighboring plots), making *adjacent* an appropriate translation. In mathematics, however, the term *смежный* is used in a specialised way to refer to *adjacent supplementary* angles, that is, angles with a common side and vertex, whose two other sides lie on a straight line. In English schools, such angles would be called *angles on a straight line*. In this case, once the linguistic issues were understood, we were able to proceed relatively smoothly to a common understanding of the topic as we all identified angles in this particular relationship as a distinct object of study for Grade 7 students.

Concluding discussion

The SET project started with an assumption that the consultants from England were 'experts' – often referred to by the NIS team as 'trainers'. Although there was agreement that the project should recognise, respect and make use of the 'local' knowledge and experience of the Kazakh authors, the location of 'expertise' initially created an asymmetry in the status afforded to the kinds of knowledge brought to the project by the two groups. On starting to work together, however, it was immediately apparent that the success of the project depended on negotiating a common understanding of the mathematical content of the curriculum that would enable teachers in Kazakhstan to make effective use of the pedagogic developments that we hoped the new textbooks would embody. Coming to this common understanding required deep and flexible engagement by both English and Kazakh team members, each reflecting on their own well-established mathematical knowledge and seeking to understand the different conceptions of others. Without a common language, the work of the interpreter/ translator has been crucial, not only in the basic sense of allowing us to speak to one another, but also making use of his expert understanding of languages in order to help us all to probe the differences in how language is used to map the mathematical world in English and in Russian.

In this negotiation the project of authoring a textbook across cultures has drawn attention to:

- how much we take for granted common understanding of words – within our own language as well as across languages and cultures;
- the fact that we are not always clear ourselves about how we use words and the wider implications this use may have.

It has also forced us to face up to and question our assumptions that our own mathematical meanings are correct. There are alternative ways of conceptualising mathematics, each of which has different consequences. Encountering and exploring these alternatives has enabled all of us to become more aware of our own conceptualisations and to develop a broader understanding of the mathematical affordances of alternative ways of speaking.

Whereas mathematics education researchers have previously considered the mathematical affordances of different languages, such studies have generally focused on non-European languages and on broad characteristics of everyday usage rather than on the details of the mathematics register (e.g. Barton, 2008). Comparison of European and non-European languages can highlight major structural differences, such as those described by Lunney-Borden (2011) for the North American language Mi'kmaq, that immediately raise questions about cross-linguistic and cross-cultural communication. Nevertheless, there is a common assumption that, once we start to deal with formal mathematics, we all share common meanings.

We can never hope to understand Euripides plays in the way they were understood by their original audiences, but Euclid's *Elements* speaks to us as clearly as it did to his contemporaries. Chinese poetry is untranslatable; but T.D Lee's lectures on particle physics and quantum field theory, originally given in Chinese, lose nothing in translation to English. (Layzer, 1989: 126)

However, the coming together of two traditions with highly developed mathematical cultures and two languages, English and Russian, both Indo-European languages with highly developed mathematics registers, challenges Layzer's claim and highlights the fact that mathematics is *not* a universal language. Cross-linguistic collaboration demands translation between languages but also close attention to possible differences in the uses of words that appear to be equivalent and a recognition that we do not all categorise or conceptualise mathematical phenomena in identical ways.

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INTEGRATING THE CONCLUSIONS OF TEACHERS' FEEDBACK INTO THE NEW MATHEMATICS TEXTBOOKS

GERGELY WINTSCHE, DÁNIEL KATONA and GERGELY SZMERKA

Abstract
We present a short overview about the current trends of the development of mathematics textbooks. Afterwards, we discuss the historical circumstances of the relatively new Hungarian textbook market. The first author has developed new learning tools within the framework of the Social Renewal Operational Programme (SROP) 3.1.2/ B-13 in Hungary, funded by the European Union between 2013 and 2016, which project involved not only the writing and printing of new textbooks and constructing their digital background materials, but also the exploration of teaching practice in connection to the usage of the new textbooks, collecting and analysing feedback, and monitoring the teaching process. The present paper focuses on the usefulness of official feedback on the mathematics textbooks, collected from teachers, and a short summary is presented about the changes in the revised textbooks influenced by the feedback.

Keywords: textbook development, mathematics textbooks, teachers' feedback, SROP project
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1. Theoretical Background – Teachers' Role in Textbook-Edition

In the textbook development process the textbook is not the only participant. As it is mainly for enhancing the teaching and learning of mathematics, being mainly used in schools and at homes, teachers and students are also decisive participants of the process. This idea is formulated in the form of the didactical triangle (Rezat 2008, p. 177, Schoenfeld 2012). It can be further expanded by a new dimension, the mathematical knowledge, into a tetrahedron, as it was presented by Valverde (Valverde et al. 2012). We suggest a little bit more elaborated model, where the didactic expert is placed in the centre of the tetrahedron (see Figure 1).
According to the traditional way of developing textbooks in Hungary, the materials have been created by small groups of didactical experts and/or teachers, and have reflected solely their attitude to and view on teaching. In this approach, mainly the textbook and the didactic expert vertices of our tetrahedron are the active participants of the development process.

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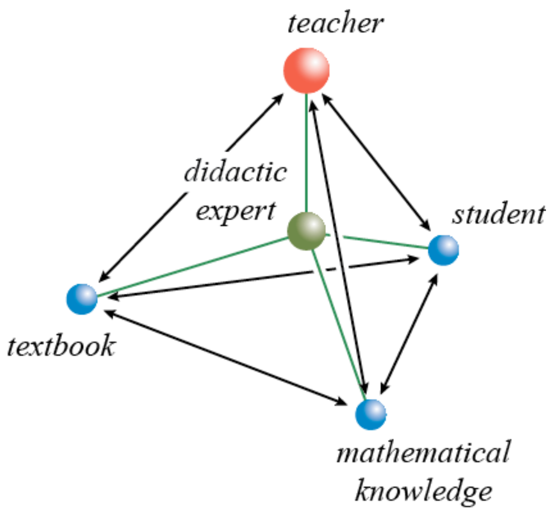


Figure 1. The expanded didactic triangle

In our project a great emphasis is put on giving a more decisive role to the teacher vertex in the textbook development process.

The literature of textbook development considers also other participants, such as parents or the government which are out of focus in this paper.

The available resources, the new tools of the 21st century, the accelerated and broadened ways of communication resulted a highly broadened set of resources and possibilities available during the textbook development procedure too than that of even only 25 years ago. That is, the new digital resources in the 21st century provide new means for designing and sharing teaching materials. (Rocha et al., 2017)

Although there are many examples of the co-working process of textbook writers and teachers, the authors usually collaborate with only 2-5 teachers who are familiar with the methods of the textbook writers, they can almost read in each other's minds. These circumstances of the organisation of the development processes are really useful for the birth of an organic and unified textbook, they lack the advantages of reflecting different approaches.

Even & Olsher (2014), Olsher & Even (2014) and Even et al. (2016) summarize how teachers can take part in the process of the textbook-edition, depending on the main goals of the project. They aim at creating an environment, in which dialogue can be born between curriculum developers and teachers. Their main goal can be to make the teachers participants of the joint-editing work of the textbook or collect possible changes suggested by the teachers.

The textbook development project in the centre of the present paper has a lot of connections with other fields of textbook research. For instance L. Fan (ICMT – 2010) similarly to our beliefs, also explores the textbook research as a new and colourful scientific field.

Partial, but fundamental parallels can also be drawn between our project and the Symposium C of the II International Conference on Mathematics Textbooks Research and Development which focused on teacher-resource use around the world. For instance, although today even the need and reason for the existence of textbooks may be questioned, our project has been based on a strong belief in the need for textbooks, similarly to projects presented at Symposium C. "As in many countries, in Brazil, teachers are heavily influenced by textbooks. They have being fundamental to teachers decision on which contents must taught as well as the instructional approach to be developed in class." (Assis & Gitirana 2017)

Our paper also focuses on the "bidirectional relationship" that was mentioned in many referenced article (Steenbrugge et al. 2017). However, our method was more than solely collecting the opinions and suggestions of teachers working in the project, and to decide what we can change in the textbook in the light of these suggestions. Our testing teachers were chosen from all around the country, to represent all types of towns, villages and schools. The teachers volunteered, they were completely independent, moreover, they got a small amount of payment for their work. The independent selection procedure of the teachers may (probably) resulted in their different view on teaching, teaching habits and also different classroom cultures. That is why we received many different types of feedback, with sometimes completely opposing opinions, from the extremely supportive to the really discouraging and perfectly honest feedback.

2. Historical Background to the Textbook Market in Hungary

Before 1989 the textbook market in Hungary was entirely controlled by the state. It has changed after the democratic transition, the market became free (Fischerné & Kojanitz 2007). As a result of this, at the end of the decade (2000/2001), in one school year there were already 5151 textbooks published from 183 publishers. This oversupply was brought under regulation by the government, and in the school year 2011/2012 there remained only 53 publishers and 3712 publications, but the 90% of this quantity has been produced by the 4 biggest publishers. It can be said undoubtedly, that after a stronger regularization which began in 2012, Hungary got in the middle with regard to the autonomy of the textbook markets in Europe. The list of textbooks, from which the teachers can choose for the students in a school year, has been significantly reduced, and now the distributor is a non-profit-making Ltd. owned by the state. (Pálfi 2016)

The history of textbooks in Hungarian mathematics education provides a huge set of various types of problems (as exercises), as well as a colourful theoretical background to the teaching mathematics. All of these determined the development of textbooks of the last centuries. The new mathematics textbooks can be seen in the light of this tradition.

3. A Textbook Development Project Supporting Social Renewal – The Srop-3.1.2-B/13-2013-0001 Project

In the frame of the Social Renewal Operational Programme (SROP), between december 2013 and november 2016, a complex research on, and development of new teaching materials were implemented in Hungary, mainly organized by the Hungarian Institute for Educational Research and Development (HIERD), including the development of new mathematics textbooks that meet the requirements of the new National Curriculum introduced in 2012. In the initial planning phase, five university teams undertook studies about the basic concepts of textbook development in the 21th century, on the basis of which the main ideas were summarized and the common goals, as well as the conceptions of the principal[1] school subjects were created.

Our main goals were:

- Support the literacy competency as a base of other competencies.
- Communicate understandable mathematics to the students.
- Build methodology recommendation into the textbooks (games, group work, outlooks, etc.).
- Organise teacher trainings.

The 3-year-long project consisted of the following phases. After a development-based research, pilot versions of the new textbooks were written, tested, evaluated, and re-edited to the final versions. One of the most decisive characteristics of these new textbooks was that a considerable number of school teachers were involved in the development process. The features and results of this contribution is the main issue of this paper; accordingly the present focus is on the testing, evaluating and re-edition phases. The SROP 3.1.2-B/13-2013-0001 project also aimed at connecting the development of ICT techniques and the printed versions, by planning, developing and testing the National Education Portal (NKP).

Our main research question in connection with this textbook development process was about the ways and usefulness of integrating the conclusions of teachers' feedback into the new textbooks. Therefore, in the present paper we survey the main phases of the development process, in several of

[1] Compulsory for all and taught in considerable hours per week.

which teachers as users and evaluators play a crucial role. We also present, interpret and evaluate the collected teachers' feedback.

4. Methods

The main universities in Hungary conducted studies in order to lay down the fundamental guidelines for the SROP textbook development programme before it was launched. Most of these studies, e.g. (Vásárhelyi, 2013) emphasized the crucial importance of collective thinking of teachers and students, as well as the significance of the cooperative development of digital support. These studies reinforced our belief that teachers and students should play a leading role in the development procedure. The main milestones of the project were the followings.

First year: Planning and creating the first, pilot versions of the textbooks and workbooks for grades 1-2, 5-6, and 9-10, according to the Hungarian school system.

Second year: Testing the created pilot books by 50 teachers for each subject, collecting feedback and creating the pilot textbooks for grades 3, 7, and 11.

Third year: Rewriting and re-editing the pilot textbooks written in the first year, based on the teachers' feedback, and creating the pilot books of the series for grades 4, 8 and 12. (Wintsche, 2015)

During the project, we received feedback in the following forms.

- Before use form (about the expectations of the teachers)
- Quick responses
- Work logs (detailed questionnaire after every lecture)
- Interviews and workshops with testing teachers and students
- Personal interviews
- After use form (about the general opinions and impressions of the teachers)

The whole SROP project covered the main school subjects, namely history, Hungarian language and literature, mathematics, biology, chemistry, physics etc. with 134 textbooks and workbooks for grades 1-12. The present paper, in the followings, focuses only on the textbooks and workbooks of mathematics for grades 5-12.

5. Results – to change, or not to change, that is the question

More than 20,000 pieces of feedback per mathematics textbooks and workbooks were collected. We classified the feedback, using categories such as misprints, errors, theoretical problems, didactical problems, constructive suggestions and good practices. Some of the proposals were useful, creative and definitely worthy to be integrated into the re-edited textbooks during the revision process, and we got an overall picture about the expectations and wishes of the teachers. Here we present some typical examples of teachers' feedback and the reactions of the textbook developers.

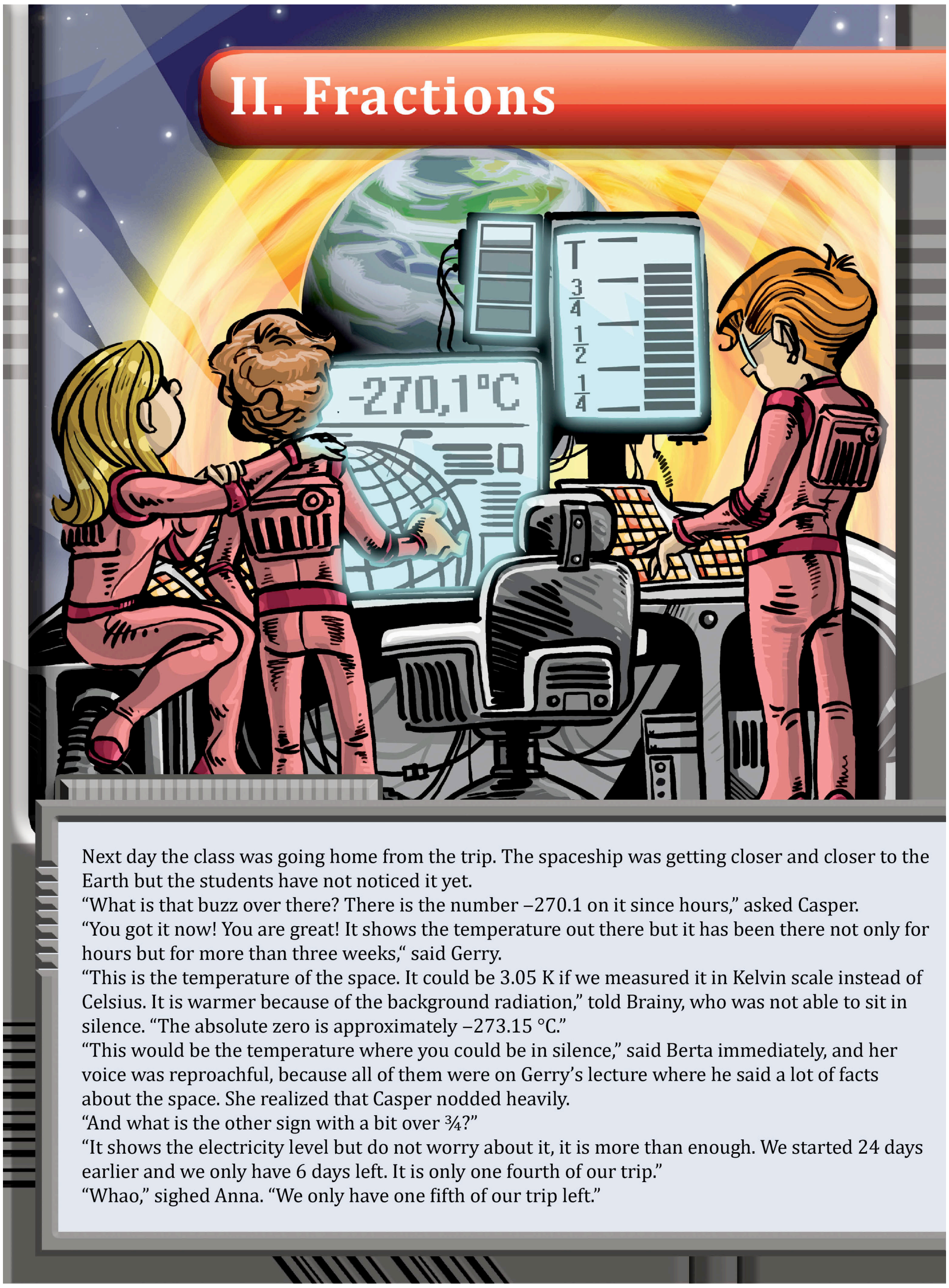
We accepted the feedback and changed the relevant part of the textbook:

Feedback 1: "The students need to meet place value earlier. It would be better to change the order of the first three lessons."

Answer 1: It could be right and there is not any obstacle for it. We changed the order of these lessons, we solved this problem. In the first version, the book introduced decimal numbers before properly defining some concepts such as decimal place value, in order for raising students' interests and building first on their intuitive understanding before introducing the formal definition. However, we accepted that it did not match the majority of the users' teaching style.

Feedback 2: "I suggest that you enter a number of tens, hundreds, thousands, tens of thousands, hundreds of thousands, which can help the students in autonomous task solving and reduce uncertainty. There should be more examples. It would also be necessary to describe more complex numbers."

Answer 2: We wrote some bigger numbers and more complex tasks into the lesson according to the wish.
We rejected the feedback and the relevant part of the textbook remained unchanged:
Feedback 3: „The story board text is not appropriate, because there are large and negative numbers in it and the students cannot read these numbers."
Answer 3: One does not have to cope with the stories completely before the chapters. Their role is only to brush up student's curiosity and develop their literacy skills. (See Picture 1.)



Picture 1. Chapter starting story boards to raise curiosity and develop literacy skills

Feedback 4: „Separate plain and space geometry, as it is in other books. It is really confusing to cope with the square and the cube at the same time."
Answer 4: We do believe that the first perceptions and impressions of the children come from the 3 dimensional space and it is natural to them. We live in this space and plane geometry is (more) imaginary and more abstract. We think that the early separation of space is one of the reasons of the

weak orientation level of the children. We separate the 2 and 3 dimensional geometry only in the 7^{th} and higher grades.
There were numerous different wishes and we couldn't satisfy everybody. Here we quote some typical contradictory wishes:

Feedback 5: "I like that the introductory tasks are simple, but interesting, like the ones with the magic squares."	vs.	*Feedback 6:* "The introductory task with the magic squares did not offer too much help for solving the later exercises of the lesson. You better omit it."
Feedback 7: "It was really useful and entertaining to meet the ancient numbers from Egypt. The students really liked drawing them."	vs.	*Feedback 8:* "It was completely boring and unnecessary to work with the Egyptian numbers that are used nowhere."

In these situations we mainly took the majority opinion into account, but in some cases we finally decided on our own didactic reasons, even if these were against the major wishes.
A great proportion of the wishes were about making the exercises easier, or at least inserting easier ones too. We could not completely take these kinds of wishes into account. We agree with those testers and evaluators who commented that easy and difficult tasks are both needed for differentiation. The textbook's aim cannot solely be serving the needs of the average students. It has to work for the gifted ones as well. Therefore, besides inserting easier tasks, we kept the more difficult problems too.
For the sake of completeness, we quote more pieces of feedback to show the wide range and different types of reflections.
Feedback 9: "The second problem of the workbook (5^{th} grade 3^{rd} lesson) wasn't clear for us. We got three different good solutions, 6232, 6333 and 6031. You should change it."
Answer: There are several mathematical problems with more than solutions. Students ought to get used to it.
Feedback 10: "The group work would be perfect but we don't have time for it. If I got everything prepared in advance, I could imagine dealing with the task. But we did not have enough time, as I mentioned."
Answer: We think it is reasonable to have optional kinds of tasks, like the group work mentioned above, which may be eliminated due to time constraints, depending on the demands and needs of the particular student group.
Feedback 11:"I had basically positive expectations, but of course you are also afraid of anything new. Applying the textbook was easy and successful, especially some sections, like the ones connected to geometry."
Answer: Many thanks.
In summary, we can state that the three most frequent and most important changes in the textbooks were the followings.

- The number of easy problems at the end of almost every lesson have been highly raised.
- The sequence of some lessons or subchapters have been changed, moreover, new lessons have been created at some particular places.
- More games, group work exercises, supplementary remarks, supplementary lessons and overviews have been created for the re-edited books.

