LESSON STUDY IN MATHEMATICS: THE CASE OF DENMARK

Lesson Study em Matemática: o caso da Dinamarca

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Abstract

Lesson Study in mathematics has a relatively short history in Denmark (about 15 years) and it is still not widely implemented. What can be learned from the first experiences in this relatively privileged small country? We outline the situation in three main institutional contexts, namely basic school (primary and lower secondary), high school (upper secondary) and teacher training in university colleges, while trying to answer the following questions: what motivated the initiatives to introduce Lesson Study to mathematics teachers? What obstacles and results are experienced? In the final section, we summarize our answers with a focus on what may be of interest in similar contexts outside of Denmark.

Keywords: Lesson Study, Denmark, mathematics.

Resumo

Lesson Study em Matemática tem uma história relativamente curta na Dinamarca (cerca de 15 anos) e ainda não é amplamente implementada. O que pode ser aprendido com as primeiras experiências neste pequeno país relativamente privilegiado? Descrevemos a situação em três contextos institucionais principais, nomeadamente o ensino básico (anos iniciais e finais do ensino fundamental), o ensino secundário (ensino médio) e a formação de professores em faculdades, procurando responder às seguintes questões: o que motivou as iniciativas de introdução do Lesson Study para os professores de matemática? Que obstáculos e resultados são vivenciados? Na seção final, resumimos nossas respostas com foco no que pode ser de interesse em contextos semelhantes fora da Dinamarca.

Palavras-chave: Lesson Study; Dinamarca; Matemática.

Introduction

Denmark is a small kingdom in the north of Europe which has enjoyed parliamentarism since 1849. Schooling has been available and mandatory for the general population since 1814, when the first “school law” was issued. Today, the compulsory school (primary and lower secondary) goes to grade 9, and mathematics is a major subject throughout. About half of the general population subsequently attend some form of academic high school (upper secondary) where mathematics is not only a major and mandatory subject, but also a subject that is increasingly afflicted with student failure, with up to 30% of the students getting exam results in mathematics below the minimal pass level. A ministerially mandated analysis of this problem (JESSEN; HOLM; WINSLØW, 2016) suggests that this can be explained by inadequate preparation from compulsory school, as well as by other factors. The need to strengthen mathematics teaching in compulsory school has also recently been underscored by decreasing performance of Danish grade 4 students in the TIMSS survey. Of course, the performance problems at high school exams have also led to discussion of how to improve teaching there.

As in other countries, whenever signs of crisis appear in the school system, politicians will consider short sighted strategies like revisions of curricula, increased testing and so on. But educational research suggests, in many ways, that the main lever for improving school results lies with *teachers*, including the c*onditions under which they work*, and the teachers’ *professional knowledge* (practical and theoretical) for teaching. Lesson Study has been pointed out in widely known educational publications (like STIGLER; HIEBERT, 1999) as a particularly efficient way to enhance both.

The first presentation of Lesson Study in a Danish publication can be dated to 2006, where it appears in a textbook for teacher education (WINSLØW, 2006). In 2008, the last author introduced Lesson Study to a about 20 university college teachers of mathematics, in the context of a national centre for the development of mathematics teaching. This led to initial experiments with Lesson Study in collaboration with mathematics teachers in compulsory school, as well as in the education of compulsory schoolteachers which takes place in university colleges. These and more current practices involving Lesson Study in Denmark will be presented in sections 2 and 3. The situation is quite different in high school, but very recently Lesson Study has also been experimented there, and we provide some first observations in section 4.

