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Measuring Long Distances With a 2600-Year-Old device: A Historical Application of Triangle Similarity by Thales of Miletus

Augusto Mendes Duarte[1]

Abstract: Triangle Similarity is a topic covered from the final years of Middle School and the early years of High School about which official documents provide only a succinct discussion of its applications. Aiming to address this need, a didactic sequence based on a historical account was applied, guiding students through the construction of an apparatus historically attributed to Thales of Miletus. This device supposedly allowed him to measure the distances of ships from the coast using Triangle Similarity. After proving its effectiveness in a miniature environment, the constructed device was applied to measure distant objects from the top of the school's tallest building. An error analysis was carried out to encourage scientific reflection on the experiment. The project also included several connections with History and Arts during the discussions and crafting of the device. The lessons were received with great enthusiasm by the students, who showed considerable understanding of the studied process.

Keywords: History of Mathematics. Triangle Similarity. Intercept Theorem. Thales of Miletus. Manipulative Materials.

Medindo longas distâncias com um aparato de 2600 anos: uma aplicação histórica da Semelhança de Triângulos por Tales de Mileto

Resumo: Semelhança de Triângulos é um conteúdo do Nono Ano do Ensino Fundamental que elabora conceitos de proporcionalidade geométrica previamente estudados. Objetivando contribuir com sugestões da BNCC, construiu-se e aplicou-se uma sequência didática centrada em torno de uma importante aplicação histórica da Semelhança de Triângulos, articulando conjuntamente diferentes abordagens de ensino. Construiu-se com turmas do Nono Ano um aparato historicamente atribuído a Tales de Mileto, que o permitia medir distâncias de navios da costa, utilizando Semelhança de Triângulos. Após comprovar sua eficácia, aplicou-se à medida de objetos distantes, a partir do topo do prédio mais alto da escola. Realizou-se, também, uma análise dos erros acumulados no processo, incentivando a reflexão científica acerca do experimento. As aulas foram recebidas com grande entusiasmo pelos estudantes, os quais mostraram considerável compreensão do processo estudado.

Palavras-chave: História da Matemática. Semelhança de Triângulos. Teorema de Tales. Tales de Mileto. Materiais Manipuláveis.

Midiendo largas distancias con un aparato de 2600 años: una aplicación de la perspectiva historica de la Semejanza de Triángulos por Tales de Mileto

Resumen: La Semejanza de Triángulos es un tema que se cubre en algunos años de Escuela Secundaria que amplía las nociones previamente adquiridas sobre proporciones geométricas. Con el objetivo de contribuir junto con las directrices de los documentos oficiales, se aplicó una secuencia didáctica basada en un relato histórico, guiando a los estudiantes en la construcción de un aparato históricamente atribuido a Tales de Mileto. Dicho dispositivo le permitiría medir las distancias de los barcos desde la costa utilizando la semejanza de triángulos. Después de demostrar su eficacia en un entorno mas pequeño, el dispositivo construido se aplicó para medir objetos distantes desde la cima del edificio más alto de la escuela. Se realizó un análisis de error para fomentar la reflexión científica sobre el experimento. Durante las discusiones y la construcción del dispositivo, el proyecto también incluyó

[1] Licentiate Degree in Mathematics and Bachelor's Degree in Science and Technology, Federal University of ABC (UFABC), São Paulo, SP, Brazil. E-mail: augusto-md@hotmail.com - ORCID: https://orcid.org/0000-0003-4952-6073.

diversas conexiones con la Historia y las Artes. Las clases fueron acogidas con gran entusiasmo por parte de los estudiantes, quienes mostraron un considerable entendimiento del proceso.

Palabras clave: Semejanza de Triángulos. Historia de la Matemática. Tales de Mileto. Teorema de Tales. Materiales Manipulativos.