Besides these major types of changes, and of course the correction of detected errors and misprints, some other important modifications also worth mentioning:

- Some pictures and figures have been changed

- Some tricky, more challenging problems have been modified
- Final tests have been created for each chapter
- More lessons have been created for annual revision

6. Conclusion

After the revision, the acceptance of the textbooks has increased. Figure 2. shows that the majority of the teachers realised the changes in the textbooks, that is why the diagram is right skewed. 48,5% (36,8%+11,7%) of the teachers considered the re-edited textbook to be better, and only 17,8% (4,8%+13%) said that it became worse. These data comes from the impressions of the teachers during the school years, when they used the revised mathematics textbooks for the grades 5 and 9.
Finally, some opinions of testing teachers are presented about the procedure of the textbook development and the SROP project.
”At last we can see an example when the opinions of colleagues do really matter. The book became better for us after the revision, and also for the students.”
”Collecting the feedback nationwide is really useful because of the various students and teachers. The changes were reasonable, thanks for you to take our wishes into consideration.”
”I am pleased to see that our works were not useless. I see that the editors preferred mostly those changes what we wished. ”

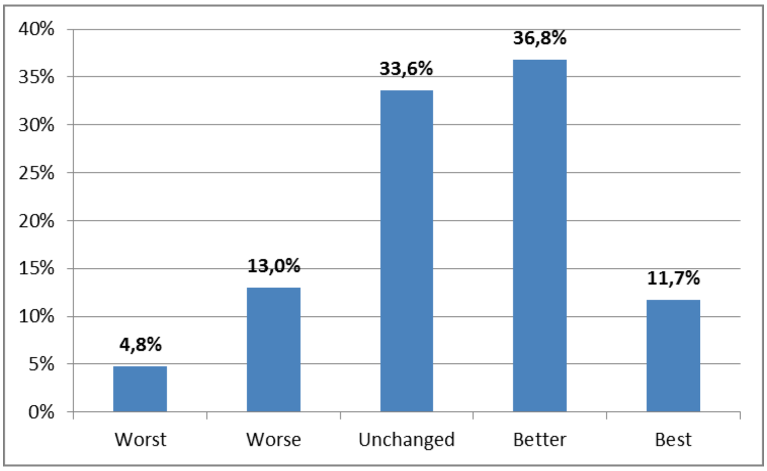


Figure 2. Changes of teacher's opinion during the school year

The data in Figure 3. come from the feedback after the whole revision process.

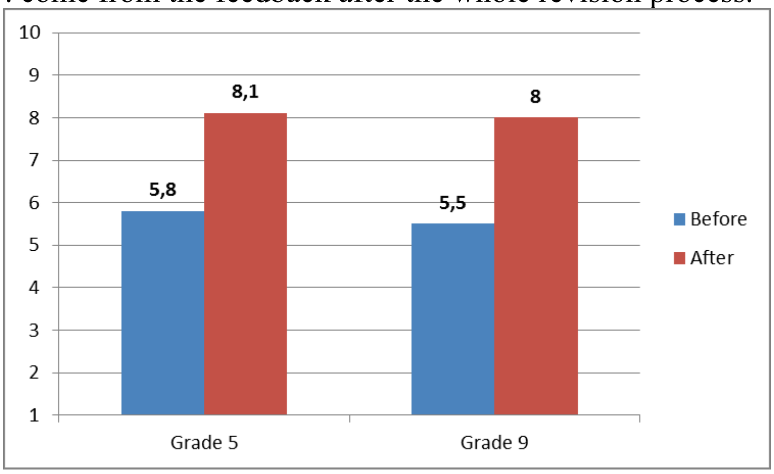


Figure 3. Changes of the teacher's opinion after the revision (1 worst – 10 best)

On the whole, the textbook development procedure, and particularly the re-edition process can be considered successful, based on the teachers' feedback. The acceptance of textbooks by the testing teachers has improved significantly during the re-edition process, resulting in more easily and more efficiently applicable textbooks, with a higher preference level among the teachers. On the other hand, considering the further development of the textbook development procedure itself, the evaluation of feedback would be more easily feasible, and could be conducted with a considerably lower budget, if the testing teachers are selected by a more precise and more purposeful method, by decreasing the number of (more or less) neutral opinions. For reaching this aim, our proposal have been accepted, and after the first two years of the development procedure, the number of testing teachers have been reduced to a selected 25 from the previous 50, for each subject.

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SECTION EVOLUTION OF TEXTBOOKS IN THE LIGHT OF NEW DIGITAL TECHNOLOGIES

AN ANALYTICAL FRAMEWORK FOR STUDYING THE IMPACT OF TECHNOLOGY ON THE USE OF MATHEMATICS RESOURCES IN TEACHING AND LEARNING

IDA MOK and LIANGHUO FAN

Abstract

We have earlier proposed to include the technology principle as one of the six principles for textbook development and argued that in studying the impact of technology on mathematics curriculum one should include three questions: What to teach? How to teach? Why to teach? (Fan 2010, 2011). We analysed how technology was reflected in the mathematics textbooks in Hong Kong and found the impact of technology in the textbooks was a result of top-down curriculum reforms, mostly guided by the mathematical content in the curriculum and around three major categories: self-regulated learning platforms, IT activities using of different software and projects using internet resources (Mok 2014).

Drawing on our earlier research and related literature in this area, this paper proposes a revised analytical framework for investigating the impact of technology in the use of mathematics resources in teaching and learning and discuss its implications in this field of research on textbooks.

Introduction

The development of technology in the field of mathematics education has been rapid and fast over the last few decades. There are significant changes with respect to the tools of demonstration, devices and display, tools for calculation, drawing and graphing, computer-assisted learning platforms and internet. Technology is obviously having an impact on different levels of curriculum, including textbooks and acting as an agent of change (Hutchinson and Torres, 1994; Gerhart, Peak & Prybutok 2015). Accordingly, we believe it is imperative to advance theoretical underpinnings, which have been largely under-developed, for research on the impact of technology on the development and use of mathematics resources. Drawing on our earlier research as well as related literature in this area, the purpose of this paper is to propose a revised analytical framework for investigating the impact of technology in the use of mathematics resources in teaching and learning and discuss its implications in this field of research on textbooks.

This paper draws upon the results of earlier studies and consisted of two parts. In Part 1, applying the framework presented by Fan (2011), the authors discuss the comparison of the textbooks in different places (China, Singapore, Hong Kong (Fan 2011, Mok 2104). The results showed the impact of technology in textbooks has been influenced by curriculum reforms in the Asian context, clustered around three major categories: self-regulated learning platforms, IT activities making use of different software and projects making use of internet resources. In Part 2, the authors discuss the findings of the project "Fundamental challenges in using Digital Technologies in Secondary Mathematics classrooms: a comparison between different paradigms, over time, and between

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places" (called DTMC later)[1] and discuss the malleable and robust features in mathematics lessons to in contrasting paradigms over time. Finally, supported with the earlier studies, we present an analytical framework for studying the impact of technology on the use of mathematics resources in learning and teaching.

Theoretical Background

As far as the impact of technology on the use of resources is concerned, we take into consideration in the theoretical background the factor of the development of technology over time that has significant changes in availability; the different development in the context of different topic areas (e.g., algebra, geometry, statistics), and the principles for textbook development (Fan 2010), for their significant influence on the epistemological and ontological nature of school mathematics.

Development of technology

Mathematics is often linked with technology. The relationship between mathematics and technology is indeed close and intertwined. The impact of technology on mathematics education, according to Roberts, Leung and Lins (2013), has gone through the stages from the slate to the web, bringing significant changes in the aspects of tools of demonstration, support of teaching and learning, the tools of calculation as well as the access of the virtual world via internet. With respect to *tools of demonstration*, there is significant enhancement and progress in the devices and the mode for display: blackboard, flip-chart, whiteboard and overhead projector OHP, computer, and all kinds of mobile devices today. With respect to *support of teaching and learning*, there are the use of different manipulatives and tools, such as, straight-edge and compasses, blocks; these can be found in the computer and tablets. With respect to *the tools of calculation*, there are: slide rule, electronic calculator in the 70s and 80s; the computer algebra system CAS in the 90s and mobile devices bringing about incredible enhancement for information, storage, display and demonstration in the 21st century. Finally, there is *the access to the Virtual Worlds*. Via the internet, we can access knowledge with convenience in all kinds of formats.

Different development in different topic areas

Algebra. When we look into the content of the mathematics per se, there are changes. For example the Computer Algebra Systems (CAS) has changed the role of algebra in the school curriculum (Heid, Thomas & Zhick 2013). CAS makes possible the symbolic manipulation linked with graphical, numerical, and tabular utilities and the symbolic links to spreadsheets and dynamical geometry programs; allowing for new explorations of mathematical invariants, active linking of dynamic representations; engagement with real data, and simulations of real and mathematical relationships.

Geometry. With respect to geometry, the invention of Dynamic Geometry Software (DGS), such as, Sketchpad, Cabri, Geogebra, has significant didactic and research implication (Sinclair & Robutti 2013). Proof and verification is no longer dominated by the traditional Euclidean approach. The dragging and measurement facility in the DGS platform has change the possibility and potential for students making conjectures, exploring properties and relationship in geometry.

Data handling. With respect to data handling, the opportunities for using real data and the availability of statistical softwares are greatly increased because of websites and apps. Students can easily get access of statistical tools and authentic data in the internet. For example, the Gapminder website (www.gapminder.org) for which Hans Roslings' project has made 200 years of global data readily accessible for exploring trends and relationship between a range of variables, related to the world poverty issues. Students can easily visualize and experience the power and joy of statistics in their exploration of global issues.

[1] Acknowledgement: The project is funded by General Research Fund, the Research Grants Council of Hong Kong, China. Mok, Ida Ah Chee, "A Change of Paradigm: A Close-up at Learning Tasks" in *the International Conference for Chinese Association of Mathematics Education*, Wuhan, China, 2016.

Last but not least, with the development of internet, sharing and exchange of knowledge in public is highly feasible and within a community knowledge can be developed and rapidly shared, hence, fixed and tangible resources are no longer considered as sustainable competitive advantage for such assets may quickly available to the others (Sharratt & Usoro 2003). As a result, *self-learning capacity* and *collaboration* skills become important.

Six principles of textbook development

At the 2010 APEC Thailand Conference, Fan proposed six principles for textbook development including Curriculum Principle, Discipline Principle, Pedagogy Principle, Technology Principle, Context Principle, and Presentation Principle (Fan 2010). Later, in his study of the impact of technology on mathematics textbooks in China and Singapore for the last 15 years, Fan (2011) offered an operational definition for ICT for studying textbooks that included: calculator, computer, internet and software, and argued that in studying the impact of technology on mathematics curriculum one should include three questions:

- What to teach (content of learning)
- How to teach (a tool to facilitate learning)
- Why to teach (an objective of learning)

The six principles can be integrated into the three questions in studying for the impact of technology on textbooks and resources, e.g. these principles may apply in an integrated manner, thus, changing the outlook as well as the depth of the students' experience of a specific content in the curriculum where technology may be applied, how technology may as a tool to facilities the learning of the content, what goals the students may achieve in carrying out the technology integrated tasks or activities, whether we want them to learn the mathematics or the technology.

Study one: Analysis of textbooks of different places (China, Singapore, Hong Kong)

(1) Fan (2011) reported a comparison of two series of textbooks: China and Singapore.

Findings from the current mathematics textbooks (China) show that there are mainly three types of use of technology:

- Use of scientific calculators to find value, to calculate, and to explore (Purposes: for what to teach, why to teach, and how to teach)
- Use of Internet as a resource (often optional only), mainly for reading, project tasks, and exploration work (Purposes: for how to teach/learn)
- Use of specific software such as excel and GSP (Sec. 3) to construct, calculate (average), and graph (Purposes: for what to teach, but also for how to teach)

Findings from the current Mathematics Textbooks (Singapore) shows that largely similar to Chinese textbooks, there are also mainly three types of use of technology:

- Use of scientific calculators to find value, to calculate, and to explore (Purposes: for what to teach, why to teach, and how to teach)
- Use of Internet as a resource (most times optional), mainly for exploration and project tasks (Purposes: for how to teach/learn)
- Use of specific software such as excel, spreadsheet programme, a graphing software, and a dynamic geometric software (but not specific name given) to construct, graph, and explore, often for In-class Activities (Purposes: for what to teach and how to teach)

(2) Extension of Fan's work (Mok 2014)

Fan (2011) offered an operational definition for ICT for studying textbooks that included (a) calculator, (b) computer, and (3) internet and software. Based on Fan's work, Mok (2014) carried out an analysis of one of the most popular series of textbook (Grades 7-9) in Hong Kong which included a CD-ROM produced by the publishers accompanying the textbook, the major types of technology are:

- Calculator: Using the calculators to find value, to calculate and to explore.
- Internet E-tutor: The E-tutor in the publisher's website providing e-guidance for the selected questions in the revision exercises.
- Internet: Additional resources and information for projects.
- Software, Internet and the CD-ROM: One type of activities uses in the Internet for exploration and these are guided by the activity sheets and files in the CD-ROM. Another set of activities can be carried out in the computer offline that use software such as Microsoft Excel, Geogebra or Animation embedded in the CD-ROM for exploring the mathematical concepts. The activities are guided by the activity sheets and the files in the CD-ROM.
- Other supplementary materials provided by the publishers: Other resources include, glossary, activity sheets, power-point presentation files and drilling program.

By comparing the work of Fan (2011) and the work of Mok (2014), there are some further development in the use of IT on textbooks: (a) providing an interactive E-tutor on the internet by the publishers, and (b) the publishers' production of supplementary materials that enhanced the display of content in teaching and provided further practice for students.

(3) Some examples from the analysis of Hong Kong textbooks

Figure 1 shows the software, internet and IT activities are identified in the curriculum document in advance and suggesting a certain direction that may *possible change the nature and process of learning* in the classroom. Some examples that may have a significant impact on what and how the teacher may teach a topic are listed here and they includes input of technology in symbolic manipulation, spreadsheets, and engagement of students in exploratory work or self-study materials.

- Activity using GeoGebra such as transformation and trigonometry (figure 2),
- Using an animation to justify the "Identity of the difference of two square" (figure 3),
- Exploring the value of the square root of 2 in a e-worksheet (figure 4),
- E-tutor: A student self-regulated interactive platform with exercises supplied by the publishers.
- Projects.

Grade 7	Grade 8	Grade 9
• Sum of all the interior angles of a triangle (Geogebra) • Rotational symmetry of plane figures (Geogebra) • Reflection and rotational transformation (Geogebra) • Order of Transformations (Geogebra) • Estimation of π (Excel) • Constructing Statistical Diagrams (Excel) • Project	• Identity of the difference of two squares (CD-ROM animation) • Investigating the graphs of linear equations in two unknowns (Excel) • The value of $\sqrt{2}$ (Excel) • Tessellation (link to activity on internet) • Proofs of Pythagoras' theorem(CD-ROM animation) • Properties of sine ratios and cosine ratios (Geogebra) • Project:	• Simple interest and compound interest (Excel) • Experimental probability (Excel) • Project

Figure 1. The topics highlighting the use of technology in the Hong Kong textbooks.

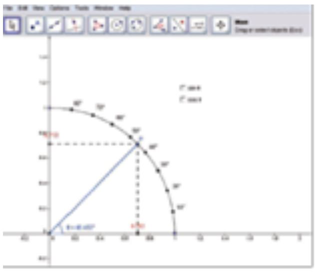

Figure 2. Example of using Geogebra

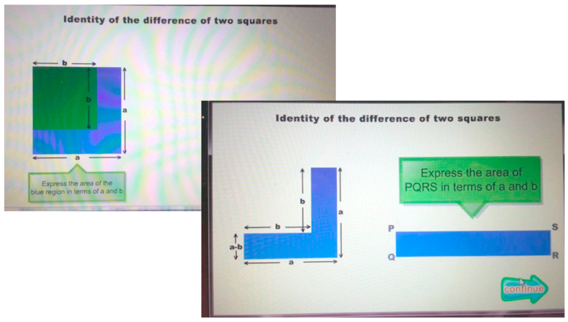

Figure 3. Using animation to justify the identity of the difference of two squares.

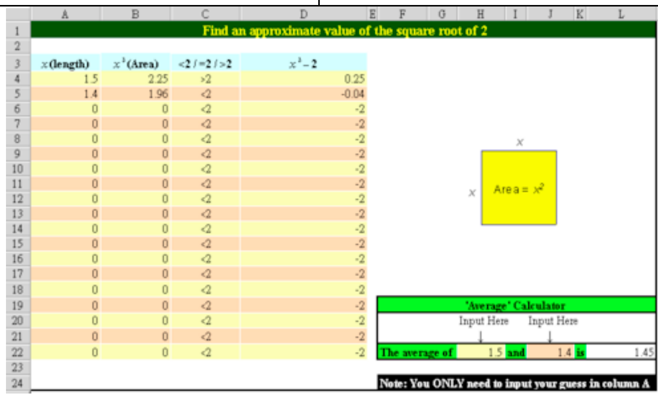

Figure 4. A e-worksheet for exploring the value of the square root of 2.

The E-tutor platform. It is worthy to mention a new feature, the E-tutor platform developed by the publishers, which was delivered to the students via the internet. At the end of each chapter, there was a revision exercise with support was provided by the E-tutor on the internet. The E-tutor provides:

- A collection of problems of varied difficulty for the topic is provided.
- Self-study guide: The e-tutor provided hints, outline of method and a list of the knowledge that they needed to solve the problem, serving a reference to a specific parts i the textbooks for further reading.
- Students' autonomy: The students might login in their accounts to use the e-tutor in the publisher's website.

Comparing with the findings of the earlier work of Fan (2011), the findings in Mok's study shows that the development of technology in textbooks and mathematics resources are picking up momentum. To conclude briefly, the publishers play a pivotal role in making suggestions and production that may influence how the teaching and learning in the lessons, especially for those teachers who are not ready to produce their own teaching materials. There is a trend for providing more opportunity for exploratory work, mathematics work that can make use the advantage of graphical interpretation and self-regulated activities.

A contrast of paradigms over time: What are malleable or robust in mathematics lessons?

Mathematics had been a long established subject, its content is pretty stable, curriculum in different places covers similar topics at similar levels (see TIMSS 2015 Encyclopedia, http://timssandpirls.bc.edu/timss2015/encyclopedia/countries/hong-kong-sar/the-mathematics-curriculum-in-primary-and-lower-secondary-grades/). For example, for some mathematics topics in the case of Hong Kong, over the past twenty years, despite the change of syllabuses, some similar problems might be used in the lessons for demonstration and student work. How may the input of technology make a change in the use of such mathematics problems? What features are malleable or robust in mathematics lessons? We seek answers from the findings of the DTMC project.

The DTMC project aims to compare the variation in paradigms of mathematics lesson with and without an application of digital technologies over two different period of time (2000 to 2002 and 2015 to 2016) in the Hong Kong context. Lessons videos of competent teachers collected with three cameras-approach (teacher camera, student camera and whole-class camera) in a naturalistic manner. The Learner’s Perspective LPS design was employed for the research methods and the video data were supplemented with post-lesson teacher and student interviews. The 2000 set of data consist of 30 grade eighth lessons of three competent teachers and the 2015 set of data consist of 16 grade eighth lessons of two teachers with both lessons using IT and not using IT. To show the contrast of snapshots over two paradigms, an illustration is given in figure 5 which shows some snapshots of the lesson events in the lesson in 2000 and the lessons in 2015 (figure 5). The snapshots were selected specially to show that there were changes but there were something that are persistent and supportive to learning, and were much appreciated by students. These features were clear explanation and demonstration in teacher-talk and the between-desk instruction while students doing problems during the lessons. In the 2015 lessons, in fact many episodes were quite similar to the picture in 2000, the activities such as between-desk instruction, clear explanation, board work mixed with Geogebra display happened very often. The picture in figure 5 was chosen for the teacher created a special lesson for the topic trigonometry, there was an activity in which the students used the apps in the ipad to measure the height of a building and this brought them doing mathematics outside the classroom. The students learn about the tasks the day before the lesson and they chose their own tools (apps) and sorted the measuring method. The findings showed that technology had given the opportunity for the teacher to create a special learning experience and the students showed great appreciation of this special experience the collaboration, hands-on and realistic nature of the activity.