In all three sections, we observe how *institutional conditions and constraints for teachers' work* are crucial to the form and results of Lesson Study, and how going beyond initial "experiments" to a sustained, long term engagement, is far from trivial. In other words, we consider that for investigating the possibility for Lesson Study to become sustainable and to yield solid and durable results in a given institution, one needs to undertake a wider analysis of the *paradidactic infrastructure* for teachers' work in a that institution (MIYAKAWA; WINSLØW, 2019). A first analysis of the paradidactic infrastructure in relation to Danish compulsory school was carried out by Østergaard and Winsløw (to appear). They also suggest that sustainability of Lesson Study should not be understood as independent from support and input from outside of the institution. On the contrary, and in line with what is found in countries where Lesson Study is well established (e.g. ISODA, 2007), continued interaction between school teachers and scholars from teacher education institutions seems to be a frequent and fruitful source of inspiration to many if not most successful implementations of Lesson Study, including those found in the teacher education institutions themselves. And these links will also be given specific attention in our presentation and subsequent analysis of Danish experiences.

Lesson Study in primary and lower secondary school

The municipality of Lyngby-Taarbæk is in the northern part of greater Copenhagen. For the past nine years, efforts have been made there to establish a new teaching culture, combining a) student centred *teaching through problem-solving* (TTP) and b) continuous professional development based on Lesson Study (LS). These two elements are, in the words of Fujii (2018, p.1), "two wheels of a cart".

Teaching in Denmark is, as in many other countries, still dominated by the teacher's presentation of facts and procedures and the students' training to remember and use these. Mathematics education research consistently confirm that teaching, which allows for the students to develop, test and validate their own ideas, is more efficient. It seems that many Danish educators and teachers believe that our teaching is based on these principles and that they lack a clear image of what such teaching could look like.

In this respect, it is not just a trivial detail, that what ignited the efforts at Lyngby Taarbæk Municipality in this respect, was a video – *live images* – of a TTP lesson ("Reflect on the meaning of 'same form'", explored in Miyakawa & Winsløw, 2009), supported by an introduction to both TTP and LS (by the last author). In lack of the Danish educational system's own investigations into what teaching, which allows for students to develop, test and validate their own ideas, could look like, the lesson of "same form" offered concrete and tangible inspiration. The need for and possible effects of such live images should not be underestimated, as we have experienced time and again.

In this section, we first present some data on the municipality and what we regard as the main drivers for its progress. Then the efforts and experiences until now will be presented in it's different phases.

Then, what teachers have and have not learned (yet) will be discussed, and lastly some advice will be presented along with some critical points which need further scrutiny and development.

The population of Lyngby-Taarbæk is about 56,000 of which the approximately 6,000 are primary and lower secondary school students, served by nine public schools (in addition there are two schools for students with special needs). In rough numbers there are about 700 students and 20 mathematics teachers at each of the nine schools.

Compared to the national average, students from schools in Lyngby-Taarbæk generally perform well. Yet, students show the same kinds of misunderstandings and mis-learnings as students from other parts of the country. For instance, some grade five students think that 3/4 + 2/4 = 5/8, which reveals that they have not understood the fundamental meaning of fractions. At the highest performing schools, more than half of grade nine students fail to demonstrate proficiency with reduction of expressions like $n^2 - (n + 1) \cdot (n - 1)$ (example taken from the national exams in spring 2021, UVM, 2021). Or, as a school leader at one of these school noted, many students in the middle grades do not realise that half a litre of milk is the same in mathematics and home economics.

Hence, despite the comparatively good-looking numerical results from mathematics tests in the municipality, there is still a need to improve. On the other hand, since the results *are* good-looking, not everyone agrees on a need to change dramatically. But a positive interest and engagement in teaching development exists among teachers and leaders at some schools. It appears that the main drivers for this movement have been (and is) i) a convincing image of a different and desirable way of teaching (TTP), and of a way to pursue that way of teaching (LS), ii) a close and consistent cooperation between the municipality, the school leadership and teachers, iii) a continuous support of resources for both leaders and teachers spanning from logistics and money to theoretical knowledge, materials for study and teaching, and facilitation of teachers' processes, and iv) a continuous supply of concrete inspiration in the form of exemplary lessons (video and live), including both lessons by expert TTP teachers and peer novice TTP teachers, and workshops by experts with deep and thorough insight into TTP and LS, comprising the rationale of and connection between both.