1 Introduction

A relevant topic for the final years of Middle School and starting Years of High School is the study of triangle similarity and the theorems of proportionality related to lines cut by transversals. In the Brazilian setting, according to the official teaching document BNCC – *Base Nacional Comum Curricular* (Brasil, 2018), the description of such skills is succinct, limited to the recognition of situations where they apply as well as to the resolution and elaboration of related problems.

Within the "Object of Knowledge" section of the document, experimental verifications of the mentioned theorems are briefly mentioned, but there are no specific guidelines or elaboration regarding their applications. In contrast, in the explanatory section on how the contents of Middle School should be approached, the BNCC stats that

> In addition to various didactic resources and materials, such as grid meshes, abacuses, games, calculators, spreadsheets, and dynamic geometry softwares, it is important to include the history of Mathematics as a resource that can spark interest and provide a meaningful context for learning and teaching Mathematics. However, these resources and materials need to be integrated into situations that foster reflection, contributing to the systematization and formalization of mathematical concepts (Brasil, 2018; translated by the author).

Thus, a didactic sequence was developed to integrate various educational resources and materials, aiming to teach triangle similarity in a contextualized and applied manner. This work seeks to serve as a resource for teachers seeking assistance in developing contextualized activities on the aforementioned topic. The goal is to expand upon and give shape to the BNCC recommendations, as its specific guidelines regarding this content are scarce in terms of how context and application can be brought to teaching this content.

The resources employed in this work were: (i) the History of Mathematics; (ii) Manipulative Materials; (iii) Application to Real World Situations; and (iv) Problem Solving, through the construction of an apparatus historically attributed to Thales of Miletus for measuring long distances. The device was used to measure distances within the school environment.

2 Theoretical Grounding

According to Brown *et al*. (2014), Mathematics is generally taught in a manner opposite to the current recommendations from the science of learning. It is common for each topic to be taught in isolation and repeated until it is "memorized". However, current research on brain function suggests that topics should be taught in an integrated manner. The inclusion of context and a diversity of stimuli is believed to enhance memorization—contrary to the assumptions of traditional teaching methods.

This perspective becomes a compelling argument for contextualized teaching: as knowledge is embedded within a real-life situation, its retention retention within the student's cognitive framework increases. Similarly, integrating Mathematics into its historical development allows for not only a better understanding of Mathematics but also of History itself, as the study of ancient civilizations is also recommended by official documents.

These principles also align with established theories of knowledge. Ausubel (2003), for example, emphasized the activation of prior knowledge as a way to connect new learning with existing knowledge. For this author, meaningful learning depends on the connections established with prior knowledge—a fact currently supported by the research analyzed by Brown.

The present project used various types of students' prior knowledge to support the learning process, including: Concepts of measurement and the different types of tools that can be employed; Intuitive understandings of the different types of errors accumulated during the measurement process and their impacts; The historical context of Thales of Miletus within the society in which he lived, as well as the historical importance of the measurement methods he developed.

Especially regarding the History of Mathematics, for decades, a line of thought has advocated for its use in the regular teaching of Mathematics. Miguel (1997), for instance, is one of the researchers who compiled diverse arguments in favor of its use: history can demystify and de-alienate Mathematics, bringing students closer to the difficulties and challenges faced by past scholars. Such contact with the past trajectory of mathematicians also serves as contextualization, placing Mathematical knowledge within a much broader social context.

Another methodology that has gained support from recent research, such as that of Brown (2014), is Problem Solving. As mentioned, engaging with content in different contexts and situations enhances learning. For this reason, contextualization was integrated both into a real problem and within the historical framework of Mathematics. According to Brown, for learning to be effective, students must recall the content in different situations, applying it to various contexts. In this case, the abstract content previously covered in the classroom is transformed into a real problem, equipped with real objects.