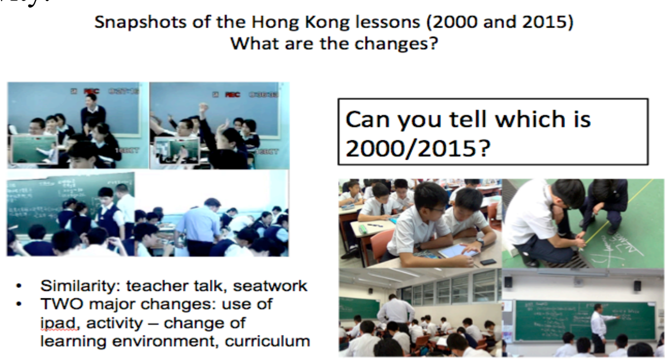

Figure 5. Some snapshots of Hong Kong lessons (2000 and 2015)

Conclusion and Discussion

The impact of technology on the use of mathematics resources was explored under the three questions framework. The investigations were carried out in an Asian context with a background of curriculum reforms. An explicit impact was found in the textbooks that often could be seen as a major resource. A quick change may be some topics and individual contents reflecting the developing technology trajectory in the mathematics curriculum. The change brings about certain direction for pedagogy and the nature of the content of the subject matter. On the one hand, the change in terms of contents were patches of insertion into the overall curriculum. These do not bring about much change in terms of content. On the other hand, the choice of the use of technology gives much room for the teachers to create special learning experience for students that may give students more opportunity for exploratory, application of realistic mathematics and self-regulated learning capacity. Figure 6 summarizes the revised framework we propose for investigating the impact of technology in the use of mathematics resources for answering the three questions: What to teach? How to teach? Why to teach?

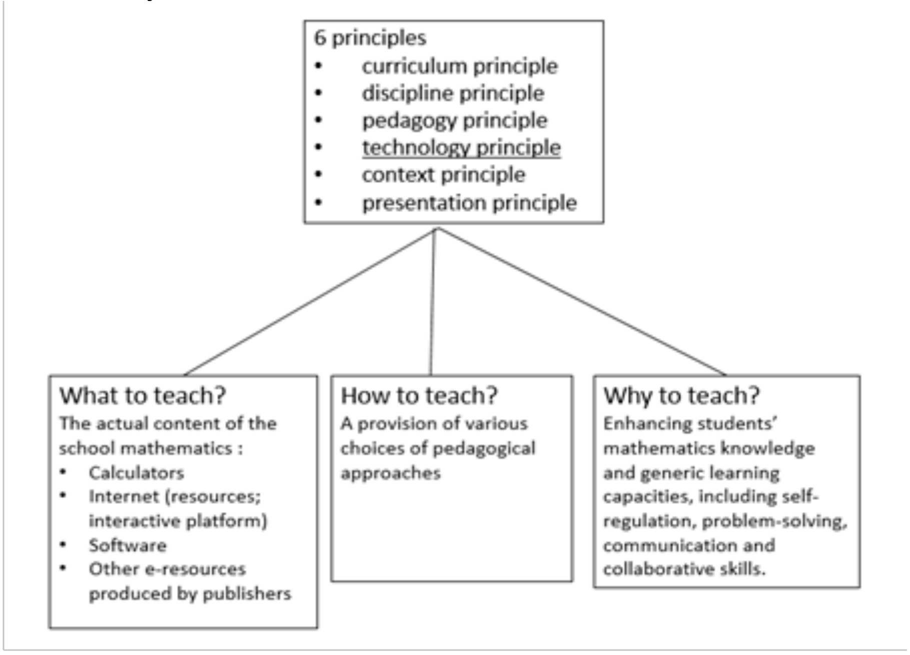

Figure 6. A proposed framework for investigating the impact of technology on the use of mathematics resources

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LEARNING MATHEMATICS WITH VIDEOS

LILIANE XAVIER NEVES, MARCELO DE CARVALHO BORBA and HANNAH DORA DE GARCIA E LACERDA

Abstract

In this paper we discuss about how students' interaction with digital videos has changed the types of educational materials used for learning. Elements that characterize audiovisual materials take educational tools to a new level and make videos, infused with visual elements, orality, gestures, and sounds, a source of educational information. The research of "Digital videos in distance learning mathematics" focuses on these main ideas and proposes, from a Freirean perspective, the joint production of videos by students and teachers to express mathematical content discussed in the classroom. Videos produced by a collective of humans-with-media may become a digital object from which others can learn. Considering that education must be connected to digital wisdom, in these times when our sense distinction is being transformed, this research focuses on an annual video festival that creates a locus for sharing these digital educational materials.

1. Digital Videos and Mathematics Education

For decades, the use of technologies as a resource for teaching and learning mathematics has motivated discussions that have influenced many researches conducted in Brazil. Borba, Scucuglia, and Gadanidis (2014) report that changes in society due to technological innovations and the democratization of the Internet have also begun to be reflected in classroom. The authors organize these changes, with attention to the use of digital technologies in the teaching and learning of mathematics, into four phases called "phases of the digital technologies in mathematical education"; the last still persists.

According to these authors, the first phase of digital technologies in mathematics education began around 1985 with the use of LOGO software, and it was also marked by the emergence of computer labs in schools and the technological training of teachers. The popularization of personal computers signaled the beginning of the second phase with the production of educational software, especially software in the area of dynamic geometry. This led to new possibilities that required a reorganization of pedagogical methods used by then, along with a change in the teacher's usual posture in classroom. The software created in this phase provided an analysis of the behavior of functions from experiments with technologies exploring the dynamic and visual characteristics of software in a mathematical research environment.

The Internet was introduced in the educational scenario initially as a tool to search for information and also as a means of communication, which Borba, Scucuglia and Gadanidis (2014) claim characterizes the beginning of the third phase of digital technologies in mathematics education. At this stage, the production of knowledge took a new seat regarding the notions of time and space related to teaching and learning, with the provision of distance learning courses for teachers.

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Mathematical thinking started to be developed in the virtual environment as well, and researchers began to question how mathematics is transformed in this new environment capable of integrating a variety of media artifacts.

In the fourth phase of technologies in mathematics education, the idea of mathematical knowledge being produced in the virtual environment intensified with the improvement of connection quality. Society had been transformed by increasingly rapid technological innovations and the democratization of access to these new technologies. At the beginning of this phase, the terms *digital natives* and *digital immigrants* emerged. Prensky (2001a) characterized "students of the twentieth-century as natives of a digital language of computers, video games and the internet." On the other hand, there were those who were not born in a digital world but who would embrace the new technologies in their lives. These the author termed digital immigrants.

> It is now clear that as a result of this ubiquitous environment and the sheer volume of their interaction with it, today's students *think and process information fundamentally differently* from their predecessors. These differences go far further and deeper than most educators suspect or realize. [...] we can say with certainty that their *thinking patterns* have changed. (Prensky 2001a, p. 1)

In fact, for students surrounded with environments permeated by digital technologies for a long period of time, the way of producing knowledge was generally different.

In this scenario transformed by digital technologies, we highlight the definition of educational technologies approved by the Association for Educational Communications and Technology (AECT) as the "study and ethical practice of facilitating learning performance and improvement through creation, use and correct application of technological processes and resources " (Leite & Aguiar 2016, p. 36). The notion of educational technologies has a broad dimension as being a resource that potentiates changes in the process of teaching and learning. Yet digital videos bring different possibilities for the development of educational activities.

Oechsler (2015) grouped the researches involving the use of videos in the educational field and emphasized, without focusing on a specific discipline, the use of cinema in the classroom. The survey also indicated the use of videos in teacher training excerpted from class films to group discussions of teachers, beginners or not, in order to predict actions and bring reflections on themes and dynamics to their classes. Also mentioned was the use of videos in qualitative research for the collection and analysis of teaching experiments, since they provide details in short periods of time that can be seen repeatedly, as well as and the visualization and production of videos in the classroom associated with the notion of Digital Performances Mathematics (Borba, Scucuglia & Gadanidis 2014).

A viewpoint of teachers using videos as digital educational objects in math classes was reported in the research of Amaral (2013). This qualitative research focused on the analysis of digital educational objects of specific textbooks with respect to the specialized mathematical knowledge of the teacher involved in the exploration of such objects. The results indicated that the use of digital educational objects, including videos, was associated with the domestication of the media and promoted little interactivity. According to Borba, Scucuglia and Gadanidis (2014, p. 41), "the domestication of the media is related to the use of technologies in the same way and anchored in the same practices that were conditioned by other media." When we domesticate a medium, we stop using its potential that surpasses the old media, such as using a video lesson in a class where the teacher is present.

Digital videos make it possible to combine visual elements, graphics, orality, gestures, body expressions, and sounds with the purpose of transmitting an idea. Jewitt, Bezemer and O'Halloran (2016, p. 411) indicate that "the term 'multimodality' was used to highlight that people use multiple means of meaning making." Thus, considering the specific potentialities of this technology, it is possible to visualize different mathematical activities that can be developed in the classroom.

The video has been implemented as an educational technological resource in this sense. Particularly the possibility of the videos to unite several forms of expression, consequently enables the students to reflect deeply on mathematical content when they produce them.

Ferrés (1995) says that the video stands out as something that is not only related to the media, but also to the language, since it requires the interlocutors to express themselves in audiovisual form, where several modes are used synchronously in an aesthetic synthesis with concordant logical significations (Wohlgemuth 2005). We believe that to carry out this synthesis in order to express a mathematical idea in a video, the student, working together with the teacher, should mobilize different aspects of the mathematical concept in question, producing a didactic resource at the end of the process that will serve as support for other students learn.

2. The reorganization of thinking in the process of producing videos with mathematics

The process of organizing ideas to express it in an audiovisual format takes us the question of how videos influence the way knowledge is constructed. Borba and Villarreal (2005) argue that thinking is collective and influenced by human-media systems. The authors emphasize the importance of thinking about the contributions of all the elements involved in learning. In addition, they consider that the media shapes the human being and, conversely, the human being shapes media as well, thus influencing the way knowledge is generated. For example, in the production of mathematical knowledge within an environment that makes use of digital technologies, everything happens in a qualitatively different way from the environment where only pencil-paper is used.

We realize that the production process of videos reveals a moment of organization of the ideas that will be expressed in the audiovisual format. This organization of the thought involves mathematical knowledge and elements that make up the video defined in a script as well as the resources used to produce the video. All are used in order to achieve a synthesis in which the mathematical idea is best expressed. In this movement the mathematical content must be seen and reviewed in different ways until it arrives in the ideal format according to the purpose of the video. This takes us to the metaphor humans-with-digital videos.

Prensky (2001a, 2001b) characterizes twentieth-century students as natives of a digital language of computers and the Internet, and affirms that today's students think and process information fundamentally differently from their predecessors. Thus, we think the collaborative production of videos with mathematical content among students and teachers as a possibility of knowledge sharing. Students (digital natives) and teacher (digital immigrant) exchange technical knowledge

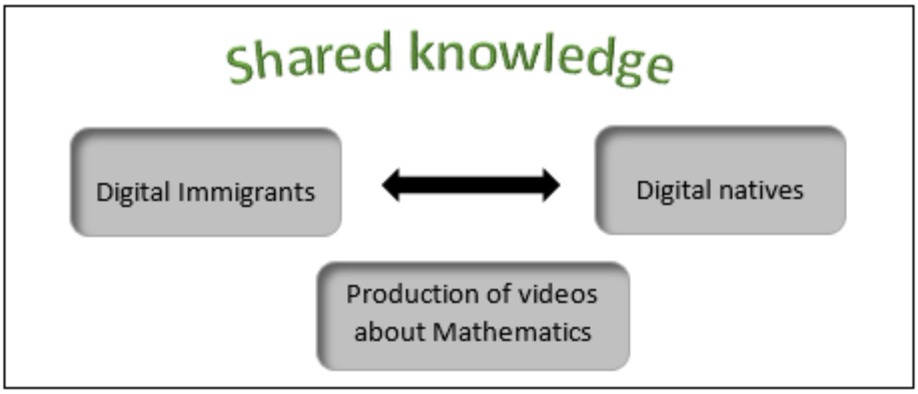

Figure 1: Shared knowledge in the production of videos on Mathematics.Source: The author himself.

about the necessary technologies required for audio-visual production and mathematical knowledge (Figure 1). In the process of video production, when seen as a collaborative activity betweenstudents and teachers, the student exposes his/her knowledge about manipulation of new technologies. Here we refer to the student as a digital native. In this process the teacher shares the theoretical knowledge and helps in the organization of ideas in order to express mathematical content in the audiovisual format, which is the purpose of this digital didactic material being produced. In addition, the student has the opportunity to share his/her practical knowledge obtained from experiences and observation of everyday situations. This enriches the student-teacher relationship, since the teacher can better understand the student's universe, his/her previous knowledge and way of learning. All this contributes to the process of teaching mathematics specifically.

The collaboration between teachers and students has been related to what Paulo Freire has for decades called a “horizontal relationship.” From this perspective, the actors involved in a given teaching activity jointly elaborate goals and procedures.

With this in mind, we think about the possibilities of collaborative construction and use of videos in the training of mathematics teacher undergraduates, understanding that collaborating is the working together, all are supporting each other and aiming to reach common objectives negotiated by the group (Fiorentini 2013). We agree with Onuchic and Allevato (2009, p. 173) that “significant experiences lived by the mathematics pre-service teachers are reflected in classroom, determining the extent of the change that their future students will experience in practice.”

The elements discussed in this section constitute objects of the research project "Digital videos in distance learning mathematics" carried out by researchers from the São Paulo State University (UNESP), Institute of Geosciences and Exact Sciences, Campus Rio Claro.

3. The research on video production

The research question for the project “Digital Videos in Distance Learning Mathematics” is *what are the possibilities of collaborative digital video production by researchers, teachers, and students?* With this question, we seek to understand the possibilities of collaborative construction and the use of videos, seen as multimodal artifacts, in the mathematics teachers’ training degrees of Open University of Brazil (UAB)[1]. In addition, the project includes actions such as mapping the way digital videos are used in UAB Mathematics degrees and actions to understand how students and teachers can generate videos that express their knowledge and serve as learning objects for others. The project also provides an analysis of the possibilities of the video festival, which was created as a locus for interaction between the university and schools that participate in the project.

Initially, the problem developed in the research project "Digital Videos in the Distance Learning Mathematics" was related to the use and production of digital videos in undergraduate courses in Distance Learning Mathematics of the UAB. There was expansion of the research field to include classroom courses at UNESP, as well the basic school scenario in the state of São Paulo.

This research project intends to intervene in distance learning mathematics courses and basic schools to encourage collaborative video production by mathematics pre-service teachers, tutors, teachers, and students. This intervention will result in digital mathematical videos that will be part of the “Festival of digital videos and mathematics education.” This action promotes the popularization of this type of production in schools and universities and favors the communication of mathematical ideas in the classroom, enabling qualitative improvements in the investigated scenarios.

This action supports and justifies the challenge proposed by the research project "Digital Videos in the Distance Learning Mathematics" to take the idea of collaborative construction of videos with mathematical content to mathematics classrooms, considering their different models. The research also stands out for appreciating the communication in mathematical learning, a theme not addressed in the Brazilian researches related so far.

We understand that research procedures such as interviews, participant observation, and intervention are conditioned by the vision of knowledge as well as by the goals themselves (Araújo

[1] “The Open University of Brazil is a system composed of public universities that offers higher education courses for the population that have difficulty access to university education, through the use of distance education methodology”. Available in http://www.capes.gov.br/component/content/article?id=7836. Access: September 2017.

& Borba 2013). Conversely, research procedures, equipment used, software, the question and goals help to shape what is meant by knowledge. It is in this sense that the research methodology is understood here as the amalgam between the vision of knowledge and the operationalization of this vision in a given investigation. The objectives presented above are associated with qualitative procedures. Knowledge originating from qualitative research is appropriate to bring silenced voices to the surface (Poupart et al. 2010).

In this research, we performed interviews, participant observations in attendance and virtual environments, and interventions in undergraduate courses in mathematics of UAB and schools of Basic Education. With the intervention in the form of a collaborative work, we proposed and, at the same time, investigated aspects of the collaboration itself to produce materials that serve as expressions of learning and objects of teaching to those who produce and/or assist them.

The project team is composed of the coordinating professor, a student of scientific initiation, two masters and six doctoral students. Within this research group, we practice what Lincoln and Guba (1985) call *peer debriefing:* researchers in the group discuss the interpretations made by a given researcher on the constructed data. In this way, knowledge is generated, though the subjectivity of individual interpretations is challenged by fellow researchers. Therefore, the analytical process is carried out in two dimensions: collective and individual. After switching individual perspectives to excerpts of raw data, we come together to analyze individual analytical work and to confront interpretations. These meetings give the collective character of analysis, because we often change or create new themes.

Among the subprojects linked to the research project "Digital Videos in the Distance Learning Mathematics" under development, one doctoral subproject is focused on investigating the potential of artistic elements in the communication of mathematical ideas developed in videos submitted to the Festival. Two others have Basic Education as a research scenario. One, a doctoral subproject, investigates how videos collectively produced by teachers and students can be used as a form of expression for learning and as methods of teaching. The other, a master's one, discusses the role of video in the production of digital mathematical narratives of basic school students, from the perspective of developing their autonomy in relation to mathematical learning.

Four other subprojects have their focus on the pre-service courses in mathematics of UAB. One doctoral student has the objective of investigating the particularities of the Festival, researching its impact on the students, teachers, tutors, and disciplines of the courses. From another perspective, based on communication theories, a master student seeks to understand how mathematics is communicated by students of one of these courses through audiovisual means. Another doctoral student investigates how the multiple representations are explored by undergraduate students in mathematics of UAB when they produce videos on Analytical Geometry. We also have another doctoral researcher investigating the use and production of videos, in the form of multimodal discourse, in the subject of Supervised Internship in another course from UAB. We are considering that the interaction between the aforementioned subjects can be enhanced in an environment promoted by the use of digital technologies and visual arts that merge as a way to enable the communication of mathematical ideas and expression of learning.

Video research began to be explored in the Research Group in Informatics, other Media and Mathematics Education (GPIMEM) in 2006, with projects funded by national and Canadian agencies (SSHRC). In addition to the undergraduate courses in Distance Learning Mathematics of the UAB, the research proposed here also develops in Basic Education. The purpose of this interaction between these two levels of teaching is linked to interest in how videos, made collaboratively by students and teachers of UAB courses, will be used as a form of expression of their learning and as an object of teaching by their users. In collaborative work, leadership is shared, with co-responsibility for the conduct of actions.

In Brazil, approximately half of the undergraduate courses use the distance learning modality (Brasil, 2013). Viel (2011) and Santos (2013) point out the limitations regarding the use of technology in the pioneering course of distance pre-service teacher learning (in the online phase) within the public institutions. Among the limitations noted, there is a lack of interaction between students and between students and teachers. This fact justify the "proposal" of this research to study the possibilities of interaction between virtual undergraduates and undergraduates and teachers by means of digital videos.

The dialogical position supported by the project highlights establishing a horizontal relationship between students and teachers, in the sense discussed by Freire in several works such as “Pedagogy of the Oppressed” (Freire, 1968). In the research carried out in the GPIMEM exploratory manner, we initially thought about the use of videos in education as a function of the teacher. Yet we quickly understood the need for video production to be a way for students to express themselves. Thus, we incorporated it into disciplinary evaluation processes and put ourselves, as teachers and researchers, in the role of also learning from students' expressions made through digital artifacts.

"Digital wisdom," understood as the ability to produce meanings within a medium characterized by plasticity that continuously combines new forms of expression, proposed by Boll and Axt (2011, 51), can become a useful concept to analyze the quality of undergraduate degrees in mathematics of the UAB.

4. First Festival of Digital Videos and Mathematical Education

As an activity related to the project, we held the First Festival of Digital Video and Mathematics Education as a way to create a space for virtual dialogue for mathematics undergraduates and basic learning students. With this festival, we also sought to connect with the virtual environment by creating an annual festival where videos are presented, using a multimodal discourse that includes usual text, usual filming, animation, and mathematical software. At the festival we had the participation of a judging commission formed by mathematicians, mathematical educators, artists, and members of the community. Virtual focus groups, interviews, and quantitative analysis of assessments were procedures used to produce the data in the festival.

The First Festival of Digital Videos and Mathematics Education began in March 2017 with submissions and ended in September that same year with an awards ceremony. A total of 379 students and 51 teacher participants submitted 118 videos. Of the 118 videos, 78 were produced by students of basic education, 31 by undergraduate students in mathematics, and 9 by students from other undergraduate courses. Of the 27 federative units of Brazil, representatives of 15 states participating in the festival, indicating the event had a significant reach.