In hindsight our work within this municipality can roughly be divided into these four phases:

<u>Phase 1</u>: 2012-2015 (3 years). In 2012 some of the teachers and municipal officials had a first introduction to TTP and LS (by the last author). As noted above, this introduction included concrete live images - in the form of a video of a TTP-lesson. During phase 1, a few teachers had sporadic experiences with LS. The desire to introduce TTP and LS to the schools emerged and grew at the municipal level during these years.

<u>Phase 2</u>: 2015-2016 (1 year). As part of the first author's Ph.D. studies (BAHN, 2018), a team of 3-4 teachers at each of three schools conducted three LS during the school-year. The LS were focused on teachers' experiments with TTP (in the image of Open-Approach Method - OAM, see e.g. NOHDA, 2000). The participating teachers were introduced to the LS and TTP, supported by concepts and models from the Theory of Didactical Situations (TDS - BROUSSEAU, 1997). Videos of TTP lessons were used as pivotal introductory elements. Mid-year, we held a *kenkyuu kai* (Lesson Study conference) with a Japanese expert teacher (Mr. Hiroshi Tanaka), who conducted an open lesson to demonstrate TTP and to illustrate the developmental potential of LS and open lessons.

Summary of phase 2: 9 LS were conducted by 11 teachers at 3 schools. All LS aimed at TTP (in the image of OAM) and each included 3 research lessons. All research lessons were free-standing lessons (i.e. unattached to a wider unit), All participating teachers (and many others) observed 1 live open lesson by an invited Japanese expert teacher. All activities were funded 90% by the Ph.D. project and 10% by the participating schools.

<u>Phase 3</u>: 2016-2020 (4 years). Based on the experiences in phase 2, two of the

schools adopted open-lessons as the fact format for professional meetings. Furthermore, the municipality decided to provide schools and teachers with the possibility to conduct LS. Both were to be organised and facilitated by the first author.

During these years, a number of teachers at most schools in the municipality gained experiences with LS and to some extent with TTP (and more precisely, OAM). From January 2018, the first author was hired by the municipality (after submitting his thesis), partially to continue the work with LS and TTP. For the following years, LS and open lessons were supplemented by various courses on topics related to LS and TTP, often involving videos, open lessons or similar activities. These activities were organised together with and often led by the last author. Each year a *kenkyuu kai* was organised, at which invited Japanese experts teachers conducted open lessons and teaching-learning workshops. The latter three years, one open lesson was also conducted by a Danish teacher and one of the lessons conducted by an expert teacher was in science.

One of the bottlenecks for using Lesson Studies strategically is the availability of - or rather lack of - qualified facilitators. Since the last year of phase 3 we have initiated the training of three teachers to become LS facilitators. These teachers receive further introduction to LS, TTP and theory on the teaching and learning of mathematics (based on the TDS). One major feature of this training is shared observation and reflection of teachers' Lesson Study activities under the supervision of the more experienced facilitator (the first author).

Summary of phase 3: 30 LS (+ 4 in science) were conducted by 116 teachers (80 unique) at 7 schools. All LS aimed at TTP (in the image of OAM) or more general experiments and each included 3 research lessons. All research lessons were free-standing lessons. All or most participating teachers attended various related course and workshop activities. At three schools, 56 open lessons were conducted in replacement of traditional professional meetings. Each year a *kenkyuu kai* was conducted with a total of 8 open lessons by invited Japanese expert teachers and 3 open lessons by Danish teachers (novice to TTP). In the last year of this phase, training of LS facilitators was initiated. All activities were funded 100% by the municipality.

<u>Phase 4</u>: From 2020 (we are now in the second year). Based on our experiences, our further studies of LS and TTP, and on advice from LS experts (e.g. Professor A. Takahashi, DePaul University, Chicago, USA), we made some fundamental adjustments to our approach.