Presenting problems to students as a way to stimulate learning is, according to Onuchic and Allevato (2011), a technique that can improve students' confidence and self-esteem, as they discover the meaning behind the problematic situation through their own efforts. Brown *et al*. (2014) place at the heart of their theory the idea that knowledge acquisition should be "effortful": what the mind has to work to build and retain is what will be remembered more easily in the future. Therefore, it is useful for students to engage in a good deal of productive struggle. Mathematical problems fulfill this function, especially when combined with the other approaches previously mentioned.

3 Organization and Methodology

In 2021, the didactic activities were carried out in four 9th-grade classes at a private school in São Paulo, after normal school activities were restored following the waning of the pandemic. Each class had approximately 22 students, all around 14 years old. A total of six of 45-minute classes were fully dedicated to this didactic sequence.

This contextualized activity was implemented after several previous classes on triangle similarity. Since it is a somewhat abstract application, it was not considered appropriate as an introduction to the topic, as students needed to already be familiar with similarity calculations. The steps of the didactic sequence are summarized below.

Activities	Objective	Methodology and Resources	Number of Classes
1.Introduction: History of Math	Discussing well-known methods for long-distance measurements; contextualizing within the History of Mathematics; encompassing their peculiarities and cultural context	Conversation about History of Math	1
2. Building the Apparatus	Using various materials to construct Thales' apparatus in the school workshop	PVC pipes; soldering iron; saw; printed ruler; hot-melt adhesive; gloves; mask	2~3
3. Testing	Testing the apparatus with short-distance objects	Thales' device; group practical work	1
4. Long Distance Measuring	Using the apparatus to measure long distances	Thales' device; group practical work	1

5. Discussion	Discussing the results	Collected data; group practical work	1

Framework 1: Summary of the didactic sequence.

The first class was dedicated to a conversation with the students about long-distance measurement methods. Therefore, emphasis was placed on recognizing students' prior knowledge, as guided by Ausubel (2003) and Brown (2014). However, it was not an expository class: the position of the teacher as the holder of knowledge was avoided, encouraging an equitable exchange of knowledge between students and the teacher. As recommended by Anastasiou and Alves (2009), this approach was part of the chosen methodology in order to allow more active participation and reflection from students.

The subsequent use of concrete materials occurred as recommended by Lorenzato (2006), who argues in favor of the playful and stimulating capabilities of such materials. This type of invention, according to Roque (2012) and Heath (1921), was part of the lives of mathematicians for many centuries, as they were almost always also inventors. Thus, the activity reported here was able to transport students to the ancient practice of Mathematics, allowing them to experience firsthand how formal knowledge intertwines with its concrete applications. During some classes in the school workshop, groups built the alleged device used by Thales of Miletus to measure the distance of ships from the coast. In the following classes, they tested the apparatus inside the classroom, verifying that it worked based on confirmatory measurements with a tape measure. Finally, they used it to measure long distances from the tallest building in the school.

In the last class, a group analysis of the obtained data was conducted: what errors were involved in the measurements? How do such errors propagate? How can the measurement be refined? Would such errors be acceptable within what the apparatus is supposed to do? Such questions aimed to expand students' scientific questioning capacity, encouraging them to interpret the activity as a scientific experiment. Since error analysis is a field of knowledge normally studied only in higher education, this conclusion sought to incite this reflection beforehand, facilitating future scientific understanding if students choose a career in exact or biological sciences.

4 Teaching Account

In this section, the directions taken by the activities within the classes are briefly outlined. A summary of the discussions and historical knowledge is included to situate the reader in what was discussed with the students, aiming to be convenient for teachers who may wish to apply a similar activity.

Firstly, a conversation was held about what the students know about ancient measurement methods. For example, what activity was carried out by the Egyptians in Figure 1? Some students were able to discern: there is evidence that the Egyptians measured land distances by stretching ropes.