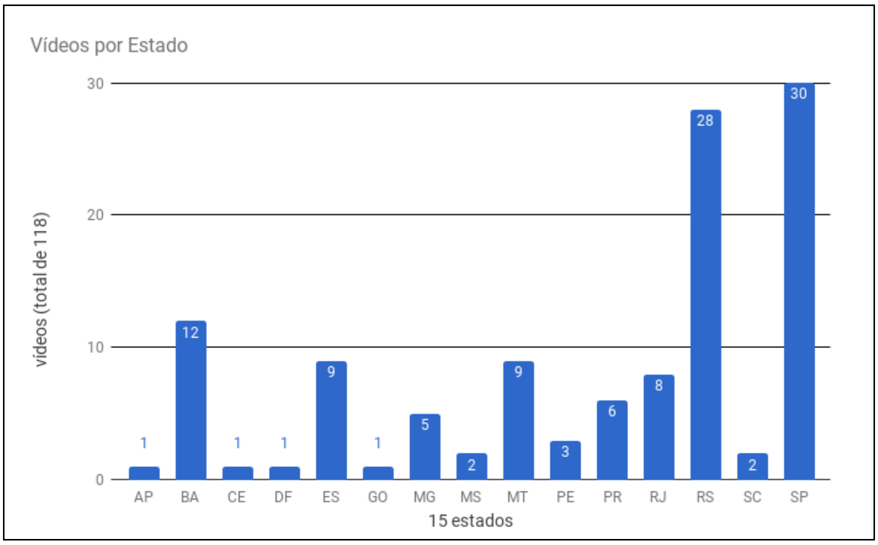

Figure 2: Number of videos participating in the event per unit of federation of Brazil. Source: Data of the 1st Festival of digital videos and mathematical education.

The videos were analyzed by a group of 8 jurors, including mathematicians, applied mathematicians, educators, mathematical educators, and artists, all who additionally discussed their impressions about the festival and the analyzed videos. These reports also constitute research data.
There were 118 videos with mathematical content produced by student groups, most in collaboration with math teachers. The participant videos dealt with 2nd degree equation, topics of plane and spatial geometry, the concept of function, and also the particular cases of linear function and exponential function, trigonometry, matrices and systems, arithmetic progression, variation rate, symmetry, topics of mathematics history, and circumference, among others.
The videos were analyzed and there was an opportunity for some correction and editing before the deadline for submissions. All 118 digital educational video materials produced for the First Festival of Digital Videos and Mathematics Education can be found at www.festivalvideomat.com. The videos were divided into two categories: Basic Education and Undergraduate Education. The Figure 3 shows the image of the Festival site with some of the participating videos.

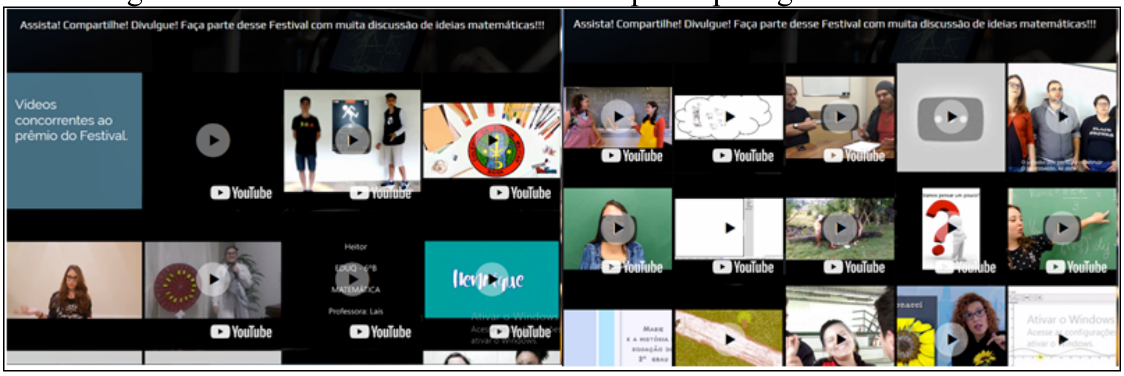

Figure 3: Picture of the videos of the 1st Festival of digital videos and mathematics education. Source: 1st Festival of digital videos and mathematics education site.

The results of the project will be evaluated at different levels that relate to the changes caused by the actual implementation of those involved. From this implementation, it is possible to obtain audiovisual materials for study, as well as disseminate this culture of video production among those involved. In order to do this, one needs to investigate how the videos were used before the project's intervention, to discuss the interactions during this process, and to verify its ramifications after the end of the project's intervention.
The videos participating in the First Festival of Digital Videos and Mathematics Education presented different themes. Some were explanatory in character, approaching the mathematical content from a problem situation that was solved during the development of the video. Others presented an informative theme in which the content was exposed without a practical example, or a theme focused on artistic manifestations.

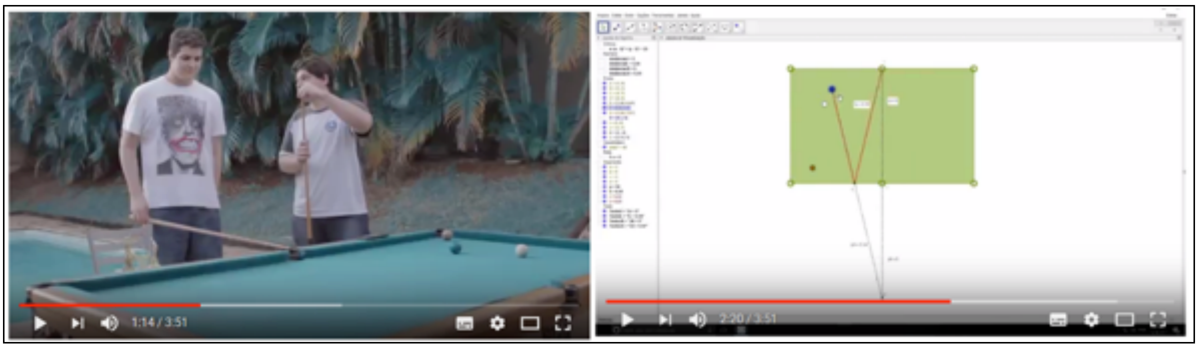

Figure 4: Video scenes "The use of geometry in the pool game". Source: Data of the 1st Festival of digital videos and mathematical education.

The video "The use of geometry in the pool game" was produced by students of Basic Education with the supervision of a teacher. In the video two students simulate a game of pool where one player makes a move based on plane geometry concepts. Due to the curiosity of the second player, the first player, using the software of geometry dynamic Geogebra, begins an explanation of the resulting movement.

The topics of symmetry and some congruence rules of triangles are covered in the video from a situation problem motivator. The video, in this case, presents itself as digital didactic material that can be used as a complement to a theoretical explanation of the concept discussed in the video.

Ball (1973) affirms that traditional teaching is characterized by the devaluation of the oral expression of students, in addition to other things, and believes that learning has, as one of its foundations, the orality that allows communication based on knowledge. In view of the scenario in which solitary work is highlighted, where one student assumes the position of spectator, the challenge is to take the idea of collaborative construction to the mathematics classroom, considering their different models. Students and teachers of Basic Education, undergraduates, university professors, and researchers relate in a way to think about and communicate mathematical ideas, having a festival of videos as a venue for the dissemination and exchange of the works produced.

5. Final considerations

The technologies developed throughout history interact with human beings in the production of knowledge. The media, be it oral, written, or multi-lingual, are leveraged by the Internet and shape the way we produce knowledge, but also the way we are constituted as humans (Borba 2012). Boll and Axt (2011) propose, supported by the creator of the term "digital natives" and "digital immigrants" (Prensky 2001), that education must be connected to a digital wisdom in these times when the distinction of meanings is being transformed.

In this scenario permeated by the digital technologies leveraged by the Internet, the work of the teacher gains a new dimension. Students also require new skills. (Borba, Scucuglia & Gadanidis 2014). The development of these new competencies cannot be considered an individual’s responsibility. A culture should be created to support and deal with creative (non-domesticated) uses of the Internet and digital artifacts.

The multiple uses of video are still little studied and are underused in the area of education, in particular in mathematics education, although there are already initiatives underway in this area. Just as the availability of paper conditioned the production of knowledge in the eighteenth century (Borba & Villarreal 2005), it is reasonable to speculate that new technological possibilities have conditioned the production of knowledge in recent years in a way that has not yet been fully realized. What seems solid--the plasticity of digital media, "inexhaustible" databases, and the mobility of information--has allowed students and teachers to express themselves in a multi-modal fashion, often freely and spontaneously.

Such relationships, therefore, are already part of the daily lives of a considerable number of students outside the classroom. One notices they immerse themselves in social networks on personal or general topics through multimodal texts, combining images, texts, and sounds, or even videos of their own or of those copied from some database of the Internet, but reproduced with certain intentionality. In this project, this means multimodal expression that is reflected in "digital video."

Thus, digital video can express certain mathematical ideas through orality, writing, gestures, body expressions and sounds. It is also possible to think of multimodality as a channel for multiple forms of expression, media, medium, and artifacts, which are used for specific contextualization, communication, formalization, or mathematical investigation.

With the development of the research project " Digital Videos in Distance Learning Mathematics," we intervene in the virtual environment with the creation of the locus for sharing videos. Our First Festival of Digital Videos and Mathematics Education was the first step and has already brought us 118 videos that present mathematical ideas from multimodal presentations that involve usual text, usual filming, animations, mathematical software, among others. Videos are available for math teachers and students who want to use them for study, research, or in class. Some of the videos produced during the development of this project are being used in a pre-service teacher course taught by a researcher from the GPIMEM research group as a form of exchange.

These interactions will enable an analysis of how the student understands certain concepts and how he or she adapts such mathematical knowledge into his or her life. We affirm this based on the observation that many videos produced for the Festival presented a mathematical idea by means of simulating a real situation. This fact, in our view, highlights the importance the student attributes to content, based on meaning and how it establishes conditions for learning.

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WORKSHOPS

READING GEOMETRICALLY: CHANGING EXPECTATIONS ACROSS K-12 FOR READING DIAGRAMS IN TEXTBOOKS

LESLIE DIETIKER, MEGHAN RILING and AARON BRAKONIECKI

Abstract

Students of all ages are asked to interpret diagrams when learning about geometry. Building on our prior work, we examine the ways in which expectations for interpreting these diagrams change by comparing tasks taken from U.S. textbooks written for first grade to tasks written for high school geometry courses. Using a framework that describes five dimensions of reading geometrically, we present the analysis of multiple textbook tasks with geometric diagrams. We hope that with this elaboration, we will enable other researchers to analyze the geometric diagrams of textbooks to advance what we know about the changing expectations of diagrammatic interpretation and to support educators by designing opportunities for students to develop strategies for reading geometric diagrams.

In this paper, we extend what we have learned from our study of the expectations of textbooks with respect to how students read geometric diagrams. In Dietiker and Brakoniecki (2014), we were inspired by Pimm (2006) to consider the question, *what does a diagram ask of its reader?* We proposed dimensions of reading geometric diagrams gleaned from analyzing the geometric tasks in multiple elementary and secondary textbooks, including traditional and reform curricula from multiple countries. These dimensions represent distinct aspects of geometric diagrams that students are expected to notice and interpret as they negotiate the meanings of mathematical tasks. Our primary concern is that the ways in which students are expected to interpret geometric diagrams appears to change; the geometric diagrams in tasks in elementary textbooks expect students to make different assumptions about the information in the diagram than those in high school textbooks.

Since then, we have continued our analysis of U.S. textbooks to learn how the expectation of diagrammatic reading changes as students progress through school. In particular, we compared the geometric diagrams found in two Grade 1 textbooks with the diagrams of two high school geometry textbooks. We developed six codes to describe the geometric reading of diagrams, identifying: (1) how the reader is expected to interpret the diagram as something (e.g., a real life object or a representation of a set of objects), (2) whether deductive reasoning with the diagram is required to solve the task, (3) if the reader needs to mentally redraw the diagram to answer the task, (4) whether the reader needs to interpret conventional markings to complete the task, and (5) if the reader is required to read the diagram at all to answer the task. We used this framework to code both the words and diagrams found in textbook tasks. We then compared how the expectations of reading geometrically differ for younger and older students. Note that this framework only describes a reader's interpretation of a geometric diagram within a textbook and does not represent the intention of the textbook's author.

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As reported in Dietiker et al. (2017), we found statistically significant differences between the expectations for reading geometric diagrams found in first grade textbooks compared to those found in the first chapter of high school geometry textbooks. In most cases, elementary readers are expected to interpret geometric diagrams as drawn with metrical assumptions (e.g., assuming that diagrams are drawn to scale), while the most common expectation of high school readers at the start of a geometry course is to interpret the geometric diagram as representing an object that may differ from the diagram (e.g., a shape whose angle measures may not match how they appear in the diagram). In addition, at the start of high school geometry, students are expected to be able to mentally redraw a geometric object represented in a diagram and interpret conventional markings within–expectations that we did not find in first grade textbooks.

In this paper, to further contextualize the differences reported in Dietiker et al. (2017), we present a comparative analysis of textbook examples to demonstrate and clarify this coding scheme for recognizing the dimensions of reading geometrically.

Comparing Geometric Diagrams for Reading Expectations across Grade Level

The elementary and high school tasks in Figure 1 differ notably in the ways that readers are expected to interpret the diagrams. First, the elementary task uses diagrams where the reader is expected to interpret the diagram *as drawn*, whereas the high school task uses a diagram where the object in the center of the diagram cannot be interpreted as an accurate representation of the mystery object, requiring the reader to ignore how it is drawn. Additionally, the diagram in the elementary task is *necessary* to the task, since without the diagram, the task is impossible to solve. In contrast, the diagram in the high school task is *supplementary* to the task, since all information needed to complete the task is included in the problem statement. Lastly, we note that the elementary task does not require the reader to mentally draw a new figure (i.e., redraw one that is not congruent to the given objects in the diagram) while the diagram in the high school task requires a reader to *mentally redraw* the potential solid in order to solve the task.

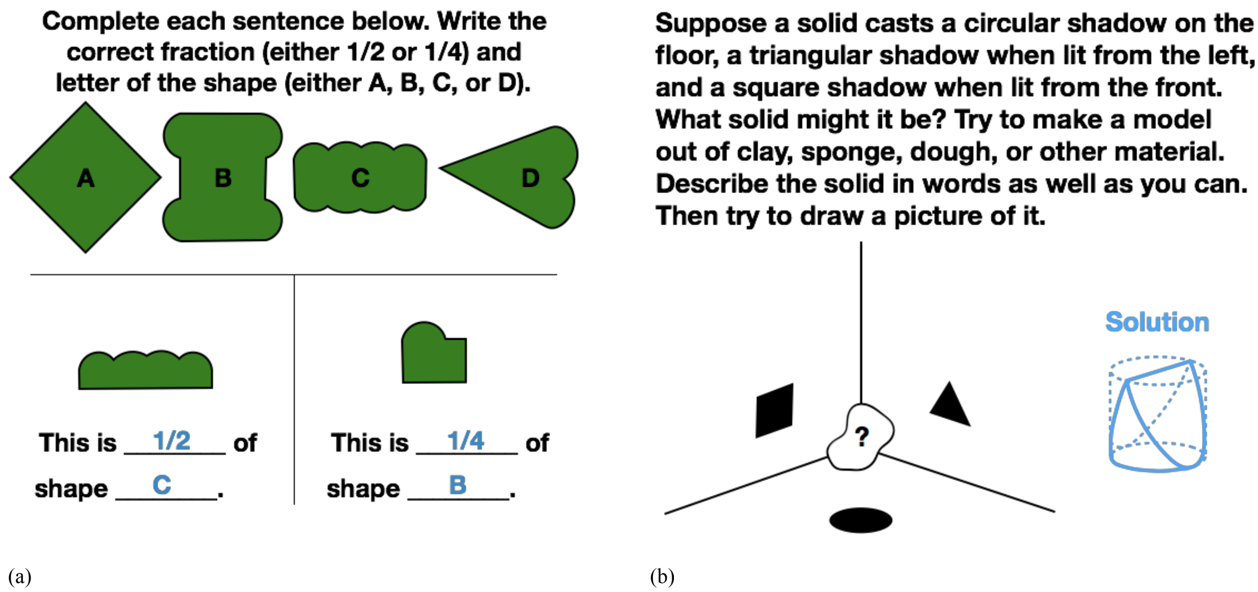
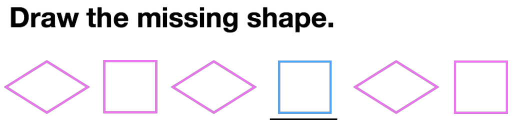
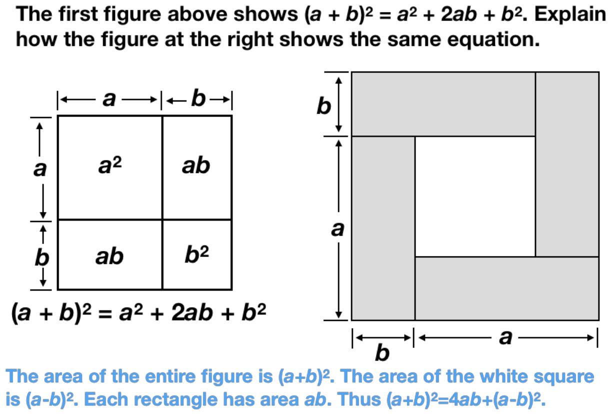


Figure 1 - Problems adapted from (a) an elementary textbook (*Trailblazers* 2008, p. 385) and (b) a high school textbook (*CME* 2009, p. 17)

Often, the same geometric figures will be present in both elementary and secondary textbooks, but their contexts mean that students are expected to interact with them very differently. For example, although both the elementary and high school tasks in Figure 2 contain squares that exist in relation to other shapes, the tasks require students to interpret the diagrams as representing different types of geometric figures. In the elementary task, the squares do not need to be *interpreted as geometric objects*. In the high school task, students need to recognize that the diagram represents many geometric figures at once, in which certain properties, such as right angles, are held constant, while the sizes of a and b change. Students must *interpret as a representation of multiple objects,* especially since the length marked a is noticeably different in both high school diagrams. High school students need to *use deduction* to determine unmarked side lengths, and then areas, in order to complete the task. Lastly, as in all elementary tasks that we found, the students do not have any *conventional markings* to interpret. The high school students, on the other hand, must interpret the side lengths correctly.



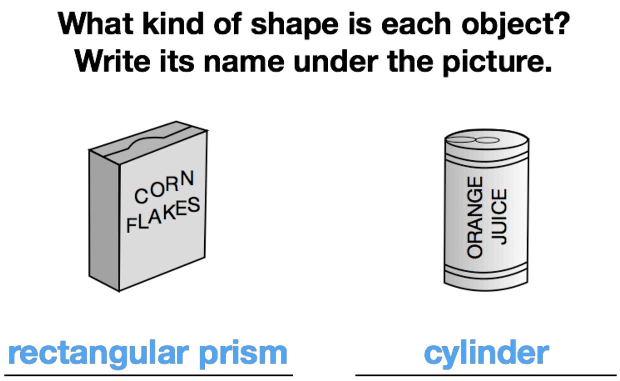
(a)



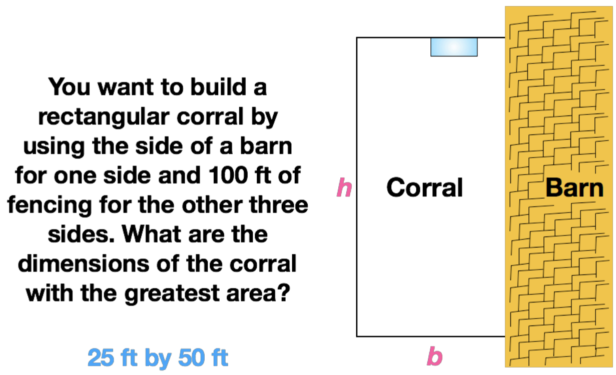
(b)

Figure 2. Problems adapted from (a) an elementary textbook (*Everyday Mathematics*, 2007, p. 285) and (b) a high school textbook (*CME* 2009, p. 15)

Figure 3 offers yet another contrast that reveals further differences for how a reader is expected to interpret a diagram. The elementary task provides diagrams of a cereal box and a can of juice. However, rather than see these diagrams as the real-world objects, students are expected to interpret these as geometric objects (namely, rectangular prism and cylinder, respectively). The students are required to read the diagrams (i.e., they are *necessary* to complete the task) and they are expected to interpret the diagrams *as they are drawn*; that is, no mental reworking of the diagrams is necessary.