First of all, we have officially adopted TTP (see TAKAHASHI, 2021) as the aim. In addition, we have introduced the concept of a model school, referring to the school's effort to become a model for how to change the teaching culture (including the development of teaching) in order to be able to adopt student centred teaching like TTP. The concept of model school involves the concept of *school-wide LS* (SW LS, see e.g. TAKAHASHI, 2017). In SW LS all teachers (of the given subject) are involved in LS. Furthermore, all teachers (of the given subject) participate in observation and post-lesson discussion of all other teams' research lessons. Currently, we have one such model school in mathematics (and one aspiring to adopt SW LS in science). It is the hope and intention that more schools will follow in the years to come.

Other important changes include the omission of multiple research lessons in each LS and (from this year) the introduction of mock-up lessons, i.e. lessons in which the lesson plan is tested and revised with teachers playing the role of students. Prior to phase 4, research lessons were free-standing lessons, i.e. they were not directly connected to other lessons. In this phase we have put (stronger) emphasis on the research lesson's place in a unit. Teachers are not just to regard the research lesson itself, but to consider it as part of a sequence of lessons leading to students' learning of a broader concept (rather than they just realise something interesting).

In phase 4, the teachers at the model school are recurrently receiving further introduction to TTP and LS by leading experts. These introductions include videos of TTP lessons, teachers' own experience as 'students' in TTP lesson-like activities and more.

Summary of Phase 4: 6 school-wide LS conducted by 24 teachers (+ 1 LS in science conducted by 6 teachers at 1 school). All LS aimed at TTP and included 1 research lesson. LS aimed at unit-based research lessons, but many turned out as free-standing. All teachers attended further introduction to LS and TTP, and to teaching-learning progression (cf. the intended focus on unit). Follow-up seminar after all LS activities had ended. Continued training of facilitators. All activities funded by the municipality. *More activities were planned but could not take place because of the corona virus situation. In most LS, the research lesson could not be conducted.*

One of the most important results of our work so far is the growing interest among teachers to improve teaching, specifically through TTP. Furthermore, though we do not have a systematic account of it, it is evident that most teachers gradually become more aware of intimate details in the connection between teaching and learning. We do not have non-subjective evidence of the teachers' teaching practices in their normal teaching, but we see a progression during Lesson Study sessions. Furthermore, many teachers report that they try to implement techniques and approaches from TTP in their teaching. These include fewer but richer problems, deeper thinking about the pertinence of teaching materials, a stronger emphasis on students' thinking and formulation, in-class sharing of student ideas, a more systematic use of the black-board and of notebooks etc.

Asked directly, some of the most engaged teachers report the results of our work until now as: it has "raised our professional discussions to actually regard teaching-learning and mathematics", it has (only) led to "small changes to normal practice", it has raised our "consciousness that what we do and how we do it are significant" and that "the way one is posing problems, asking questions and responding to students' ideas", is crucial. Two of the most experienced teachers expressed that "In LS we work with teaching and learning at a level which I have never experienced in my 30 years as a mathematics teacher, including my 10 years as a supervising mathematics teacher" and "I have been a teacher for 20 years. This is the first time I have tried something that makes me a better teacher".

In general, in LS teachers often realise that their mathematical knowledge is either insufficient or imprecise. It is also a typical observation, that the students know and are capable of much more *or* much less than what the teachers expect and hence, what they base their teaching and evaluation of students on, on an everyday basis.

In the end, our work seems to have had a very positive effect on (many) teachers but we do not yet know enough about what it has done - if anything - to their normal teaching, or to the learning of their students in general.

Also, there are some expected learnings that have not taken place. For instance, it is still very difficult for teachers to point out a precise learning goal for a lesson or a unit. The learning goals we see are still broad and vague and tend to be more about an activity than a learning goal. Related to this, the teachers also struggle to identify specific pieces of mathematical content and their possible progression in and across a unit. This issue persists, even if it has been addressed repeatedly by the facilitator (the first author) during the years. With regard to this question, it appears that our teacher education does not prepare teachers to be able to identify and work with neither precise learning goals and concise teaching-learning progressions. Also, neither our national standards (common goals) nor our textbooks and teaching guides address these issues precisely.