Figure 1: People measuring distance by rope stretching ropes in Ancient Egypt

Source: Creative Commons Attribuition Share-Alike 3.0 Unported and Wikimedia, from https://commons.wikimedia.org/wiki/File:Rope_stretching.jpg

Although much debated, Roque (2012) argues that there is no current evidence of continuity or influence between Egyptian and Greek mathematics. Nevertheless, we have discussed how ancient historical records credit the Egyptians, in particular, as the origin of the mathematics that later flourished in ancient Greece. In the 5th century, a book by Proclus, a Greek born in Constantinople (modern-day Istanbul), describes how geometry was created in Egypt for land measurement. This view is shared by Herodotus, a famous historian of the same century.

According to Heath (1921), the first Greek mathematician is considered to be Thales of Miletus. He is believed to have traveled to Egypt to study mathematics and later disseminated this knowledge in Greece. After this, he is said to have formulated many propositions in his studies, bringing both theoretical and empirical teachings to his contemporaries. One of the most famous stories about him recounts that he was able to measure the height of a pyramid. To do this, it was enough to notice that the size of people's and objects' shadows varies throughout the day. At noon, when the Sun is directly overhead and no shadow is cast, there must be a moment, as shadows lengthen, when the shadow's length equals a person's height. At this moment, he is believed to have measured the size of the pyramid's shadow.

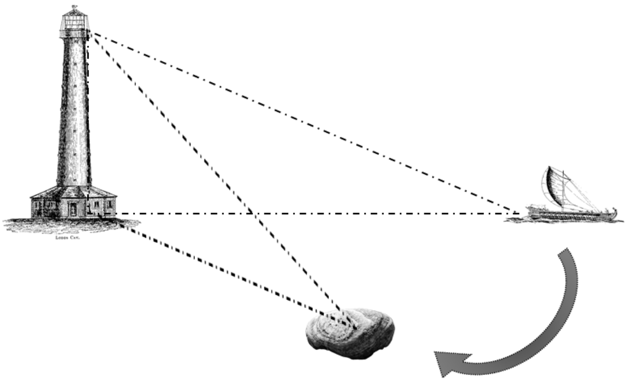
This story was first recorded by Diogenes around 400 BCE. Later, other authors reformulated it by adding additional details, such as stating that instead of measuring his own height, he measured the height of a stick fixed in the ground. According to Roque (2012), today it is debated whether this actually happened, or whether Thales even existed. All that can be certain about this legend is that studies on triangles were indeed relevant. If the legend was fabricated, it would serve to show this fact.

The procedure described was discussed with the students. It is important to note that the Sun’s rays come in at the same angle and therefore form similar triangles when intercepting the top of the pyramid and the person’s head. This notion can be inferred from the thinking developed in another geometry skill of the 9th Year. According to the BNCC, students should: “(EF09MA10) Demonstrate simple relationships between the angles formed by parallel lines cut by a transversal” (Brasil, 2018. Translation by the author).

Once this was understood, we proceeded to explore other applications of triangle construction with the same angles. There is also the legend that Thales was capable, using some method, of measuring the exact distance of ships from the coast, a technique that is very useful for both military operations and peacetime, as it allows one to estimate the time required for ships to reach the shore.

Heath (1921) suggests a method for which there is historical evidence from 1st-century geometry and from reports about a Napoleonic engineer measuring the width of a river. As in the previously discussed case of the pyramids, the incidence of light rays was used. The students were asked: if Thales were to stand in a lighthouse or **a** tall tower on the beach and knew the distance to certain objects on the coast, how could he know the distance of a ship? One answer is that he would use an instrument such as a compass, with one leg fixed in the ground and another adjustable. Aligning his sight with the adjustable leg, he would aim it at the ship, and then, without changing the angle, he would aim it at one of the objects on the beach whose distance is known.