(a)



(b)

Figure 3. Problems adapted from (a) an elementary textbook (*Everyday Mathematics* 2007, p. 651) and (b) a high school textbook (*Prentice Hall Geometry* 2004, p. 57).

Similarly, the task from the high school textbook in Figure 3 expects students to *interpret a geometric diagram as a representation of a real-world phenomenon* (i.e., farm with a barn and corral). Yet this diagram is expected to be read differently. Rather than being interpreted as *the* geometric object of focus, which is how the juice can or cereal box can be interpreted, the high school diagram instead represents one of many potential representations of the corral. For example, students are expected to accept that the dimensions of the corral will vary and can even contradict the image in the diagram (e.g., when $b > h$). In that way, we say that the high school diagram expects students to be able to “see” the diagram as one of many, by *mentally redrawing* it to correspond to the variety of possible dimensions. In addition, since the problem statement of the task contains all the necessary information, the high school geometric diagram is *supplementary*.

Discussion

As shown in this paper, reading geometrically involves connecting objects with real-world contexts, recognizing if the diagram is to be taken as drawn or as an abstract representation of a single or multiple objects, interpreting with deduction, mentally redrawing, and interpreting conventional markings. In addition, we showed how some tasks that include diagrams can be completed without consulting the diagram at all. Future research is needed to determine whether the change in expectations is abrupt or gradual. Moreover, these differences raise the question of how mathematics educators can help students understand how to read diagrams and navigate the transitions between the different expected ways of reading. We hope that if educators make these expectations explicit and offer opportunities to develop reading strategies, students may gain fluency in reading and interpreting mathematical diagrams, which may further enhance their mathematical understanding.

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ANALYSIS OF BRAZILIAN TEXTBOOKS

LUISA RODRÍGUEZ DOERING, CYDARA CAVEDON RIPOLL and ANDRÉIA DALCIN

Abstract

In this text we present a synthesis of the "Analysis of Brazilian textbooks" workshop offered to primary education as well as undergraduate teachers that was idealized by the three authors and presented by the first two at the II International Conference on Mathematics Textbook Research and Development.

Key words: Textbook analysis. Brazilian Textbooks.

Introduction

The analysis of Brazilian primary education textbooks constitutes a relatively recent practice in Brazil. Although the policies that regulate the choice and offer of textbooks for public school students goes back to 1929, it was only in 1996 that the *Programa Nacional do Livro Didático* (PNLD)[1] started, assessing the submitted textbooks, and collecting the approved ones in an official Guide. Nowadays, with the implementation of the PNLD, each public school, oriented by its teachers, indicates three collections of textbooks among the approved ones. The government then sends to each school, for free, one of the three indicated collections to be used along the following three years. As a consequence, it became part of the reality of the public school teachers in Brazil to have available in their classrooms a mathematics textbook that may not have been the one that she or he previously suggested. Moreover, very often, this is the only material resource available to the students. This shows how important it is nowadays for Brazilian teachers to develop the ability to analyse textbooks and to criticize them, as well as the ability to adapt the activities or the textbooks texts, creating alternative ones. The same ability applies to educational resources available on free websites.

This was the context in which the authors idealized this workshop. It had three goals: i) to discuss the list of items to be analysed in textbooks that exists in the official PNLD Guide and improve it with other items suggested by the audience with emphasis on the content, the language and the images used in the textbook under analysis; ii) to practice critical analysis of textbooks, working on excerpts previously chosen by the authors; iii) to stress the importance of a continuous critical textbook analysis, even with PNLD approved textbooks.

There were nine participants in the workshop, with quite different interests as well as viewpoints, which enriched the workshop proposal and contributed to its success: two Elementary School teachers who also work for Publishers, three graduate students, three undergraduate students, and

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[1] The Programa Nacional do Livro Didático (PNLD) is Brazil's textbook assessment program, which includes mathematics, and selects the textbooks that are distributed for free by the Brazilian Ministry of Education.

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finally an University teacher who is responsible for an undergraduate course which has the same goal of this workshop.

The workshop was structured on four moments. In the first 20 minutes, the question *What to analyse in a textbook?* was offered to the audience. Then some slides were presented with the criteria put forth by the Brazilian government in the PNLD Guide. For example, books that present conceptual errors or sentences that induce to errors or that suggest prejudice or discrimination of any kind are nowadays not accepted in the official Guide. During the discussion about the criteria defined in the PNLD, one participant called the attention to the lack of a specific item that evaluates the existence of activities or didactic situations that could provoke students' formulation of conjectures, motivated for example by the search for patterns that stimulates abstract thinking. This was one of the items that were added to the list of items to be analysed in textbooks.

The second moment of the workshop lasted 40 minutes. During this time, four different one page excerpts of textbooks for the second segment of Elementary School[2] were chosen for analysis by the authors. The nine participants were organized into four small groups, and each group analysed one excerpt for 15 minutes.

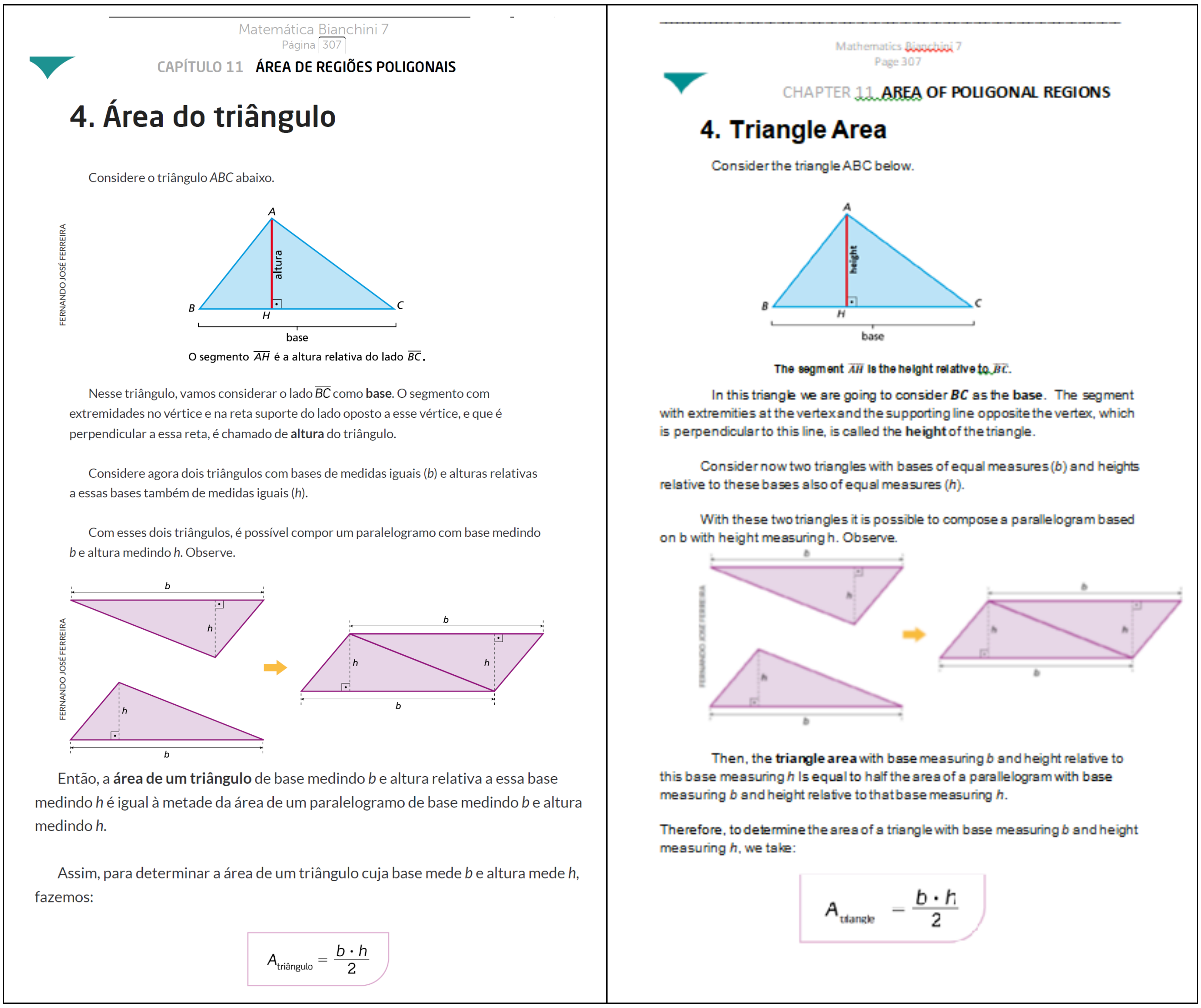
The third moment lasted 50 minutes, and was devoted to the presentation of the analysis carried out by each group. Each presentation was accompanied by a discussion of all the participants about the content covered in each excerpt and the analysis carried out. We report below two of the four recorded presentations related to the excerpts in Figures 1 and 2.

Figure 1 contains an excerpt from a 7th grade textbook. The group started calling the attention of the participants for the first paragraph, emphasizing that a definition for base of a triangle is not presented in the excerpt, and that it is not possible to deduce, simply analysing this page, whether the term base was previously defined or not. The group also criticized the definition of height of a triangle, observing that it is presented in a confusing and inappropriate way for a 7th grade student. "In fact, the use of the term 'supporting line' does not belong to the mathematical vocabulary of 7th grade students", pointed out one of the members of the group. In addition, "the paragraph begins considering the side BC as the triangle's base, and the definition of height does not even mention the term base at all", pointed another one. Another aspect that called the attention of the group was that it is not made explicit that a triangle has three bases, hence three heights. Finally, it was considered by the group and the other participants that height was not sufficiently explored in other triangles, for example, in triangles that have an obtuse angle.

In the second paragraph, considering the objective to be achieved in the third paragraph, the group pointed out that there is a slip of the author: the triangles not only need to have the same base and the same height, but they must also be congruent, something which is not emphasized neither in the text nor in the image. This was considered by the audience an error that may induce the misconception that two triangles of the same base and the same height are always congruent. The audience concluded that this situation would justify an exclusion of this textbook by the PNLD, as required in the PNLD Guide: there is a statement, supported by an image, which leads to a conceptual error.

The third paragraph presents the deduction of the formula for the area of a triangle, "instead of suggesting exploratory activities that allow the creation of strategies by the students (involving cut-outs of figures, for example)" remarked one of the members. The group considered this approach inadequate, since it does not stimulate the development of mathematical thinking and problem solving.

[2] The second segment of elementary school is from the 6th to the 9th grade.



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CAPÍTULO 11 ÁREA DE REGIÕES POLIGONAIS

4. Área do triângulo

Considere o triângulo *ABC* abaixo.

O segmento $\overline{AH}$ é a altura relativa do lado $\overline{BC}$.

Nesse triângulo, vamos considerar o lado $\overline{BC}$ como **base**. O segmento com extremidades no vértice e na reta suporte do lado oposto a esse vértice, e que é perpendicular a essa reta, é chamado de **altura** do triângulo.

Considere agora dois triângulos com bases de medidas iguais (*b*) e alturas relativas a essas bases também de medidas iguais (*h*).

Com esses dois triângulos, é possível compor um paralelogramo com base medindo *b* e altura medindo *h*. Observe.

Então, a **área de um triângulo** de base medindo *b* e altura relativa a essa base medindo *h* é igual à metade da área de um paralelogramo de base medindo *b* e altura medindo *h*.

Assim, para determinar a área de um triângulo cuja base mede *b* e altura mede *h*, fazemos:

$$A_{\text{triângulo}} = \frac{b \cdot h}{2}$$

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CHAPTER 11 AREA OF POLIGONAL REGIONS

4. Triangle Area

Consider the triangle ABC below.

The segment $\overline{AH}$ is the height relative to $\overline{BC}$.

In this triangle we are going to consider *BC* as the **base**. The segment with extremities at the vertex and the supporting line opposite the vertex, which is perpendicular to this line, is called the **height** of the triangle.

Consider now two triangles with bases of equal measures (*b*) and heights relative to these bases also of equal measures (*h*).

With these two triangles it is possible to compose a parallelogram based on b with height measuring h. Observe.

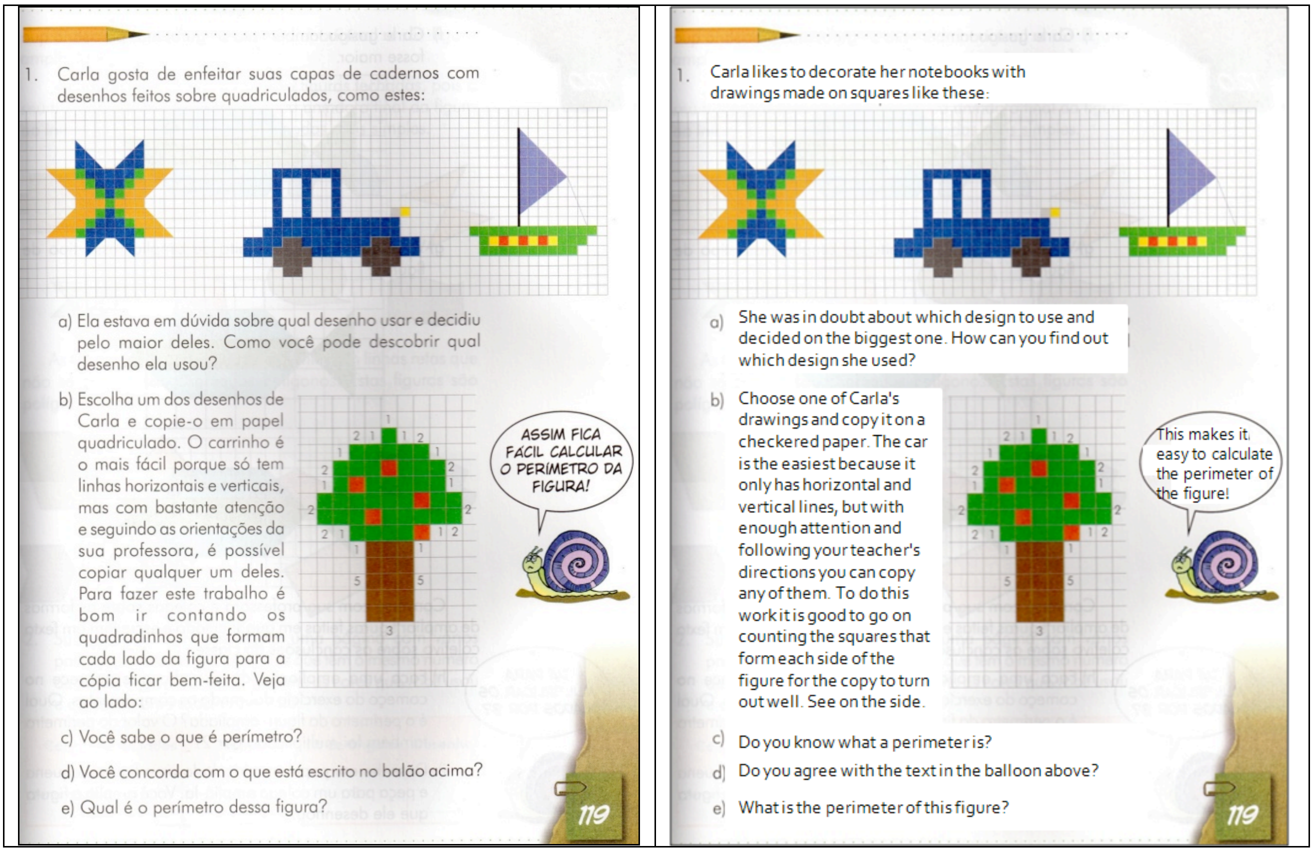
Then, the **triangle area** with base measuring *b* and height relative to this base measuring *h* is equal to half the area of a parallelogram with base measuring *b* and height relative to that base measuring *h*.

Therefore, to determine the area of a triangle with base measuring *b* and height measuring *h*, we take:

$$A_{\text{triangle}} = \frac{b \cdot h}{2}$$

Figure 1: on the left, an excerpt from Bianchini (2014), and its translation on the right

Another group presented the analysis of a 5th grade textbook excerpt (Figure 2), claiming that it has many problems. One of them is that the excerpt consists of an activity “that suggests several different interpretations, which even hinder its solving. For example, the word ‘biggest’ in the sentence ‘She decided on the biggest one’ is mathematically ambiguous” remarked one of the group members. The group also called the attention of the audience to item (b): the command “Choose one of Carla’s drawings…” is followed by the sentence “The car is the easiest ...”, suggesting that there is actually no choice for the student; also the final commands of this item may confuse a 5th grade student, especially the sentence “... it is good to go on counting the squares that form each side of the figure in order that the copy turns out well made”. The group pointed out that this statement may elicit misunderstandings between area and perimeter: how does a square (two-dimensional) form a side (one-dimensional)? The audience agreed with the group that, after all, this activity aims to conceptualize perimeter, which becomes clear in item (c) but in the beginning it is suggested to the reader the counting of squares, which indicates an aim to introduce area. It was also stressed by the group that the question in item (c) (“Do you know what a perimeter is?”) deserves attention from the teacher, since it is very likely that the students’ answer will be “no”, once the term perimeter is not self-explanatory.



1. Carla gosta de enfeitar suas capas de cadernos com desenhos feitos sobre quadriculados, como estes:

a) Ela estava em dúvida sobre qual desenho usar e decidiu pelo maior deles. Como você pode descobrir qual desenho ela usou?

b) Escolha um dos desenhos de Carla e copie-o em papel quadriculado. O carrinho é o mais fácil porque só tem linhas horizontais e verticais, mas com bastante atenção e seguindo as orientações da sua professora, é possível copiar qualquer um deles. Para fazer este trabalho é bom ir contando os quadradinhos que formam cada lado da figura para a cópia ficar bem-feita. Veja ao lado:

ASSIM FICA FÁCIL CALCULAR O PERÍMETRO DA FIGURA!

c) Você sabe o que é perímetro?

d) Você concorda com o que está escrito no balão acima?

e) Qual é o perímetro dessa figura?

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1. Carla likes to decorate her notebooks with drawings made on squares like these:

a) She was in doubt about which design to use and decided on the biggest one. How can you find out which design she used?

b) Choose one of Carla's drawings and copy it on a checkered paper. The car is the easiest because it only has horizontal and vertical lines, but with enough attention and following your teacher's directions you can copy any of them. To do this work it is good to go on counting the squares that form each side of the figure for the copy to turn out well. See on the side.

This makes it easy to calculate the perimeter of the figure!

c) Do you know what a perimeter is?

d) Do you agree with the text in the balloon above?

e) What is the perimeter of this figure?

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Figure 2: On the left, an excerpt from Isolani (2005), and its translation on the right.

The excerpt also presents images which the group considered problematic: regarding the image of the tree, it was pointed out that this image does not help in the understanding of the ongoing process. Moreover, the indication of the counting in the illustration is chaotic: there is a “2 1” that the student needs to interpret as “2 and 1”, and not as twenty one, remarked one of the members. “Also, if the student tries to use as a hint the saying of the snail (“Thus it is easy to calculate the perimeter of the figure”), she or he will not be able to calculate the perimeter of the star or the boat, since parts of those figures are formed only by portions of squares. In addition, and this was considered a serious mistake by the group and the audience, students are induced to choose the car, whose perimeter is not easy to be determined, “because it is not a simply connected picture”, remarked one of the participants; hence, in order to calculate the perimeter of the car, it would be necessary to discuss with 5th grade students whether the windows are part of the perimeter of the car, or not. All the participants agreed that all the chosen drawings were inadequate to introduce the concept of perimeter with 5th grade students.

The group ended its presentation saying that the idea behind the sentence “For the copy to be well made” may discourage students to try any other strategy of their own. The opportunity to stimulate students to outline their own strategies is once more lost.

In the fourth and final moments of the workshop, a reflection was carried out concerning the process of textbook analysis. The participants concluded that both excerpts needed reformulation. Two aspects were then stressed by the authors: i) it is imperative that the teacher examines critically the textbook adopted for the classroom; ii) critical analysis is a practice that needs to be developed.

The objectives set for the workshop were attained, in the opinion of the authors. The most important conclusion of the participants was about the need for specific activities involving the analysis of textbooks (or even specific courses with this aim), which should be part of the curricula of pre-service undergraduate teachers’ education. This would allow future teachers to go through the process of textbook analysis, enabling them to develop a critical and flexible look on the materials offered by institutions, publishers and Internet.