Based on our focus on the phases of TTP (grasping the problem, working on the problem, discussing solutions to the problem, summarising) and inspired by a number of videos and live lessons, many teachers have become aware of the counter-productive effect of them intervening too early in students' work on a problem. Japanese teachers use this phase to collect data on the students' understandings of and ideas about how to solve the given problem(s). While the teachers increasingly tend to interfere less, many will just stand by the blackboard and ignore the crucial task of collecting data.

TTP fundamentally conflicts with our traditional teaching practices. Breaking with these, and adopting TTP is a long and demanding job which involves many steps. While there are many aspects of establishing a LS-based structure for professional development outside Japan, which need further scrutiny, one of the crucial ones seem to be how to best support teachers' learning. More specifically we need to understand better: A) teachers' possible learnings and their possible progressions, B) how we best support these learnings and their progression, and C) how we can educate facilitators to support teachers learnings.

Based on our experiences, we advice others who wish to establish a new teaching culture to: I) Decide on and study the aim of the efforts (e.g. TTP), II) study others' experiences with LS and the aim, III) ask for help by those more experienced and knowledgeable, IV) provide concrete images of the aim and inspiration for progress, V) start by establishing small 'enclaves' of LS-based culture for professional development, VI) be persistent and focused, and VII) adjust rather than abandon, when encountering difficulties (remember to ask for help).

Lesson Study initiatives originating from university colleges

In the previous section we have seen how LS have been initiated by actors on a municipal institutional level to attempt changing the continued professional development (CPD) at the school level. In the Danish context however, municipalities are not the only source of initiatives to cater for CPD amongst schoolteachers. The biggest stakeholders in terms of providing the manpower for providing CPD support are the university colleges. These institutions, while primarily tasked with pre-service education of teachers, also provide in-service training or CPD to schools. This service can be asked for (and paid for, directly by schools or municipalities), but more often the CPD provided are part of research or development projects funded by grants given by the government or private foundations.

University Colleges thus have the opportunity to influence teaching and learning at schools (e.g. by using LS) not only indirectly as part of pre-service teacher education, but also directly by engaging with in-service teachers. University Colleges often serve as an intermediary between foundations, government bodies, universities, national research centres, municipalities, and the schools themselves. University Colleges only have very modest internal funding for research. Consequently, all projects run by the University Colleges need external funding. In the following section we present lessons learned from one such research project carried out with in-service teachers.

A general background to the design of this project is the realisation that several attempts to secure funding and school cooperation to carry out research projects involving "all" the elements of "proper" Lesson Study, have failed. A "full" package of Lesson Study (e.g. with several cycles of substantial "kyouzai kenkyuu", detailed lesson plans, research lessons observed by colleagues, attended by knowledgeable others and dissemination of findings) is simply too overwhelming for school leaders and most teachers. From an organisational point of view, many teachers find it difficult to implement in the reality of a busy school, and cognitively demanding as most teachers have practically no experience with inquiry into their own practice.

The above outlined constraints have led us to experiment with picking out only a few characteristic elements of Lesson Study, making these the central notion of a CPD effort, during the SeSam-LS project ("looking together at mathematics lessons").

In SeSam-LS, five teachers[1] were given a partner from the University College to act as *knowledgeable other*. The partner also acts as a guide or facilitator in a process where the teacher prepares a *lesson plan* (preferably with a clear research objective) for a research lesson, which is

[1] The teachers voulenteered to participate in the project by responding to an open invitation.

video recorded. The teacher and the partner have a *joint reflection* about the lesson recorded. With only these three elements as "must have" it is up to the collaboration between partner and teacher to navigate the institutional constraints at the schools. The partner is well versed in other facets of LS, but should only introduce these in accordance with the affordances of the teachers' working environment and professional ability. SeSam-LS is thus primarily an individual experience between teacher and partner, that may not foster the sense of community and joint learning which is among the frequently emphasised benefits from LS. But this most individual setup resonates well with the existing paradidactic infrastructure, in which teachers rarely watch each others' lessons, and much less have the inclination to constructively comment and criticize. The Danish maths teacher's practice is very much a private matter, and collegial feedback tends to be a delicate issue, that most teachers are uncomfortable with. Hence the sparring and reflection with the partner who is external to the school context, are less unsettling, and could be perceived as more "objective".

