Jeremy Kilpatrick on Pólya — An interview

Henrique Manuel Guimarães
hmguimaraes@fc.ul.pt

Some notes about the interview

In February last year, I had the opportunity to visit the University of Georgia for three months, during my sabbatical leave, to do some research on and study of George Pólya with the support and collaboration of Jeremy Kilpatrick. I was mainly concerned with Pólya's ideas about mathematical discovery and activity, about problem solving and its teaching, and about mathematics teaching in general.

In fact, I chose the College of Education at UGA precisely because Jeremy Kilpatrick was there. I knew from previous talks with him and from his work that he was a student of Pólya's at Stanford University, and early in his academic career had been Pólya's assistant and collaborator. He was one of the very first people I thought of for guidance and collaboration for the work that I wanted to carry out. I thought that his first-hand experience with George Pólya was valuable, not only for understanding Pólya in another way but also for getting vivid testimony from one who had worked with him. So, I was very interested to hear from Kilpatrick about his direct and living experience with Pólya.

I interviewed Kilpatrick in April, twice in the same week, just after his doctoral class on the Advanced Study of Mathematics Curriculum that I also attended, which ended at 14h. For the conversation, we agreed to meet at my room 502 in Aderhold Hall, which was in the same hallway as Kilpatrick's office and almost opposite. That room was kindly placed at my disposal during my stay by the Department of Mathematics and Science Education.

(Photos: H. G. Guimarães)

J. Kilpatrick, after the interview Aderhold Hall, room 502, April 22, 2010

The talks went very well, in a smooth and fluent way, with both about 50 minutes long. Kilpatrick was very attentive and with great openness - and patience - tried to answer my questions, which were not always well formulated.

The following text comes from transcripts of the two interviews in their original sequence. Some pieces of the transcripts were excluded for text economy and global unity, and some corrections were made to avoid excessive repetition and to favor a better reading. However, I tried to keep

RIPEM, V. 1, N.1, 2011
10.37001/ripem.v1i1.1127

the informal and colloquial tone of the conversation. Kilpatrick read and reviewed the final version of the text, giving me help in some of the notes I decided to include. I am thankful for his gentle and careful assistance.
Kilpatrick begins by telling how he came to 'find' George Pólya, and how he decided to go to Stanford University where Pólya was, first being his student in an academic year institute, and then being his assistant in the following institute. Kilpatrick gives us some flavor of how Polya used to teach problem-solving courses in those institutes - "by far the most interesting [ones]" - with some examples and interesting details.

The interviews came to focus mainly on problem solving and the teaching of problem solving, but also on teaching in a broader sense. Kilpatrick's warmly spoke of how influential it was to be in Pólya's classes - "I saw problem solving in a different way" - and how Pólya also influenced him as both a teacher and a person: "I never had a teacher who was as influential on my way of thinking about teaching and being a human being than Pólya."

Jeremy Kilpatrick: a brief biographical note

Jeremy Kilpatrick is, since 1993, the Regents Professor of Mathematics Education at the University of Georgia, Athens GA, in the United States of America, where he currently teaches master's and doctoral courses on curriculum and on research. He jointed the College of Education in 1975, and before that he taught at Teachers College, Columbia University, as Assistant (1967-70) and Associated Professor (1970-75).

Jeremy Kilpatrick graduated in mathematics at the University of California at Berkeley (1957), and received his Master's degree in education from the same University (1960) while he is teaching in a junior high school in Berkley (1957-60). Early in the 60's, he went to Palo Alto, California, and received from Stanford University another Master's degree, this time in mathematics (1962), and a doctorate degree in mathematics education (1967). While at Stanford University Jeremy Kilpatrick attended George Pólya's seminars on problem solving, and became his assistant. George Polya was in his PhD dissertation committee.

Doctor Honoris Causa from Gothenburg University in Sweden (1995), Jeremy Kilpatrick is a widely, nationally and internationally, recognized leader in mathematics education. He taught courses in mathematics education at several European and Latin American universities, and has received Fulbright awards for work in several countries, namely, Colombia, New Zealand, Spain, and Sweden.

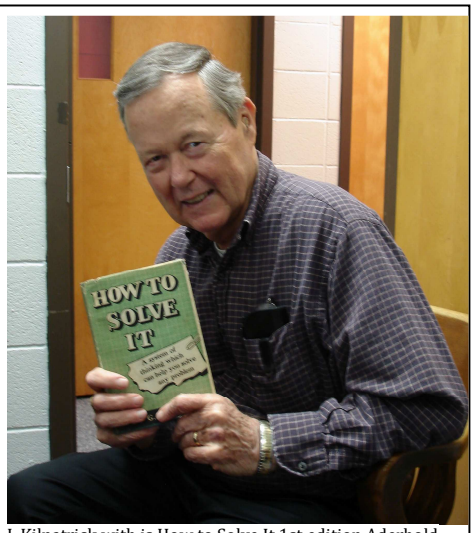
In 2003, Jeremy Kilpatrick was honored with the Lifetime Achievement Award for Distinguished Service to Mathematics Education received from the National Council of Teachers of Mathematics. More recently, in 2007, Jeremy Kilpatrick received from the International Commission on Mathematical Instruction the Felix Klein Medal, honoring lifetime achievement in mathematics education.