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MATHEMATICAL LESSONS IN A NEWSPAPER OF PORTO (PORTUGAL) IN 1853: A SINGULAR EPISODE IN TEACHER TRAINING

HÉLDER PINTO

"Não se deve exigir da infancia mais do que ella é capaz, também se não deve exigir menos: é perigoso resumir tanto que deixem de saber o que devem aprender; é enganal-os, é fazer que mais tarde lhes custe muito; sem esforços não ha saber."
[One should not demand more from childhood than they are capable of, nor demand less: it is dangerous to summarize so much that they do not know what they should learn; it is deceiving them, and making it harder later; without effort there is no knowledge]
António Luís Soares (Soares 1853, p. 245)

Abstract
In this workshop in ICMT-2017 we presented and analyzed the lectures published in 1853 by António Luís Soares in detail. Since it is impossible to do so in these pages, we just highlight some parts dedicated to the metric system and its comparison with the ancient Portuguese units.

The Industrial Association of Porto was founded in 1849 in the city of Porto (Portugal). This association was formed by the high society of the city (608 members in 1853): 192 artisans/artifices, 36 manufacturers, 174 traders, 48 goldsmiths, 32 landowners, 30 medical doctors and chemists, 84 public servants, 5 farmers, and 7 militaries (Soares 1853, p. 307). The industrial education was one of its first priorities, having created the Industrial School of Porto in 1852, with the following graduations: factory director, overseer of public works, overseer of machines (steam engines), overseer of mines, telegrapher, master of public works and master of chemistry. There was also a factory worker graduation that was preparatory for all the others courses. In 1853, António Luís Soares, professor of this school, published several arithmetic lessons for primary education in the newspaper of the association, a biweekly newspaper whose first issue came out on 1852 – *Da exposição dos elementos da arithmetica na aula de instrucção primaria da Associação* [Lecture of the arithmetic elements in the Association's primary school grade].
António Luís Soares (Porto, 1805 – 1875) was a professor of the Polytechnic Academy of Porto since 1836 (First lecture topics: Arithmetic, Elementary Geometry, Trigonometry and Elementary Algebra) and a professor of The Industrial School of Porto since 1852 (Arithmetic, Algebra and Geometry) (Pinto 2013, pp.139-140). There is a lack of information about this professor and, except for the text that we present here, there is only a reference (Carvalho 2017, pp. 9-10) to another text (Exposição dos elementos de Aritmética para uso dos estudantes do Colégio de Santa Bárbara na cidade de Pelotas, Pelotas, Brazil, 1849), an arithmetic book published in Brazil. No copy of this text was found (Scipião says that António Soares was in Brazil between 1847 and 1851, but no justifications for his stay are pointed out). The Polytechnic Academy of Porto (created in 1837, replacing The Royal Academy of Navy and Trade Affairs of the City of Porto from 1803) and the Industrial School of Porto (1852) were connected in various ways like, for instance, many professors taught simultaneously in both schools and both shared the same building until 1933 (about 80 years). There were also proposals to fusion both in one single institution. However, there was a major reform of the Polytechnic Academy of Porto in 1885 (in 1884 the Academy received

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the most important Portuguese mathematician at that time: Gomes Teixeira) and, at that point, the difference between the two institutions was definitely settled: one more theoretical (university level studies) and the other more technical (intermediate studies). One of the most important characteristics that both schools shared was the fact that both were created to attend effective necessities of the city (sailors, trade men, engineers, industrial and commercial workers,…) and both were funded by the initiative of private institutions of the city (Alves 2006, Serra 1989, and Pinto 2013, pp. 107-179).

Even before the creation of the Industrial School, although primary education was not the main focus of the Industrial Association, a course of «reading and writing» opened, which was attended by 117 students (many of them were destined to the Industrial School). This group included 25 individuals who attended these lectures in order to propagate this knowledge for several villages around the city of Porto. António Luís Soares was intended (Soares 1853, pp. 243-244) to address to these 25 "future teachers" in order to present some works on teaching of arithmetic. Just a few could "attend the invitation" (why?) and so the alternative was to publish such works in the newspaper of Industrial Association of Porto (figure 1).

JORNAL DA ASSOCIAÇÃO INDUSTRIAL PORTUENSE.

NUMERO 16. SEXTA FEIRA 1 DE ABRIL. ANNO 1853.

Figure **1.** Header of the newspaper of the Industrial Association of Porto (number 16, April 1, 1853).

The author makes, in the first issue, several considerations about the importance of propagating the basic math instruction, either for industry either to the trade workers, two important activities for the city at the time. In the following table, the structure of these lessons is presented (table 1).

Date	Number (pp.)	Sections	Lessons
April, 1	16 (pp. 244-248)	Section 1: Formation of the numbers	Introductory observations 1. numeration system
			2. spoken numbering
May, 1	18 (pp. 277-281)		3. written numbering
			4. observations about quantities
June, 1	20 (pp. 307-312)		5. metric system (units, multiples and submultiples)
			6. metric system (written abbreviations)
July, 1	22 (pp. 339-345)		7. the "big" numbers; roman numerals
		Section 2: The first arithmetic operations	1. addition
July, 31	24 (pp. 374-383)		1. addition (cont.) 2. multiplication
		Tables of units conversions	
August, 1	1 (T2) (pp. 2-3)		
December, 1	9 (T2) (pp. 138-141)	How to convert the ancient Portuguese units to the metric system and vice versa	

Table 1**.** Structure of the lectures.

In the fourth lesson, the author made several observations about quantities. He observes that the numbers ("a number is a set of things with the same name") learned are good to count, for instance, men and trees. And what about a big quantity of grains of wheat? It was impossible and inutile to count each single grain… There are things that we cannot measure the quantity only using numbers. So, it's necessary to have another type of measurement units for dry volumes (grains and beans for instance), surfaces (lands and fields), liquid volumes (milk, wine), weights (reference to the scale of two arms), time and money (coins). At this point, the author does only a first introduction to this subject and a reinforcement of the need of other measurement units for the everyday live/industry/commerce.

The metric system is formally presented in lesson number 5 (units, multiples and submultiples). The author explains that, historically, the first measurement units were, naturally, the *Palmo* (hand) and the *Pé* (foot), but this kind of units, over time, had proven to be difficult to work with and a source of problems and errors. Afterwards, he made reference to France and to the difficulties in the implementation of the metric system. But he thinks that the context in Portugal in 1853 was different:

> "However, there was hope that this time was possible to introduce the metric system in the plenitude. The artisans didn't fight against this system anymore, because they are acquainted with the new units by the visits of the academic/theoretic people to their shops."
>
> Jornal da Associação Industrial Portuense, 20, June 1, 1853, pp. 307-308
>
> [translated by the author of this paper]

The next step, in his opinion, was to propagate this system for the common retail trade, because it would facilitate the commercial transactions in everyday life. Then he presented some common old Portuguese units in a way to highlight two major problems: first of all, it's difficult to memorize all the relations between them. For example, he presented the following length units: 1 *braça* = 2 *vara*; 1 *vara* = 5 *palmo*; 1 *palmo* = 3 *pollegada*, and weight units: 1 *quintal* = 4 *arroba*; 1 *arroba* = 32 *arratel*; 1 *arratel* = 16 *onça*; 1 *onça* = 8 *oitava*. On the other hand, it's very difficult to operate with them (for instance, what is the relation between *quintal* and *oitava*?).

Afterwards, the author presented, finally, some metric units: the meter (linear); the are (surface; 100 square meters; note that the square meter is too small to measure fields…); the liter (capacity) and the gram (weight). Then, the multiples of these units were presented («Deca» means 10 primitive units, «Hecto» means 100 primitive units and «Kilo» means 1000 primitive units) and submultiples («Deci» means 1/10 primitive units, «Centi» means 1/100 primitive units and «Milli» means 1/1000 primitive units). The author does now an important warning: for surfaces (1 m^2 = 100 deci-m^2) and capacities (1 m^3 = 1000 deci-m^3); we must be very careful when working with multiples and submultiples.

The lesson number 6 was the continuation of the presentation of the metric system. He teaches the written abbreviations and presents various tables comparing the old Portuguese units with the "new" metric system (units for big lengths, small lengths, agricultural, small surface, liquid volumes, dry volumes, solid volumes (like wood), weights and small weights).

Afterwards, when he teaches addition, he returned to emphasize that the metric system is preferable to the old Portuguese units. At this point, he noted that adding numbers with decimal parts is essentially the same as adding integers and presented several examples. It's only needed to put the decimal points vertically aligned and the method is exactly the same (add 1,23 meters to 6,94 meters is not harder than add 123 to 694). In fact, this is an important advantage from the metric system: it's easier to work (in this case, add) with the sub-units of the metric system than with the old Portuguese subunits. And to emphasize this point of view, he presented a very difficult example (table 2) using linear units (note that: 1 *p(olegada)* = 12 *l(inha)*; 1 *P(almo)* = 8 *p*; 1 *B(raça)* = 10 *P*).

B.	P.	p.	l.
30	3	7	5
12	8	2	11
13	2	2	4

$5\,l. + 11\,l. = 16\,l. = 1\,p. + \boldsymbol{4\,l}.$

$7\,p. + 2\,p. + 1\,p. = 10\,p. = 1\,P. + \boldsymbol{2\,p}.$

$3\,P. + 8\,P. + 1\,P. = 12\,P. = 1\,B. + \boldsymbol{2\,P}.$

$30\,B. + 12\,B. + 1\,B. = \boldsymbol{43\,B}$ (there is a typo).

Table 2. Fourth example from the Lesson 1 ("The first arithmetic operations").

For students' homework, the author suggests more examples with old Portuguese units, even more complicated, to convince everyone that it was a mess to work with the old units and it was necessary and easier to adopt the «new» metric system.

In conclusion, the lectures presented here are intimately connected with the socio-economic context of the city of Porto: an industrial and commercial city and the second city in population number of the country (Pinto 2013, p. 20). On the other hand, these lectures were sponsored by an Industrial Association, which explains the fact that they were lectures with a very practical goal (teach the basic arithmetic always with the aim of using it in the industrial/commercial trade). Note also that the metric system was implemented officially in Portugal in 1852 (the first attempt was in 1814 but with no success), so it was a very important and new subject when these lectures were published. All these factors explain the (excessive?) focus on the metric system and its relation with the old Portuguese units. It should also be highlighted that it is quite peculiar that these lectures have been published in a newspaper and not, for example, in a textbook to be acquired only by students, which seems to be indicative of the intention to propagate this basic knowledge of mathematics by several target audiences (not only for children but also to adults).

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DEVELOPING OPEN-SOURCE CURRICULUM IN BRAZIL: THE *LIVRO ABERTO DE MATHEMATICA* PROJECT

MERIL RASMUSSEN and FABIO SIMAS

Introduction

The *Livro Aberto de Mathematica* project is a Brazilian initiative to create a low-cost, digital mathematics textbook series for high schools through a process designed to empower practicing teachers both by providing them with high-quality didactic materials and by allowing them a voice in their creation and ongoing revision.
In early 2016, a volunteer-based team of mathematics teachers and teacher-educators created a pilot textbook, *Frações no Ensino Fundamental I* (Bortolossi et al. 2016), which serves as the prototypical production model for current efforts.

In addition to a concurrent workshop on the Fractions book offered in Portuguese for Brazilian high school teachers and an overview of the *Livro Aberto* project presented as part of the conference's oral communications, we held a workshop in English entitled: "Adapting Sociocultural Research Frameworks for Brazil," which targeted international participants. Our plan was to offer up the ongoing *Livro Aberto* project as a case study through which to discuss our challenge of how to create a meaningful, ongoing feedback loop between practicing teachers and our mostly-university-based author teams. In particular, we were interested in recent sociocultural and developmental research involving teacher empowerment (Even & Olsher 2012; Gravemeijer & van Eerde 2009; Ruthven 2012).

We also hoped to share and discuss the mechanics of our open-source style of collaboration. To this end, we set out guidelines for an open-source-inspired strategy of how to manage the divergent opinions that might arise within the group discussions. Thus, rather than just talking about how we had chosen to organize our project, we hoped to offer workshop participants a participatory experience of open-source decision-making.

As it played out during the workshop, discussion focused on this open-source aspect of our project and little attention was given to developing our research plan. Thus, in the act, the focus of the workshop shifted from *how best to conduct developmental research* to *how best to accommodate difference and dissent within a collaborative initiative*.

The Open-Source Approach

The history of the open-source software movement is tied to the collaborative development of the Linux operating system initiated by Linus Torvalds beginning in 1991. The open-source system of collaboration was further refined by Torvalds in 2005 as the Git version control system (Hamano & Torvalds 2005). Git goes beyond the open sharing of source-code to address the dynamics and mechanics of collaboration and it is this that the *Livro Aberto* project has adapted to create our own collaboration platform.

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Git relies on a system of *forks* and *pull requests* designed to maximize collaborators' independence while ensuring accountability. Git maintains clear, local hierarchal control while offering the possibility of free, non-hierarchical splintering designed to promotes diversity and innovation.

A diagram of the evolution of the Linux operating system (Singh 2013) makes evident its obvious parallel with biological evolution.

Figure 1 : **The Evolution of Linux**

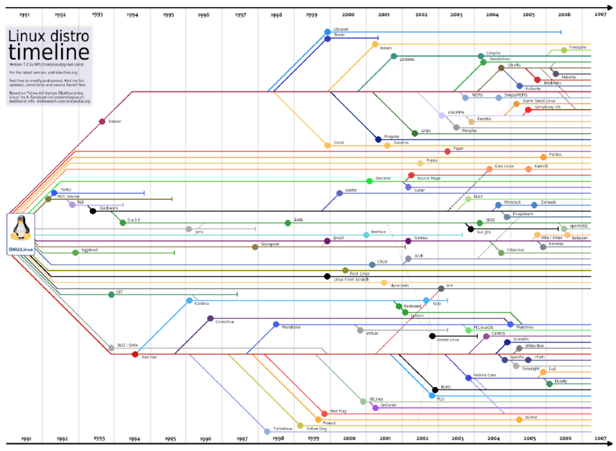

image source: fair use of (Singh 2013) http://techpp.com/2013/02/20/evolution-of-linux/

In the open-source process, differences result in forks, while cooperation remains entirely voluntary and optional. If an author wants to contribute back to the collective project, they request a pull to the custodian of the master line who examines the suggested changes and decides whether to accept them back in.

Figure 2: **Wikipedia Process**

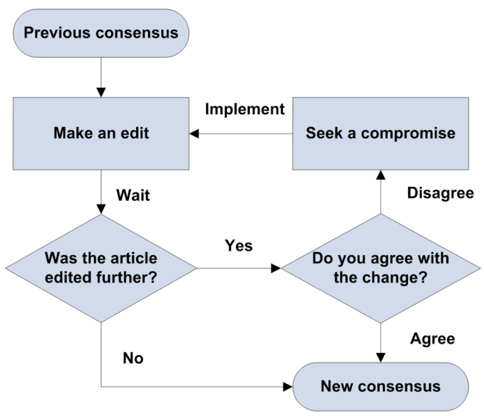

image source: by Locke Cole at English Wikipedia via Wikimedia Creative Commons

This open-source approach to collaboration stands in stark contrast to the consensus-based approach favoured by Wikipedia.

On a spectrum of decision-making strategies, open-source extends out beyond a democratic approach which ultimately prioritizes unified action. In an open-source system, participants who disagree with decisions taken are always free to forge their own way separately. In fact, the open-source approach most closely resembles free-flowing social interaction. In conversation, people typically agree and/or disagree and ultimately move forwards either together or apart. Any formal insistence on reaching an agreement can feel oppressive. On the other hand, agreeing to disagree, while liberating, can tend to circumvent cooperation. What the open-source approach ultimately facilitates is a flexible hierarchy, one that may tend towards a meritocracy, but that at least flows with people's individual inclinations, a peaceful coalition of the willing.

The Workshop

The workshop was attended by a dozen participants. We began by sharing the story of the *Livro Aberto* project and describing the kinds of choices we were making and the collaborative strategies employed. We also explained the open-source-inspired workshop format: respect and listening were required but agreement was not, and divergent sub-groups were welcome to break away and lead separate conversations at any point.

Early on, in the conversation this strategy came to a head, as two participants voiced their skepticism in regards to the viability of our open-source approach. After several minutes of intense discussion, and with no easy resolution in sight, we invited them to lead a breakaway faction in order to pursue their preferred strategy and leave us to pursue ours. It was a poignant moment, on the edge of social awkwardness. In such a small group, meeting together for a short duration, it was perhaps unrealistic to expect that a subgroup would actually form. However, even entertaining the possibility of amicably splitting the group carried with it a liberating effect by dignifying each individual's autonomy.

As it happened, there was also an experienced open-source programmer in the group who was able to offer a third-party opinion attesting to the functionality of the open-source approach, and we proceeded cognizant of an emerging social dynamic within the group that allowed space for skeptics, who were free to lurk and listen, disagree, and depart when they wished, as well as for participants with common ideas to share and develop together.

Overall, the workshop discussion focused on the open-source strategy and how it can be adapted to curriculum development and on the details of the *Livro Aberto* project. The discussions elaborated on the particular situation in Brazil (Watts 2016), the regulatory framework of the National Program for Educational Textbooks (PNLD), and the history of the MatDigital project (Giraldo, Rangel, Ripoll & Mattos 2014).

We tentatively explored the question of how to best bridge Brazil's digital divide -- how to improve mathematics education for students with limited access to computers and connectivity -- and how to conduct meaningful research in Brazil's low-resource context. However, the answers to these questions seemed beyond the purview of researchers working elsewhere.

Conclusion

In planning the workshop, we had hoped to elicit a master-class in how to translate developmental research methods to our low-resource context. In the end, discussions revolved around the potential of the open-source approach already embedded in our process. As experienced in both the workshop and the larger project, an open-source approach foregrounds power dynamics within a group. Not only does it highlight differences, it also makes evident existing and evolving hierarchies. As a strategy designed to mobilize participants, it offers an interesting compliment to sociocultural and developmental research approaches (Even & Olsher 2012; Gravemeijer & van Eerde 2009; Ruthven 2012) which also tend to lead researchers to examine their own roles as actors within educational systems, each with their own relationship to power and authority, and how these inform research strategies and perspectives.

Our experimental workshop format led to a rich participatory discussion. It also offered participants who had minimal familiarity with open-source collaboration an experience of this style of interaction.

For *Livro Aberto*, the workshop served to reconfirm that the issues we struggle with in Brazil do not have easy answers. Little progress was made during the workshop towards devising a research-based feedback loop to support further development of our newly-created materials. However, networking at the conference served to help us to connect with existing materials in Portuguese on the Japanese Lesson Study Approach, which may hold a potential key to our needed feedback mechanism. (Conferences, by their nature, facilitate open-source sharing.)

As we push ahead with our project in the face of a turbulent economic and political environment in Brazil, our open-source approach is also intended to make a space for others to contribute and to help us carry forward the work of devising better mathematics textbooks as they can.

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AN APPROACH TO THE HYPERBOLE CONCEPT BASED ON THE ANALYSIS OF HIGH SCHOOL TEXTBOOKS

NORA OLINDA CABRERA ZÚÑIGA, MARIANA LIMA VILELA AND NAYARA KATHERINE DUARTE PINTO

The proposal of this workshop for teachers was an unfolding of an experience realized with the third year of high school classes in the Technical College of the Federal University of Minas Gerais (Coltec-UFMG), in the year 2015. One of the authors, who is a Mathematics teacher in this institution, together with three trainees from the undergraduate programme in mathematics at the UFMG, currently teachers, have planned and developed a dialogued expository class to introduce and explore the concept of hyperbole in the classroom.
For this class, one of the aspects initially considered was the analysis of the proposed content in the textbook adopted by Coltec. This analysis led us to perceive the need to adapt and complement this content, aiming to introduce the concept of hyperbole comprehensively to high school students. We emphasised that the textbook was a support for teachers to prepare and perform the teaching, and a reference for the students.
Considering the positive returns we had from the students and teachers who experienced the report of Pinto et al. (2016), we developed this workshop at the II International Conference on Mathematics Textbook Research and Development (ICMT-2). The workshop sought to familiarise high school teachers with an approach that prioritised the comprehension of the concept of hyperbole, little explored in textbooks of this level of education.

First, we dialogued with the participating teachers about the learning of conic sections during high school and graduation, and about teaching in the secondary school. The participants remembered that the content of hyperbole was taught them predominantly by direct application of formulas and some sketches, but there was no memory of any approach that explored the concept of hyperbole in itself.