Participating teachers in SeSam-LS prepared, conducted, recorded and reflected upon 4-6 lessons each. The lesson plans, video recordings and post-interviews with each teacher were reviewed qualitatively in order to assess the presence of desirable LS elements. This is methodologically a tricky issue, as it depends on a lot of complex evaluations of data, which also reflect a development process of each teacher. What we did was to compile a (longer) list of LS elements, hypothesised to become evident in the data, besides the three explicitly "must haves". The list is by no means exhaustive, but we believe it broadly represents didactic and paradidactic infrastructure inherent to LS. This analysis was carried out by the third author of this paper, assisted by the two SeSam-LS partners who worked with the teachers. For each of the elements in the list, we reviewed the data for each teacher as a case, discussed and reached a judgement of whether the element was satisfactorily present[2]. In **Erro! Fonte de referência não encontrada.** below the list of elements and their presence can be seen, as well as the number of cases where the element were favourably assessed by the teacher in an interview conducted after the experience.

Table 1: Presense of didactic and paradidactic infrastructure elements of Lesson Study in teacher cases[3]

List of didactic and paradidactic 'elements' which could/should be part of a Lesson Study process (cf. Asami-Johansson, 2011, 2021; Shimizu, 1999)	Number of cases where element was satisfactorily present.	Number of cases where teacher expressed element as beneficial
A long or substantial 'kyouzai kenkyuu' phase	0	0
A clear research purpose	1	0
Using/following the lesson plan	5	5
Anticipation of students' solutions processes	1	3
Structured problem solving approaches or problem oriented lesson structure	0	0
"hatsumon" – get students 'hooked'	2	3
"kikan jyunshi" – purposeful monitoring	0	0
"kikan shido" – instruction during student work	3	5
"neriage" - comparison discussion	2	0
"matome" – summing up	0	0
Reflection with colleagues	0	0
Input from discussions with knowledgeable other used/integrated in lessons	5	5
Revising the lesson plan	0	0
Re-teaching the lesson	1	1
Dissemination of LS findings	0	1

Source: prepared by the third author.

It is quite apparent that the explicit 'must have' elements are present, and although this may be regarded as trivial, we do not think that is necessarily the case in the teachers' normal practice. Danish teachers have little, if any, experience with preparing lesson plans that are intelligible to others. Also, the evidence of teachers making use of input from the reflections with the knowledgeable other, points to a

[2] We understand this method could be strengthened in many ways, and therefore the conclusions should not be taken for more than an indication of what we estimated was satisfactorily present. However, we claim to have expert opinion in the matter.

[3] The element of "reflecting with the knowledgable other" was trivially present, and therefore not included in the list.

common accord between teacher and partner to value the joint exploration of teaching and learning.

The SeSam-LS setup clearly does not compel teachers to do any extended study of materials and other sources of knowledge in connection to preparing the lesson plan or research lesson. Teachers seem to work primarily with the knowledge they already have, using "just" the same amount of working time and teaching resources, as with regular preparation for lessons. This may lead to better sustainability of the activity carried out in the project, as it seems unlikely that the paradidactic infrastructure will change to provide extra (paid) time for preparation.

SeSam-LS seems to orient teachers towards student-centered teaching, as the teacher's active role during "seat work" was evident in most cases. This appears connected to an appreciation of being able to anticipate student solutions and building upon students' suggested solutions when conducting whole class discussions. Two teachers became quite adept at this, while they did not express this as particularly noteworthy when interviewed about their gains from the experience.

One teacher had the opportunity to re-teach one of his research lessons, as he had two classes at the same grade level. He did not change the lesson according to experiences from the first teaching, but was able to get a bigger perspective on the range of students' solution strategies, while considering the impact of quite different social dynamics in the two classes.