Figure 2: Triangles formed by the visual process described above

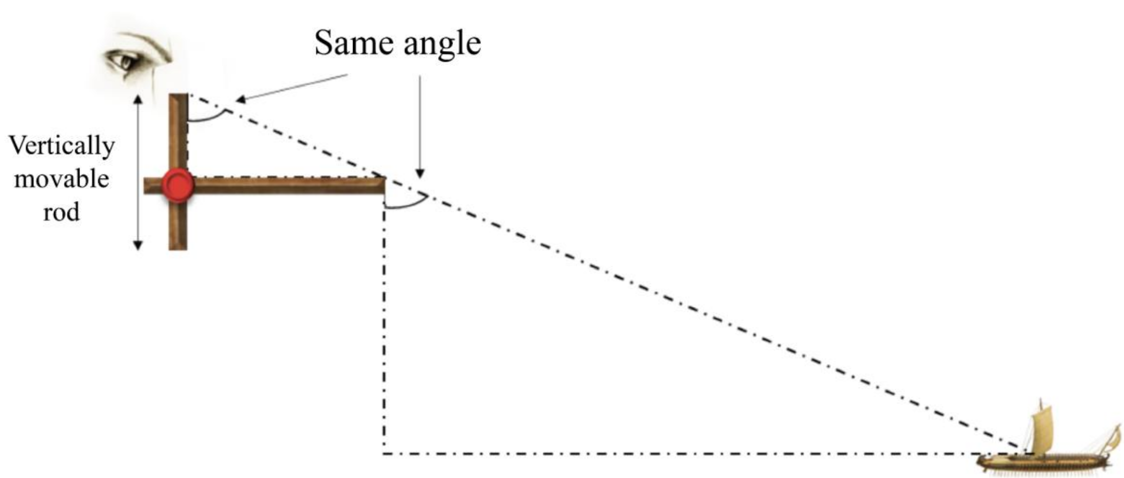


Source: Public domain CC0 License (tower), Creative Commons 4.0 BY-NC (rock) and CC BY-SA (greek ship). Composition by the author.

This process makes use of a triangle formed by the sight angle between Thales (in the tower) and the ship. The triangle was then rotated to align with another object whose distance is known. As a result, the distance from the tower to the ship is equal to the distance from the tower to the object, because the triangles are congruent.

It was mentioned, however, that an even better possibility to determine the distance to ships was proposed by Eudemus. According to him, Thales would have a device similar to a square constituted of rods forming a 90° angle: a vertical rod of adjustable height and another horizontal one parallel to the ground. By adjusting the vertical rod so that, when looking over its tip, the ship is seen at the tip of the horizontal rod, one obtains a miniature version of the triangle formed by the tower and the ground leading to the ship.

Figure 3: Thales's device for measuring long distances



Source: Public domain licence CC BY-SA (ship) e CC BY-NC-ND (eye). Composition by the author.

With the aim of contextualizing the situation in triangle similarity problems similar to those the students were accustomed to solving, an example in these terms was provided.

> "By observing the previous figure, suppose the tower measures 20 m. The horizontal rod of our device measures 20 cm. To be able to see the ship at its tip, we have to make the height of the vertical rod as long as 0.5 cm, from the red joint up to its tip. What is the distance from the ship to the tower?"

The students discussed different ways to solve the problem. As shown in Figure 3, this configuration forms a triangle formed by a movable vertical rod attached to a fixed horizontal rod of known length. The user must be located on a tower of known height to make the measurement: when the observer's line of sight connects to the vessel, the height of the tower and the distance from the tower to the ship describe a triangle similar to the triangle formed by the two rods. Drawing on their previous experience from earlier problems and discussions, the students solved the problem without much difficulty. The next step was to replicate the legendary device in the school environment. The school where the project was implemented has a workshop where students can build objects using various tools and materials. Available materials were investigated, culminating in the construction of the object using PVC pipes.

The construction steps were:

(i) Drill a hole in the joint of the pipes using a soldering iron and a saw. This hole allows the user to control the vertical sliding of the graduated tube using their thumb.

Figure 4: Hole for thumb control of tube sliding

Source: Author.