After this dialogue, we described and analysed approaches of the concept of hyperbole proposed in different textbooks, in an attempt to indicate the possibilities suggested in the student's book and in the methodological guidelines in the teacher's manual. Next, we had an activity to introduce the concept of hyperbole using manipulative material, taking as reference the work of Pinto et al. (2016, pp. 6-8). Finally, we held a plenary discussion.

The participation of teachers throughout this workshop in the ICMT-2 and the evaluation of it by the teachers indicated that this activity gave them an experience in which the concept of hyperbole

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was carefully developed using manipulative material. In this sense, we present below three of the statements that were part of the final evaluation by the teachers.

Participant A: “I loved the workshop. It was very didactic, with good ideas for classroom and reflections on textbooks. I am going to do the string activity with my students.”

Participant B: “The workshop was excellent; it highlights the importance of concrete material in building concepts, helping the educational process together with the concept from the textbook.”

Participant C: “The workshop is very good to apply in hyperbole teaching because it allows the student to see how the hyperbole and its concept are built. It is the manipulation of the material what facilitates students to understand the concept of hyperbole.”

We hope this workshop had motivated the teachers to think about their pedagogical practices and the importance to analyse the textbooks used in the educational institution, moreover to inspire them to adapt approaches of mathematical concepts proposed in the educational materials.

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POSTER SESSION

TEACHING PROBABILITY IN EARLY SCHOOL YEARS: THE APPROACH IN BRAZILIAN TEXTBOOKS

MICHAELLE SANTANA and RUTE BORBA

Introduction

Teaching probability concepts in school is highly essential for today´s citizen that has to constantly deal with certainties and uncertainties. Therefore, it is critical for teachers to promote the development of a wide range of probability concepts through experiences, enabling students to observe and then draw conclusions, awakening scientific thinking critical to their education. In this sense, textbooks are very important because they can direct teachers to contents to be taught in each school level and year and show how to broaden the understanding of each concept, in particular those concerning probability.

Literature Review

According to Carvalho and Lima (2010), textbooks contain choices on: the contents to be studied; the methods to be used for students' better understanding; and the curriculum organization to be adopted throughout schooling years. Care on choices, concerning probability, is, thus, necessary in order to address appropriate probabilistic concepts in the classroom.

According to Novaes and Coutinho (2009), when introducing probability, it is necessary to explore some indispensable concepts, namely: *randomness*, *random experiment*, *sample space*, *event* and distinction between the *different approaches* in determining probability. Teachers of early years need to know and understand, and also to be aware of ways in which these notions can be worked on – not necessarily using formal terms, but in activities appropriate to children in early schooling.

These activities may be suggested in textbooks from early years and may also be sought from other sources. Researchers (such as Santana 2011) point out that the various basic notions – such as *randomness*, *chance*, *determinism*, *possibility*, *prediction*, *trial*, *sample space*, *event*, *equiprobability*, *frequency*, *conditionality* – have not been adequately addressed by teachers, students and the resources available, such as textbooks. In this way, probability teaching is limited and, as a consequence, students' understanding in this content may be impaired.

Aims

The research question of the study was: How do Brazilian elementary school mathematics textbooks approach the concept of *probability*? In this sense, the specific aims were:

a) Observe how the concept of probability is constructed in 5th grade textbooks and in the respective teacher manuals;

b) Check the notions addressed (perception of chance, idea of random experience, notion of chance, concept of possibility, among others); and

c) Identify what types of activities (interpretation and construction of tables, bar charts, tree diagrams, among others) are suggested by the authors to work with students.

Procedures

The present study aimed to analyse how Brazilian elementary school mathematics textbooks approach the concept of *probability* and how authors consider the dimensions pointed out by

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Vergnaud (1986) in his Theory of Conceptual Fields: *situations* that bring meanings to a concept; properties and relationships (*invariants*) of the concept; and symbolic *representations* used to represent and work on the concept. With this aim, 11 5th grade textbooks of the Brazilian National Textbook Program were analysed. Analyses considered the way the concept of *probability* was introduced; notions dealt with; the types of activities provided and symbolic representations used.

Analysis and Results

It was observed that the commonly used ways to introduce probability, such as shown in Figure 1, are linked with the ideas of *percentage*, *fractions* or *combinations*. In this example is requested the chance, in fractions, of obtaining heads and tails when tossing a coin.

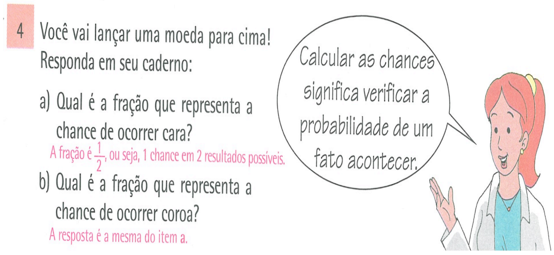

Figure 1. Example of the introduction, in Collection F, of the concept of probability associated with the idea of chance and fractional representation

Furthermore, it was noticed that five notions, namely *chance*, *probability*, *experiment*, *randomness*, *prediction and trial*, were largely covered, with *chance* the most frequent, although no textbook dealt with all these notions. Regarding the types of activities, word problems were the most frequent with 50% of them only with instructions and no type of auxiliary symbolic representation. Other problems, such as the one in Figure 2, used other forms of symbolism to represent the situation posed – in this case the tossing of dice.

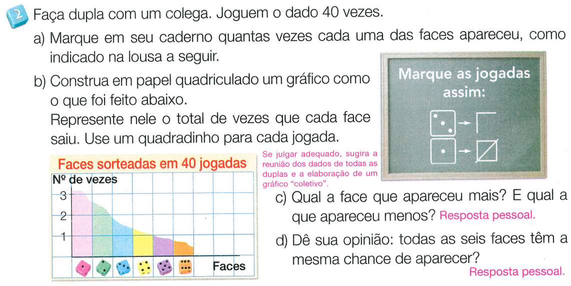

Figure 2. Example of activities, in Collection B, which uses a graph as a symbolic representation

In general, the textbooks analysed do not deeply explore the concept of probability and the proposed teaching in the teachers' manuals is fragmented. Thus, improvement is necessary to better motivate early school students in their development of probabilistic thinking.

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THE PRESCRIBED CURRICULUM FOR COMBINATORICS AND WHAT IS PRESENTED IN 5TH GRADE BRAZILIAN TEXTBOOKS

GLAUCE VILELA and RUTE BORBA

Faced with all the variety of material offered to the teacher, the textbook still occupies a prominent place in teaching and learning processes in the school context. In the last decades in Brazil, this resource is becoming more and more the target of research, especially in what concerns the approach of specific contents from the official curricular guidelines. For Sacristán (2000), the official curricular guidelines, defined by him as prescribed curriculum, acts as a starting point in ordering the curricular system, and becomes a reference in the elaboration of didactic materials.

In this sense, Sacristán (2000) defines textbooks as presented curriculum that translate and interpret the meaning and contents presented in the prescribed curriculum.

The present study aims to analyze the guidelines given concerning combinatorics in the official documents for elementary mathematics education, the relations of these orientations with the approach to the problems involving combinatorial reasoning in 5th year textbooks and the guidelines presented in teachers' manuals, taking into account varieties in the three dimensions of concepts proposed by Vergnaud (1986) (meanings, invariants and symbolic representations). For the analysis, we used as official documents the national curricular parameters - PCN (Brasil 1997), eight 5th grade mathematics textbooks and their respective manuals. These textbooks were chosen randomly and the 5th grade was chosen because previous studies (Borba, Rocha & Azevedo 2015) showed that, in elementary school, more combinatorial situations are presented in this school year.

The prescribed curriculum analysed presented indications of paths to be followed for the teaching of combinatorics. This approach becomes explicit when the document emphasizes the need of activities with different meanings of multiplication, dealing explicitly with one of the combinatorial situations, that is the Cartesian product, and pointing out possible symbolic representations such as drawings and tree diagrams. The national curricular parameters (PCN) states: "Having two skirts - one black (B) and one white (W) and three blouses - one pink (P), one blue (B) and one grey (G) - in how many different ways can I dress? Analyzing this problems, it is seen that the answer to the question asked depends on the possible combinations. Students can obtain the answer, in a first moment, by making drawings and tree diagrams, until exhausting the possibilities" (Brasil 1997, p.69)".

According to the PCN: "With respect to combinatorics, the objective is to lead the student to deal with problem situations involving combinations, arrangements, permutations, and especially the multiplicative principle of counting" (Brasil 1997, p.36). However, in this prescribed curriculum we observed the absence of specific orientations for the teaching in the first school years with different combinatorics situations, their meanings, their invariants and varied forms of symbolic representation.

Regarding the results obtained in textbooks, we observed that the meanings with the highest total percentages of presentation were combination (such as the example presented in Figure 1) and

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Cartesian product (such as the example presented in Figure 2). There was little variation of the symbolic representations used in the presentation and in the request for solving the problems.

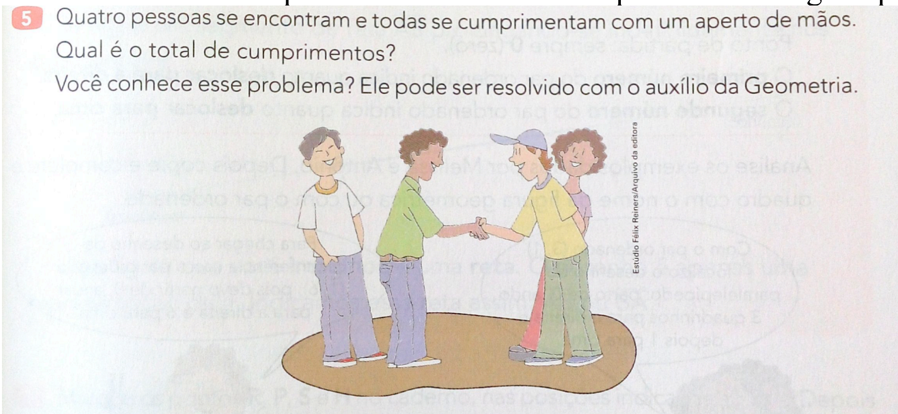

Figure 1 - Example of a combination problem:
"Four people meet and shake each other's hands. What is the total of greetings?"

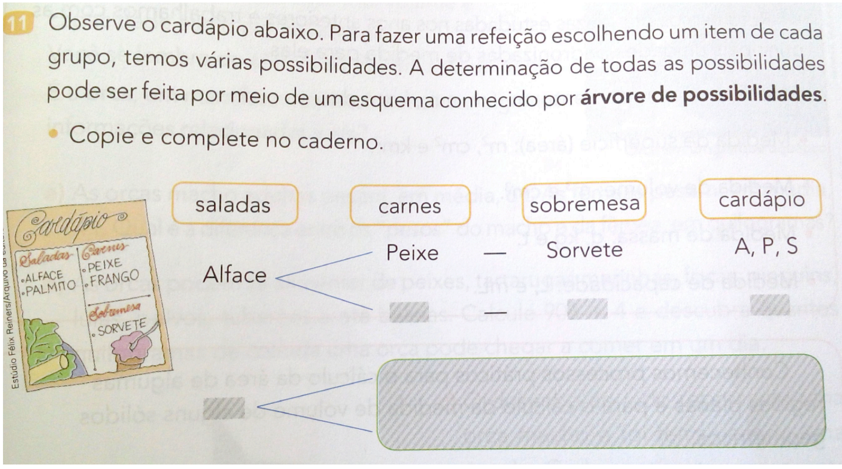

Figure 2 - Example of a Cartesian product problem concerning the choice of a meal with the options between 2 types of salad, 2 types of meat and 1 type of desert.

The results show little orientation in textbooks to the approach of combinatorics in the early years of schooling and that the books attend, only to some extent, to what is pointed out in the prescribed curriculum.

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ADDENDUM TO SYMPOSIUM C

DOCUMENTATIONAL TRAJECTORIES AS A MEANS FOR UNDERSTANDING TEACHERS' ENGAGEMENT WITH RESOURCES: THE CASE OF FRENCH TEACHERS FACING A NEW CURRICULUM

KATIANE DE MORAES ROCHA, LUC TROUCHE AND GHISLAINE GUEUDET

Abstract

This article contributes to the symposium coordinated by Janine Remillard, Hendrik Van Steenbrugge and Luc Trouche, that discusses teacher-resource use around the world. We address the following issue: how might we understand the processes by which teachers engage with curriculum resources to design instruction? We situate our work in the documentational approach to didactics and propose two new concepts: documentational experience and documentational trajectory, aiming to analyze teachers' professional development over time through their interactions with resources. Our methodological choices are inspired by the reflective investigation leading teachers to reflect about their work. We analyze the case of two middle school teachers, Anna and Viviane. Our preliminary findings evidence that these teachers' documentational experience developed quite differently. Anna's documentation is strongly supported by her collective work outside of school, helping her to face new curricular changes. Viviane's documentation work is strongly supported by her interactions with her colleagues and the use of institutional resources, helping her to face the same changes.

Keywords: Documentational approach to didactics, documentational experience, documentational trajectory, reflective investigation, professional development.

Introduction and context

The central theme of our work is the engagement of teachers with curriculum resources. Our study is situated in France, in a context marked (at least) by three features.

The first one is the profusion of available digital curriculum resources. Digital resources provide new means for designing and sharing teaching materials. In France, many such resources are available on the Internet. A good example is the Sésamath association (http://www.sesamath.net) that designs online resources, among them free mathematics e-textbooks.

A second important feature is the introduction, in September 2016, of a new curriculum, for grades 1 to 9 introducing deep changes: the curriculum is designed over cycles of three years instead of one year; new topics, mainly algorithmic and programming, are introduced within mathematics. The French Ministry proposed a set of resources that aim to support teachers' implementation of this new curriculum: new tasks for working with students, with respect to new topics to be taught and to competences to be developed and some methodological advice to do it. In general, teachers never taught, and, most of the time, never learnt these new topics. Thus they need to learn these

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topics and learn to teach them using new digital resources. Moreover, they should design new resources in this perspective.

The third feature is a national research project, ReVEA (Living Resources for Teaching and Learning, 2014-2018, http://anr-revea.fr) that seeks to understand *which* resources are used by teachers and *how*. This project acts as an incubator of concepts (Trouche 2016) and addresses many research questions. In this article, we will discuss the issue *how might we understand the processes by which teachers engage with curriculum resources to design instruction?*

This article is divided into four sections. Firstly, we present our framework to analyze teachers' work with resources and we will introduce the concepts of *documentational trajectory and experience*. Secondly, we discuss our methodological choices focusing on some tools as *reflective mapping of documentational trajectory*. Thirdly, we analyze the case of two middle school teachers with two different profiles. And finally, we present some considerations contrasting these two case studies.

Theoretical Framework

In this section, we present our theoretical framework, combining the *Documentational Approach to Didactics* (DAD) and the *Professional Didactics* mobilizing particularly the concept of *experience*. We then introduce two concepts - *documentational experience and trajectory* – that we use for analyzing teachers' professional development over time. Finally, we propose an articulation between individual and collective work thanks to the work of Fleck (1935).

Our work is grounded in the Documentational Approach to Didactics (Gueudet & Trouche 2010), analyzing teachers' activity and professional development through their interactions with resources, named their documentation work, consisting in choosing, collecting, interpreting, designing, and adapting their resources. This approach distinguishes two important notions: *resource* and *document*. The term "resource" is considered in a broad sense, as everything that teachers use to do their work (Adler 2000). The document is a hybrid entity, composed of *resources* and *schemes* developed by the teachers for reaching a given teaching goal. A scheme is defined by Vergnaud (2009, p. 88), as "*the invariant organization of activity for a certain class of situations*". It is constituted by four components: *goals* and anticipations that guide the activity, *rules* that retain "*sequences of actions, information gathering, and controls*", *operational invariants* that is the "knowledge in action", and *inferences* that allow to take in account the singularities of the situation. The resources components of a given document are not isolated but become a part of the teacher's *resource system*.

In the framework of *Professional Didactics*, Pastré (2011) defines professional learning as a development in action for facing different situations over time. He proposes to analyze teachers' professional development by[1] "a focus on activity in the broad sense, that is, including learning; [...] a focus on the importance of the cognitive dimension present in the activity, particularly in the form of conceptualization in action" (Pastré 2011, p. 48). He mobilizes the notions of scheme and situation to analyze it, and for him the development of the subject is understood "as [the] construction of oneself, as appropriation of all the events experienced by a subject to give them meaning for oneself"[2] (Pastré 2011, p. 118). These events will constitute the experience of the subject which is defined as the accumulation and appropriation of the past by itself (Pastré 2011).

Building on these frameworks, we consider that teachers develop their experience *for* and *from* designing resources for teaching. It leads us to define the *documentational experience* of a given teacher as her accumulation and appropriation of her past documentation work. We hypothesize then that over time teachers meet some events that trigger their documentation work. We name

[1] « une centration sur l'activité au sens large, c'est-à-dire incluant l'apprentissage ; […] une centration sur l'importance de la dimension cognitive présente dans l'activité notamment sous la forme de la conceptualisation dans l'action ».

[2] « comme [la] construction de soi, comme appropriation de l'ensemble des événements vécus par un sujet pour leur donner du sens pour soi »

documentational trajectory the set of events, both individual and collective, grounding teachers' documentational experience. Understanding how teachers develop their documentational experience allows us to analyze their *professional development*, defined, as proposed by Gueudet and Trouche (2008), as an interplay between the design of new resources, the development of knowledge (curricular, pedagogical and content), the evolution of their relationships with other actors of the educational system and the evolution of their professional beliefs.

Teachers interact continuously with their students, colleagues, trainers… Teachers' documentation work is then culturally and socially situated (Gueudet & Trouche 2008). This reality leads us to take into account the role of collectives in teachers' documentation work. And our interest for analyzing how teachers develop their documentational experience over time leads us to consider all kinds of collectives that teachers can participate in: informal vs. formal, institutional vs. associative stable vs. momentary, among other features. In this perspective, we mobilize the broad notion of *thought collective* proposed by Fleck (1936, p. 44) existing as soon as "*two or more people are exchanging thoughts"*. Thought collective leads participants to develop a *thought style*, "*characterized by standard features in the problems of interest to a thought collective, by the judgment which the thought collective considers evident, and by the methods which it applies as a means of cognition*" (Fleck 1936, 99). The collectives that we are interested in are the collectives that contribute to resource design, particularly, to support teaching mathematics. The problems of interest to these thought collectives are related to the production of resources to teach mathematics. And we try to identify the *judgment* about mathematics and teaching mathematics and the *methods* applied by the collective. These judgements and methods are influenced by the members' individual *schemes*. In this paper we analyze how teachers work together and how it influences their individual work. Understanding the individual/collective dialectics of schemes requires further work.

Finally, we analyze teachers' documentation work taking in account their individual work (a teacher interacting with resources), their collective work (a teacher meeting thought collectives and styles) and keeping one historical point of view (a teacher developing her documentational experience and trajectory). In the next section, we will present our methodological choices.

Methodological Choices

In this section, we present our methodological framework rooted in the principle of *reflective investigation* (Gueudet & Trouche 2012). Then, we present tools that we developed in our work with teachers, focusing on a tool named *reflective mapping* (Rocha 2018). Afterwards, we present our criteria for following the work of two middle school mathematics teachers. Finally, we present the way we analyze the thought style of a given collective.

The *reflective investigation* proposes four principles for following teachers' documentation work. Firstly, a *long-term follow-up* aiming to grasp an expanded view of teachers' work with resources, helping to identify stable elements (operational invariants, particularly) and unstable (changes of practice) in teachers' documentation work. Secondly, a teachers' *in- and out-of-class follow-up*, because teachers' work is not only when they teach their students; there are many occasions where teachers interact with resources, occasions that are essential for understanding their practice (teachers' training, lesson preparation, informal interactions with colleagues, among others). Thirdly, a *broad collection of the material resources* used or designed by teachers. Finally, a *reflective follow-up*, grounding our design of methodological tools. This last principle has also one formative intention leading teachers to reflect about their work.

Our methodological tools include three categories for three purposes:

(1) Face-to-face follow-up, that means follow-up of teachers preparing then implementing a given lesson; we record these moments (audio and video) and collect every resource used or designed in these circumstances;

(2) Distance follow-up, that means here the development of an online folder shared with teachers, in which they can store their resources and interact with the researcher;

And (3) a reflective follow up, that means we develop interviews that lead teachers to reflect about their work with resources. In this article, we explore one tool developed for this purpose, the Reflective Mapping of Documentational Trajectory (RMDT). Teachers design this mapping during a reflective interview. Teachers are asked to represent on a time axis events that were remarkable to them for having some impact in their documentation work and to list some resources associated to them. This tool is inspired by the schematic representation of resources system (SRRS) proposed by Gueudet and Trouche (2012). We chose the word *mapping* instead of *representation*, because it gives some more dynamic aspects (Rocha 2018): the map results from the exploration of a territory, and it develops over the exploration itself. This map gives to us a panoramic view of the teachers' resource system developing over the time, and in particular the major events.