Only one of the teacher cases was considered by us to present clear research purposes. This does not mean that teachers had no purpose. It was just often a diffuse purpose like "how can I teach X", not linking it to any specific experienced challenges with student learning. The concern (and purpose) of the teachers were to create a 'good activity' in which the students could work on understanding a "piece of mathematics". This provided quite substantial consideration of what, in a Japanese structured problem solving "language", would be called "hatsumon", which Danish teachers usually associate closely with motivation and engagement: A good activity should engage the students in the mathematical activity.

In the performed analysis, there was no consideration of what specific mathematical and didactical knowledge the teachers acquired by participating in SeSam-LS. This is symptomatic of much research, outside Japan, on attempts to utilise Lesson Study. Perhaps researchers (including ourselves) become overly concerned with the didactic and paradidactic infrastructure and fail to report and disseminate findings about teaching and student learning which Lesson Studies actually produce. This may lead Danish teachers to view Lesson Study as mostly "form" and not so much "content".

Another noteworthy lesson from the SeSam-LS project, is that funding agencies are mostly concerned with outcomes for the students, and only indirectly interested in what knowledge is acquired by the teachers, about student learning. Therefore, the SeSam-LS project tested a setup with control groups, using national Danish achievement tests, to gauge if students in classes where the teacher participated in Lesson Studies, performed better. This demand by the "noosphere" for external validity or proof of effectiveness, is still not satisfied, and SeSem-LS is currently in a process of scaling up, in an attempt to provide such "proof".

Regardless of what we, as researchers, may think about these internal and external demands for justification, it is something we must consider in every institutional ecology where we want to introduce Lesson Study.

Lesson Study in upper secondary school

Upper secondary school in many countries, including Denmark and Japan, is more selective than primary and lower secondary school, and the teaching of mathematics is more oriented towards preparing students for high stakes exams that regulate the access to higher education. This means that mathematics teaching focuses much more on developing students' computational skills and, to varying extents, on mastery of theoretical contents based on rigorous definitions and proofs as covered in a text book. Even in Japan, LS focusing on engaging students in problem solving situations, such as OAM or TTP, is more rare, although recently significance experimentations with such LS has occurred (NISHIMURA; KOBAYASHI; OHTA, 2018). Such experiments, on the other hand, could be motivated by the widespread failure (particularly with low achiever students) of traditional methods of teaching, even when it comes to have students succeed with traditional exams. Moreover, the increasing spread of computational tools, both in society and in secondary schools, seem to reinforce the needs for students to develop higher level capacities for modelling, reasoning and problem solving, and thus lend motivation for at least partial shifts towards TTP also at this level. In fact, curricula in many countries (including Japan and Denmark) now increasingly emphasize that students should be capable to engage in mathematical enquiries, and not just in solving standard tasks while using standard techniques.

In Europe, these general tendencies have motivated a large number of development projects, supported by the European Union, to further “inquiry-based teaching” in the STEM subjects, including mathematics (e.g., ARTIGUE; BAPTIST, 2012). The last author has been involved in two of them. First, from 2016 to 2019, the MERIA (Mathematics Education: Relevant, Interesting, Applicable) project, where a team of university researchers and high school teachers from four European countries (Croatia, Denmark, Netherlands and Slovenia) designed and experimented a series of inquiry-based mathematics lessons, with TDS and Realistic Mathematics Education (RME) as explicit theoretical foundations. This means that the designs were based both on epistemological analyses and modelling tools rooted in RME, and in didactical analysis and design methodology coming from TDS. All products from the projects, including the handbook (WINSLØW, 2017) and all teaching modules developed, were first developed in English, and then translated and made widely available for teachers in the four countries.