Henrique Guimarães – I think that one way to begin is to begin before [your stay at] Stanford. How did you 'find' Pólya?

Jeremy Kilpatrick – Well, the first contact I had, which was… — and actually I didn't know that it was from Pólya — [it] was when I was in my final year of secondary school, and there was that Stanford mathematics exam[ination][1]. At that time, if you did well on that examination you got a scholarship to Stanford. Later they decreased the prizes, but at that time I am pretty sure you got a scholarship if you did well. And so I took the exam in my senior year. And I had never taken an exam like that in school at all. It was three hours, and I believe the year I took it there were only three questions, but some times there were four, sometimes even five. But to spend that much time on one problem was something I had never done. […] But it was my first contact with something… related to Pólya. And then… before I went to Berkeley, I was for two years in a junior college in California, Southern California. […] I went to Chaffey College, which is a junior college, two years, [and] I took courses that allowed me to transfer to Berkeley. While I was still at Chaffey, I remember going to a large department store and seeing this book called *How to Solve It*[2], and I…

HG – It was in nineteen?…

JK – It would be nineteen … fifty … three, probably, or possibly fifty-four. I graduated from Chaffey in fifty-four. So it was fifty-three or fifty-four.



J. Kilpatrick with is How to Solve It 1st edition Aderhold Hall, room 502, May 9, 2010.

HG – And you were studying mathematics?

JK – I was taking mathematics courses, studying mathematics preparing to— … At that time then I had decided to go to Berkeley, so I was preparing to transfer to Berkeley. And I found this book in the store, it was just on the shelves in a … in a store in a city nearby. It was a big department store that had a book section. I went into the book section, and I found *How to Solve It.* Not the paperback, but the hardback, the Princeton … the original Princeton [University Press edition].

HG – Do you still have it?

JK – I still have it; I still have it, yes, yes. So I read that book, and I thought "such a wonderful book". I didn't know [it] was connected to the examination. I didn't know that, but I was interested in *How to Solve It.*

HG – Let me interrupt you just to talk about How to Solve It. *You picked up the book; you read the book, and as you have just said "that's a wonderful book"* (JK – Yes). *What made you say that?*

[1] The Stanford University Competitive Mathematics Examination was a competition at Stanford University for high school seniors modeled on the well known Hungarian Eötvös Contest. George Pólya and his colleague and friend Gábor Szegö started the Stanford competition in 1946, and it was given annually until 1965.
[2] *How to Solve It: A new aspect of mathematical method* (Pólya, 1945). This book was published in Portuguese in Brasil (Pólya, 1977), and in Portugal (Pólya, 2003).

JK – Because for someone who is trying to do mathematics, it has, I think, such good suggestions, such good ideas. It comes from Pólya's …

HG – [Good ideas] to do mathematics?

JK – Yeah. It comes from Pólya's experience as a teacher trying to help someone else learn how to do mathematics, and so it has a lot of good suggestions.

HG – Suggestions to do mathematics?

JK – To do … to solve problems.

HG – Why do you say they are good?

JK – Because they are good suggestions. They are useful if you, if you—. I think they are difficult to learn. I mean, one of the issues that comes up later is [that] … … Alan Schoenfeld[3] argues that these [suggestions] are too general, that they are too difficult to teach, and that they are not specific enough for … programming a machine to solve problems. But I would argue that if you see enough examples of how these heuristics work, you gradually come to adopt them in your own problem solving. And you can see … you can see how they would operate, and Pólya gives a lot of examples of that. Actually, *Mathematical Discovery*[4] includes even more examples — [Pólya's] book *Mathematical Discovery* — and I learned all about that later. But even with *How to Solve It,* which has … fewer examples of the kind of problems [from] secondary mathematics, even *How to Solve It* gives a lot of good advice on ways of how to think about problems.

HG – At that time you still had the idea to major in mathematics?

JK – Yes. Well, I was just beginning to think I would major in mathematics. I wasn't sure when I went to Berkeley that I would, but I decided—. I'd taken so many good mathematics courses in my first two years at Chaffey that I had more mathematics then anything else, and I enjoyed it. So, I knew I was going to be a teacher, and—. I liked psychology, and I think that's why some of what Pólya was saying in *How to Solve It* was in a way psychological. And so the combination of psychology and mathematics was interesting to me. I finally decided to major in mathematics, and I found Pólya's book really quite helpful in thinking about mathematics, and doing mathematics.

HG – That's interesting because this book is [often] seen as directed to the teaching of mathematics.

JK – Yes, [but] I didn't think about it that way. I was not thinking about teaching.

HG – At that time.

JK – No. Although in my last two years before I graduated at Berkeley, I was thinking about … obviously becoming a teacher, becoming a mathematics teacher. And I got a job teaching mathematics in the Berkeley schools, teaching in a junior high school, [grades] seven-eight-nine. And I taught science, and I taught mathematics. And then I could see how useful Pólya's ideas

[3] Alan H. Schoenfeld, professor at the Graduate School of Education, University of Califórnia, Berkley.
[4] *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*, vol. I & vol. 2 (Polya, 1962 &1965; 1981, combined edition).

were for teaching. But before that … my first attraction to Pólya was more about the learning of mathematics and learning how to do mathematics.