We had five criteria for choosing the two middle school mathematics teachers to be followed, Anna and Viviane. First, we looked for two middle school teachers who work in different schools, to have different contexts. Secondly, teachers with contrasted profiles in terms of documentational experiences: for this purpose, we chose one teacher participating intensively in collectives outside of her school and another participating in collectives only in her school. Thirdly, teachers who have the same textbook in class, for analyzing how the same resource can nourish different documentation work. Fourthly, teachers having more than fifteen years of experience, looking for long trajectories. Finally, we chose teachers who have a closely work with one colleague in their school; Wang (2018) named such a colleague a *documentation-working mate*. This criterion is very helpful for analyzing teachers' documentation work, because when teachers produce resources together we have access to a rich dialogue between them. This dialogue gives to us more information about the springs of their documentation work.

The criterion "documentation-working mate" also corresponds to our interest for analyzing influences of teachers' collective work on their work with resources. The teacher and her mate form a thought collective. And for analyzing teachers' collective documentation work, we identify their thought style considering two points coming from Fleck's definition: their *judgments* and *methods*. We will consider the collective's *judgment* about mathematics and teaching mathematics as the pedagogical assumptions, i.e. its points of view about teaching mathematics and using curricular digital resources, among others. Also, we will analyze the collective's *methods* for creating resources as the functioning mode, member status, and type of interaction, among others.

Anna was followed since March 2015 and Viviane since July 2016. This follow-up happened in a moment of curricular changes. For this reason, we followed their documentation work to prepare and implement a new subject, algorithms and programming in 6th, 7th, 8th and 9th grade, in the design of new resources. In the next section, we present our preliminary analysis of these two cases.

Data Analysis

We split our analysis in two parts, one for each case study. For each part, we first present the teacher's RMDT, giving means for understanding how they develop their documentational experience. Then we present some elements of thought style appearing in the preparation of one lesson to teach algorithms and programming. Finally, we infer one scheme structuring their lesson preparation and implementation. This analysis is ongoing and for now we identify only this scheme, but we will identify other schemes in future research.

Anna's case

We have divided this section in two parts. In the first part, we explore Anna's documentational trajectory and evidence some elements that structure her documentational work. In the second part, we analyze her lesson preparation about algorithmic and programming and its implementation.

Evolution of teachers' documentation work over time

In Figure 1 we present our digital transposition of Anna's RMDT. This map presenting many information, we focus here on three aspects.

i) The stable collectives. We start by analyzing the events represented above the time line, related to (our categorization): institution, collectives outside school and collectives inside school. Most of them are related to collectives outside of school. Among them, three are stable collectives: Sésames[3] (gathering teachers for reflecting about teaching Algebra); APMEP[4] (professional association of mathematics teachers); IREM[5] (a university structure, gathering primary, secondary and university teachers, and researchers, for reflecting about math teaching). And the three other projects in this category are related to her work in Sésames.

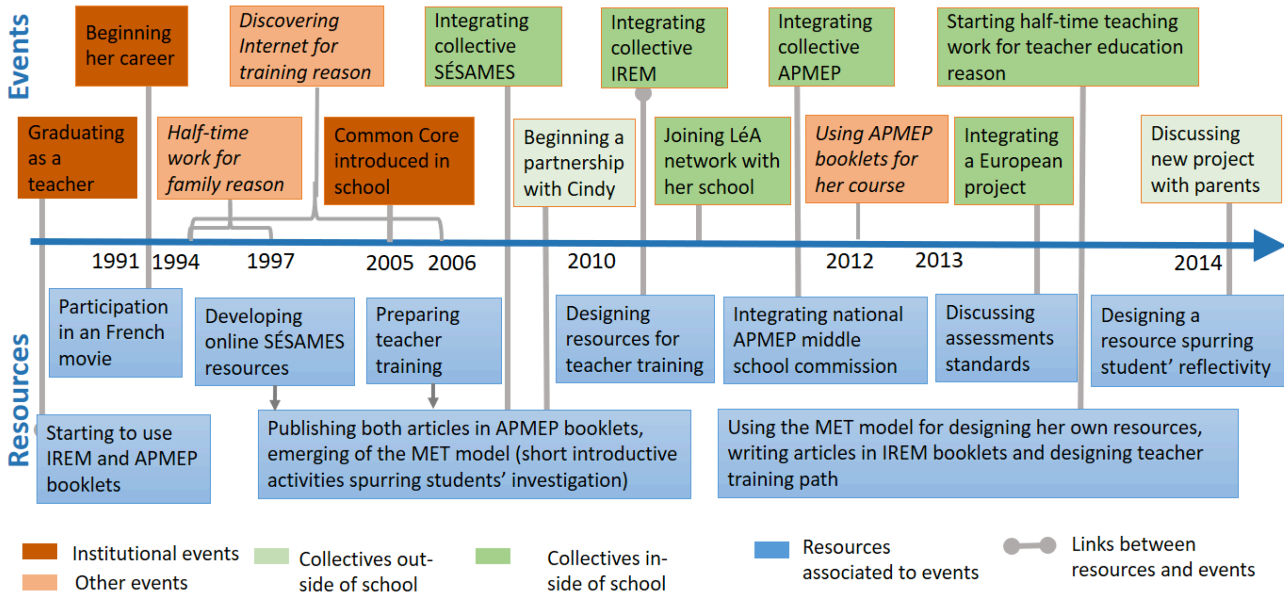

Figure 2- Our digital transposition of Anna's RMTD

ii) Among the stable collectives, the collective appearing as the most *resource-influencing*. Sésames occupies indeed an important place. This is a collective coordinated by one researcher that proposes to teachers take a role of researcher to rethinking their practices. Its members discuss teaching algebra in college and high school. Using Sésames website we inferred some elements of their thought style: (1) *judgments about teaching mathematics*: providing rich and open problems; stimulating student research; stimulate the development of strategies, among others; (2*) methods to create resources*: teachers work deciding everything together with researchers and test their activities in class. The last aspect in Anna's RMDT is her work with Cindy (her documentation working mate): they work together in the same school and in Sésames, and they also conduct some teachers' trainings using Sésames activities.

iii) For the most influencing collective, the resource appearing as the most *resource-structuring*. Sésames members created a resource that was very important for the group and for Anna, the "Mise en train" (MET). MET is a set of principles guiding a part of their documentation work since the lesson preparation until its implementation: first, the activities need to be short for warming students in the first 15 minutes in class. Second, the activities are created to teach one mathematical notion, dividing this teaching in short parts over time. Third, the activities use open problems for bringing students to research. Finally, the end of the 15 minutes has to be dedicated for institutionalizing the new knowledge in a collective moment. MET grounds the thought style of Sésames. However, Anna and Cindy extended these principles for all domains of mathematics teaching. They published articles for disseminating this resource and they use it for teachers' training.

[3] Science Education: Modeling Activities, Assessment, Simulation (http://pegame.ens-lyon.fr/).
[4] French national mathematics teacher association (http://www.apmep.fr/).
[5] French research institute about teaching mathematics in Lyon (http://www.univ-irem.fr/).

Crossing data from Anna's RMDT and our interviews, we evidence structuring aspects in her documentational experience:

i) She does not like to work alone; she always works with colleagues for designing her resources.

ii) For creating her resources, she prefers to start with resources specifying didactical intentions of the authors. This demand affects her relationship with textbooks, and then she uses them only for giving additional exercises to students.

iii) For renewing her lessons, she stays connected to many social networks for getting new ideas and resources. She developed her resource system in including new tools to save online information as *Padlet* (https://padlet.com/), that is an application to save, organize and share each one's favourite resources.

iv) For supporting her work in many collectives, she developed her resource system supported by digital tools for sharing resources in 'the cloud'.

v) She is in a moment of her career where she wants to share her resources with her colleagues. For example, she promotes MET resources in IREM and APMEP. She develops resources in one collective and leads them to other collectives.

These aspects structure also her implementation of the new curriculum; this is the purpose of the next section.

Implementing new curricular purposes

We follow Anna's lesson preparation and implementation of one lesson to teach algorithms and programming in 6th grade. She prepared her lesson with her documentation-working mate, Cindy. Their thought style for creating resources appears clearly:

- *Judgments:* they looked for resources to teach algorithmic as thinking about mathematics and putting students to research; they did not want to base their teaching on only one software (to be noticed: the software Scratch is strongly recommended by the inspectors);
- *Methods*: they prepare their lessons taking decisions for each task together; gathered all possible resources coming for the ancient curriculum that could still be interesting to use (many textbooks, sites, etc.). They do not like so much to use textbooks (Aii).

For this lesson preparation, Anna used a *padlet*. She saved during the year before the curricular reform resources that she thought relevant for this teaching (Aiii). Anna and Cindy started their lesson preparation in reading curricular propositions. They discussed a lot about many activities that they have seen or heard of, and they surfed over many textbooks. They did not choose textbook resources; they selected some open online resources. We present here just one resource that we follow during their documentation work, entitled the *crépier psychorigide*[6]. The objectives of this resource are *introduction of algorithmic thinking and work with algorithmic writing*. A team of IREM-Grenoble, giving many didactical advices that are taken into account by Anna, proposed this resource. In our observation, Anna incites and gives a lot of time to students for research and discussion of results.

In our initial analysis, we identify one scheme: *choosing tasks to work with students*. The aim in this scheme was to find tasks for teaching algorithmic and programming. We identify two operational invariants: *working with algorithmic favors working with situations that put students in research; put students in research makes learning more meaningful*. These operational invariants are linked with many rules of action: proposing students to work in groups, proposing them open problems, stimulating students' research, among others. We can see that this lesson respects some principles

[6] http://www-irem.ujf-grenoble.fr/spip/IMG/pdf/fiche_prof_crepier_psychorigide.pdf

of MET, but it is not a MET. However, theses similarities lead us to infer the influences of Sésames thought style in Anna's lesson preparation of algorithmic and programming.

Generally, Anna's documentational experience is developed through her work in many thought collectives. These collectives support her reflection about curricular changes. And her work with Cindy for preparing lessons helps her to interpret curricular purposes and to link them with her documentation work.

Viviane's case

We have divided this section in two parts. The first one explores Viviane's documentational trajectory evidencing some elements that structure her documentational work r facing curriculum changes. The second one analyzes her lesson preparation on algorithmic and programming and its implementation.

Evolution of teachers' documentation work over time

In Figure 2 we present our digital transposition of Viviane's RMDT. We use the same categories that in Anna's case for classifying events (institution, collectives outside and inside of school). We observe 14 events and we focus on three points.

(i) We observe that most of them (six events represented in brown in Figure 2) are institutional ones: two of them are related to changes of curriculum that lead this teacher to integrate and create new resources; two related to her change of school giving her the opportunity to have access to new software and to exchange with new colleagues; one other is her change of status in school being the main teacher for grade 6 class; and the last one the emergence of needs for students with a particular profile.

(ii) We observe that her collective work is concentrated on her exchange with colleagues in school. These collectives are more informal and momentary than in Anna's case, and then it is very complex to identify their thought style. However, we observe that they have a central function: exchanging experiences about lessons in class and resources (booklets, software, activities, among others).

(iii) Her implication in teachers' training two times by year. This training is for her a moment for looking for new practices and new resources to use with her students. These events are thus very important to her. Another important aspect about her participation in this training is that all mathematics teachers of her lower secondary school were involved in it.

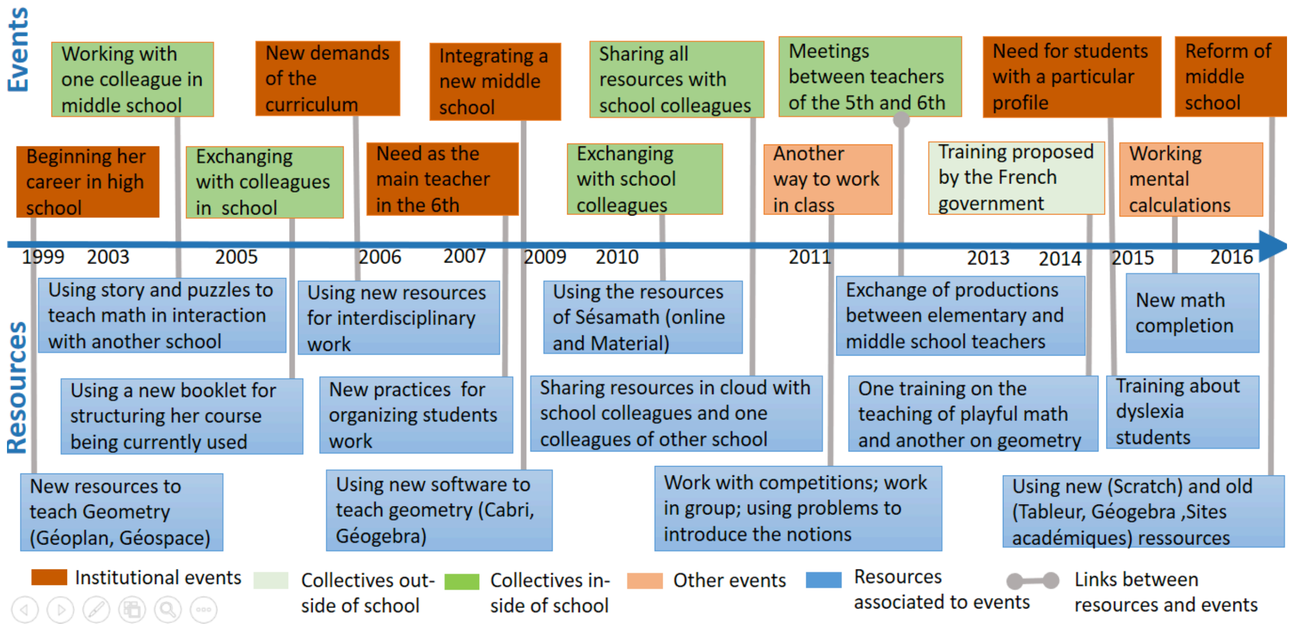

Figure 3-Our digital transposition of Viviane's RMTD

These stages led Viviane to incorporate new resources and new practices. She presented during her interview five different teachers' training that she remembered in her career. One training that led her to 'playful math' to introduce or apply mathematical notions. In this training, she found new resources as APMEP booklets and one new institutional website giving new ideas. One other training gave some ideas for introducing in class challenging problems from mathematics competitions. She had one training about teaching geometry, in which she learnt about different tools and problems to introduce new notions. And the other two trainings are related. In one year, she participated to a training given by Anna, in which she heard about MET to teach mental calculations. And then she implemented some MET in her class. She used short activities (15 min) for warming up students, but not necessarily leading her to put students to research. In the other year, she participated to one other training given by Anna and Cindy, where they discuss MET design for working on all mathematical notions. In this training, Viviane and her school colleagues prepared activities with teachers of other schools. They chose to work with introduction of perimeter using MET. They experimented activities in their class and they discussed afterwards in the following training meeting. To Viviane, it is a good occasion for designing collectively resources and for exchanging their results.

We also confront data from Viviane's RMDT and our interviews, and we observe structuring aspects in her documentational experience:

i) She considers important to exchange experiences and resources with her colleagues in school.

ii) For creating her resources, most part of her lesson came from textbooks, institutional resources and official collective's booklets.

iii) She renews her lessons and practice through participation to teachers' training;

iv) Many changes in her documentation work is linked to institutional changes. They led her to look for new resources and practices;

v) In her resource system, digital resources are used to support her exchange with colleagues, some institutional sites and some software.

We suppose that some of these aspects structure her implementation of new curriculum too. We will explore this hypothesis in the next section.

Implementing new curricular purposes

We followed Viviane's lesson preparation and implementation of one lesson about algorithms and programming in 7th grade. She prepared her lesson exchanging resources with her documentation-working mate, Jessica. About their thought style for creating resources:

- *Judgments:* they looked for playful activities that keep students autonomous and mobilize mathematics concepts; they wanted to use Scratch to work on mathematical concepts and to connect them with algorithmic and programming.
- *Methods*: they prepared their lessons and activities throughout a work dividing tasks, Viviane prepared activities for the 7th grade and Jessica for 6th; They found their activities reading textbooks; they never used textbooks activities like they are, they changed and created their own student's file.

Before the curricular reform Viviane participated in institutional training for using Scratch and she exchanged informally with other colleagues in her school meetings (Viii). For generating ideas, she used government regional site, textbooks and exchanges with colleagues (Vi, Vii). Most of the activities prepared were found in textbooks and adapted in the student's sheets (Vii). And the lesson that we observed was about: *(1) the mobilization of the properties of quadrilaterals and triangles and (2) the work with the programming language.*

In the same way, we identify the same situations class than in Anna's case: *choosing tasks to work with students*. This scheme comprises operational invariants: working with programming allows to propose activities in which students work in an autonomous way; students working in an autonomous way are responsible for their learning. We identify many rules: the tasks proposed for programming are chosen to allow the application of already learnt mathematical knowledge, students work in pairs, enough time is given to students for answering the questions, etc. We observe similarities between the thought style of her work with Jessica and her work in class.

Generally, Viviane's documentational experience is strongly linked with changes in her institution and with interactions with her colleagues. The interactions in her school support her reflection about curricular changes. And her work with Jessica helps her to exchange experiences and ideas about the new curriculum.

Final considerations

We start our considerations by some results of our cases studies. We observe that Anna and Viviane's documentational trajectories present some common points and some differences.

- For both of them, collective work is important for their documentation work, and it is a way for validating their work in class. However, their way to work collectively is quite different, Anna's collective work develops more outside of school and Viviane's inside of school;
- Both of them work with one documentation-working mate, but in a different way. Anna looks for establishing a coherent work with the teachers inside her school, for having the same vocabulary and similar resources. Viviane tries to share the work for facilitating resources production and exchange resources;
- About their resource system we observe two points. Textbooks are used differently (for Viviane the textbook is an important resource and for Anna only one exercises' source for complementing her lesson). Digital resources are central to Anna and complementary resources to Viviane;
- About teaching algorithmic and programming, Anna starts to prepare for curricular changes before their implementation in many collectives. Viviane exchanges some informal ideas with colleagues before, but her work was more intense after curricular implementation.

We could observe influences of collective work in teachers' design decisions.

Anna and Cindy design their lesson and search for resources together; their work with resources is coherent with Sésames' thought style; they start to think about the new program before the curricular changes in diverse collectives (using temporary version of the curriculum), and accumulating resources found during that.

Viviane and Jessica divided their work for creating their resources and after they exchanged their activities; they did not work collectively outside of their school; they started to prepare their lesson during reform implementation, based on resources extracted from the textbook. Analyzing teachers' work with their documentation working mate was interesting for a better understanding of their documentation work, because their discussion during lesson preparation was very rich and natural.

We go back to our research question: how might we understand the processes by which teachers engage with curriculum resources to design instruction? We draw, from our study, five elements of answers. Firstly, the work with the documentation mate appears as a good frame for analyzing collective thought style, because teachers' exchanges are very rich; secondly, we consider that curricular changes were a good occasion for analyzing the features of teachers' documentation work and development of operational invariants, because it was a moment of learning and creating new resources for teaching new contents. Thirdly, for us, analyzing the history of resource design by teachers helped us to understand the operational invariants guiding their choices. Fourthly, the articulation between the teachers' reflectivity and their analysis in action helped us to analyze the process of knowledge development. And lastly, the development of the concept of documentational trajectory was relevant for analyzing interactions between resources, collective work and teachers' practice.

Many questions related to our conceptual propositions and methodological choices stay open, especially in relation to the concepts of documentational experience and trajectory. We proposed definitions that are in development, that need to be more refined. However, it appears very important to consider teachers' appropriation of their past work with resources through their reflection about their work. In fact, analyzing teachers' experience helps us to understand the springs of their documentation work. In this sense, reflective mapping was an efficient tool to collect their point of view about their documentation work. It was important too to take into account the accumulation of past documentation work. It evidences indeed a set of knowledge and resources that teachers developed over the time.

For the future, we will address at least three important issues. Firstly, it seems interesting to analyze other teachers' documentational trajectories for better understanding teachers' professional development over time. Secondly, working with teachers from different disciplines could contribute to this reflection, evidencing eventual features depending on the discipline. Last, we could question the effects of developing the teachers' reflectivity on his/her practice in class.

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