(ii) Glue the T-shaped joint to the horizontal (fixed) rod by inserting it into the connection opposite to the hole that was just opened.

(iii) Create a small hole at the end of the movable rod, through which a clamp can be fixed. This clamp allows the vertical rod to slide but not fall off the apparatus.

Figure 5: Hole in PVC pipe

Source: Author.

(iv) After the group construction, the students glued a printed ruler cutout onto the movable tube and covered it with plastic to prevent friction from tearing it off. We also provided colored paints so that each group could paint the device as they wished, promoting an integration with the Arts. One of the classes achieved the result shown in the figure below.

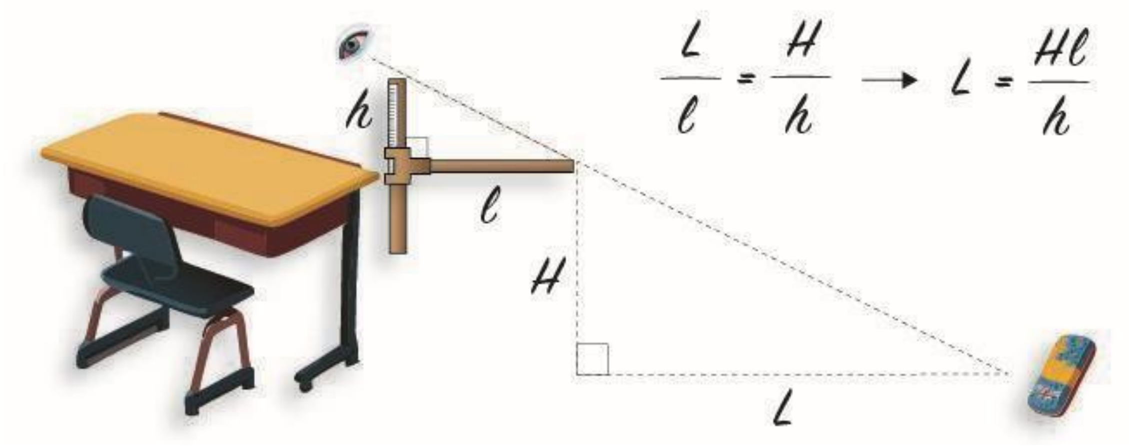
Figure 6: Complete device on the left. The graduated tube is the movable rod, while the other one is fixed, being used to align the sight. Result painted by the students on the right

Source: Author.

The following classes were dedicated to the practical application of the device. Before the activity, we conducted a brief review of how the device works, emphasizing the required idea of similarity. With the participation of the students, we derived the formula from the similarity equation for this case. Aimed at exercising scientific thinking, the students first tested it on nearby objects to verify the effectiveness of the apparatus.

This stage was carried out by measuring the distance to three erasers that were spread across the empty classroom, taking a desk as the starting point. By placing the device on the edge of a desk, the groups measured the distance from the desk to the eraser using the formula, as shown in Figure 7. The height of the desk was obtained using a tape measure.

Figure 7: Diagram of measurements taken in class



Source: Author.

Each member of the group conducted their own measurement, ultimately using an average of the results. The actual distance from the desk to the eraser was measured again with a tape measure for verification of the effectiveness of the apparatus.

Guiding questions were provided:

a) Measure and record the distance to the three objects using the device and the formula we developed. Each student should perform at least one measurement for each object.

b) Organize the values in a table as in the example below. Note that there is a field in which the average distances for each object must be calculated.

Table 1: Space for students to organize their findings

	OBJECT 1	OBJECT 2	OBJECT 3
Student A			
Student B			
Student C			
Student D			
Mean			
Real Distance			
Deviation			
Percentual Deviation			

Source: Author.

c) Measure the actual distance with the tape measure for each object. Then, record the values in the table.

d) Compare the calculated averages with the actual measurements obtained. Find the absolute difference between them and record it in the table within the “deviation” field.