The same partners then got funding for a second project, starting in 2019 and still ongoing, called TIME (Teachers’ Inquiry in Mathematics Education). In this project, the tools from MERIA are implemented through LS, as teachers now do all the design work within this framework, with the support of researchers as “knowledgeable others” (TIME, 201?). In spite of previous, small-scale attempts to carry out LS on upper secondary mathematics in some of the countries (e.g. VERHOEF; TALL; COENDERS; VAN SMAALEN, 2014), it is probably fair to say that this project represents the first large-scale experiment with LS at upper secondary level in Europe. Some of the materials – including a guide to Lesson Study and templates for creating lesson plans and practice reports – are already available on the project website (TIME, 201?) in five languages, while some of them have been translated into Portuguese for explorative use in Brazil.

It may be too early to conclude from these experiences, in particular because the Covid situation affected the work from 2020-2021, leading to many of the research lessons to be taught online. Nevertheless, some interesting observations can be shared. It is true that teachers experience the previously mentioned external pressures on high school mathematics as a partial obstacle, as both students and colleagues may expect rather traditional forms of teaching to prevail at this level. At the same time, the 2-6 LS teams set up in each country have worked with considerable enthusiasm and creativity, and managed to design research lessons with solid, high school level contents and, at the same time, with much more challenge for students in areas such as modelling real life situations and investigating more theoretical problems in mathematics. The sharing of lesson plans and practice reports among teachers from all four countries has been particularly fruitful, as have “open lessons” with onsite or online participation of teachers from other teams (in the same country, or from other countries). We have seen many of the same challenges for teachers who engage in their first LS as are known from primary and secondary schools outside Japan, and also some of the same pleasant surprises –

like the virtual absence of distraction of students caused by observing teachers, and the massive development of shared teacher knowledge achieved, particularly during reflection sessions.

We have also noticed some specific advantages of carrying out LS at this level. Here, the fact that European upper-secondary teacher usually has a strong, university based mathematical background, is a real resource both in the preliminary analysis of the knowledge at stake in lessons, in the design of creative and innovative lessons, in the management of unexpected student productions, and in the analysis of observations from research lessons. At the same time, collaboration with university researchers (generally with specialties in mathematics and mathematics education), has been very fruitful in all phases, and the "knowledgeable others" have learned just as much as teachers from the collaboration. One can even say that, compared to the more traditional set-up in the MERIA project, the collaboration has been more fruitful for both teachers and researchers, since teachers were leading what pertains to the core of their profession (lesson design), with researchers in a more appropriate role as "assistants" whose knowledge of mathematics could be activated at selected spots – and as "learners", for instance, of the delicate task of making relevant observations during a lesson. For them, gaining more intimate knowledge of the conditions and constraints of contemporary high school teaching – for instance, in relation to the use of computer tools, and the actual contents currently taught in high school – was also of particular interest for them as teachers (of students with a mathematical background from high school).

Conclusions

We have considered a variety of recent experiments with LS, carried out in Danish schools (from primary to upper secondary level) and as a means for professional development of teachers (in the same schools, and carried out with support from university colleges and universities). The overall impression is that LS is indeed feasible as an activity, with various obstacles and potentials being visible, that depend both on the actual set-up of the experiments, and on the institutions involved. A common trait is that LS seems to necessitate not only a strong organisation, backed up by school management and the engagement of leading teachers, but also the collaboration between teacher teams (observing each others' lessons) and with external "knowledgeable others". This is not surprising in itself, given that such paradidactic infrastructure is also common and, it would seem, necessary in Japan and other countries where LS is just part of the normal practice of teachers.

It is also a shared experience that LS furnishes opportunities for teachers to share observations, reflections and innovations which touch upon a wide range of topics that are central to their profession: from specific, purely mathematical ideas, over knowledge about how students are able to develop mathematical problem solving, to more generic aspects of teaching and learning.

Perhaps the potential of LS to foster teachers' own inquiry into the mathematics they teach, is a main potential which is both to be deliberately pursued and actually observed in such activities: only when teachers remain mathematics learners themselves – and curious investigators and designers of mathematical problems, naturally related to teaching – can we get beyond the sad discourse about teaching as a mere matter of "delivery" and "teaching styles". Certainly, teachers' own knowledge of mathematics is crucial – but even more so, their knowledge of how to investigate and explore it, with their students, and with each other.

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