- **e)** Considering the data obtained in (d), do you think the measurement error with the device is too large? What sources of measurement errors can you identify in this experiment?

As per the questions, a brief analysis of errors and reflection on their meanings was conducted. The discussion on question (e) was held in a plenary session with the entire class, and then each student constructed their personal response. In every class there was at least a group with excessive errors, leading to insightful reflections. The general conclusion was that the percentage deviation is indeed small when considering the range of human vision to the horizon, thus providing a good approximation of the actual distance.

In the next class, the students were taken to the High School building, a three-story structure made of large brick blocks. The height to the parapet of the third floor was already known, as a school staff member measured it by dropping a long tape measure from the top. However, the students were not informed, as it was expected that the groups would attempt to solve the problem in their own way. For those who needed it, the hint of counting the bricks was offered—reckoning their rows could be used to estimate the height.

Three cones were positioned in the courtyard below. Students were supposed to measure the distance to the objects from fixed points on the parapet, using the estimated height of the building in the calculation. Everything went satisfactorily, as the weather was also good. Finally, in the classroom, the actual height of the parapet was revealed for them to compare with their measurements. The last question of the script asked them to estimate the expected error of this measurement based on the error obtained in the test measurement.

The wrap-up was conducted by diving back into the history contained in the situation. As Heath (1921) informs, besides being the first Greek geometer, Thales was also an astronomer. It is said that he was able to predict an eclipse at the time, which requires advanced astronomical and mathematical knowledge. He possibly obtained such knowledge from the Babylonians and Egyptians. There is a famous tale that says that, because he was so focused on the sky, Thales ended up falling into a well.

At this point, students can be led to think that Thales was a great "genius." A very illustrative example, however, is the fact that despite being an incredible astronomer, Thales did not believe that the Earth was round. Instead, it is said that he saw the Earth as a disk floating on a great ocean. The first Greeks to have considered the Earth round apparently were the Pythagoreans, who were the most important mathematicians right after Thales. Such a fact was offered to the students as a good example of how even the greatest sages made mistakes, and theories being reassessed or surpassed is a common trend in the History of Sciences.

5 Conclusion

In general, the classes proceeded appropriately and as planned. The only problem was organizing the rotation of groups using the tables, since the teacher also had to help the other students who were engaged in calculations. Three of the classes completed the activity successfully. One of them had a great lack of concentration, so that even after two classes they were unable to complete the itinerary. As a result, they did not carry out the long distance measurement activity.

Regarding the content, it is known that history books sometimes provide descriptions of the achievements of ancient mathematicians as mere anecdotes, distant from the world we live in today. Such a view ends up portraying Mathematics and its calculations as something exotic and mysterious, incomprehensible to people nowadays. Sir Thomas Heath's book (on which this project is based) is almost a century old, and the description of the apparatus we built did not extend beyond a few lines.

However, once transmuted into an object, such a vague description could bring the students closer to its reality, illustrating how concepts like triangle similarity have facilitated human life for over two thousand years. Through simple calculations, measurements that would not be feasible otherwise suddenly become possible.

In this sense, I consider the didactic experience a success, as the majority of students expressed enthusiasm in using the tools to build the object, as well as in using it to measure long distances. The error analysis conducted also allowed the students to understand the type of deviation present in all human measurements, verifying how even in the face of errors inherent to the methods, the measurements still hold great practical value. The students also exhibited a profuse assimilation of content, as shown by the positive assessment results.

This experience is shared here with the aim of inspiring mathematics educators to explore the articulation of different resources, highlighting the value obtained by combining, for example, problem-solving, history, and manipulative materials. Although it is desirable for teachers to increasingly know the history of mathematics through the mentioned sources, it is acknowledged that not all teachers have the time to conduct this type of research. This article, therefore, attempts to fulfill the role of transmitting information that was tested in practice, sharing different ideas for teaching triangle similarity and the intercept theorem.

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