HG – How to solve problems…

JK – Yeah. So, when I was at Berkeley — I don't know which year it was either, but it probably was fifty-five or fifty-six, I don't remember — my roommate who went on to become a doctor, a medical doctor, he was a freshman. And he was taking a course in mathematics, and Pólya was a guest speaker. And then of course I knew the name Pólya, so when he told me that they were going to have Pólya come into to their class and give some lectures—

HG – You went there…

JK – I went there. It was my first opportunity to see him give a lecture, and he was very good. I don't remember the topic; I don't remember that.

HG – There is anything special you can recall?

JK – No, I can't recall. I know I went there, that's all I know. And that was what, that was what … encouraged me to go to Stanford. I mean—. Well, there were several things. One is [that] … while… … in one of the years I was teaching at Berkeley, they had a summer institute. The General Electric Company had a summer institute at Stanford and I went for that summer institute. Pólya was not teaching that summer, but I think that I got to know him there.

HG – But you haven't met Pólya there?

JK – I am not sure. I don't really remember. I knew he was there; I knew he was there. I decided after teaching some time in Berkeley that I wanted to go back [to Stanford] and get an advanced degree. I was getting a master's degree [in education] at Berkeley, but I wanted to get a doctoral degree and become a teacher educator. […] And so I thought, "Well, maybe I'll go to Stanford instead[5] because Pólya is at Stanford, and because Stanford had what was called in those days an academic year institute. This was a National Science Foundation program which allowed you to go … to spend an entire year studying with somebody. And Pólya was going to be … giving the main class as part of that academic year institute. I think that there were three classes that we all took, but Pólya gave the most interesting one.

So I applied for that institute, and I also applied for a summer [NSF-supported] internship. So I went to Stanford … the summer before the academic year began in September. I finished my master's degree in Berkeley, which was about problem solving actually, but word problem solving, the difficulty of word problem solving. But it didn't really make much use of Pólya, [of] Pólya's ideas.

But I went to Stanford and started work on a master's degree in mathematics. […] I got a fellowship to study mathematics in the summer, so [I] studied mathematics in the summer, and then when the fall term began I studied in the academic year institute with Pólya. And that was

[5] After getting his master's degree in education at Berkeley, Jeremy Kilpatrick, as he told me, thought about going to Yale to get a doctorate in education — "I'd like, I'm thinking about going to Yale, where Ed Begle is", he said — but a former teacher of his at Berkeley told him that, at that time, Yale had no program in education.
Edward Griffith Begle (1914 – 1978) was mathematician at Yale who led the School Mathematics Study Group (SMSG) — the largest New Math project in USA — from its beginning in 1958 until its end in 1972. He and SMSG moved to Stanford in 1961, Jeremy Kilpatrick was already there, and Ed Begle came to be Kilpatrick's major professor for his doctorate; Pólya was on his dissertation committee.

my first introduction to Pólya as a teacher of a course. So he taught that course all year, and then — I think it was the following year; yeah, it was the following year — they offered another academic year institute, and I became his assistant for that.

HG – After the course?

JK – Yeah. After one year of taking the course, then I was the assistant for the course. I graded the papers and did that sort of thing.

HG – That course was on problem solving?

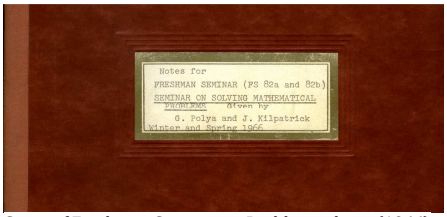
JK – Yes, it was a course was on problem solving. I still have the notes from it. Actually a lot of the ideas in that course eventually were incorporated into *Mathematical Discovery*. The two volumes of *Mathematical Discovery* were written from Pólya's experience in putting together this academic year course. He used a lot of problems from the Stanford competition[6]. By the way, when I was at Stanford, I had the experience of marking the competition. … So I took it when I was in high school, and then I marked it when I was at Stanford to help Pólya. Then, eventually, it was Jim Wilson's[7] idea, Jim said, you know, that all those Stanford mathematics exams should be collected. So we collected them. We put them in the [*The American*] *Mathematical Monthly*[8], but they didn't want the solutions. They figured that everybody who reads the *Monthly* can solve them. We put them in the Monthly, but we put the hints and the solutions in the book[9]. I did it after I left Stanford, but that was a project that really began at Stanford.

HG – Let me ask you one thing. You have used the words "Pólya [gave] the most interesting course."

JK – Yes, yes, it was *by far* the most interesting course.

HG – By far (JK – Yeah). *How can you describe it, focusing in the content but also in the teaching?*

JK – It was actually called a seminar, and the content of the seminar was really … problem solving. It was really to illustrate problem solving and to show different techniques for problem solving pretty much along the lines the way it is organized in *Mathematical Discovery*. … There were a number of issues. … It was very interesting to me to be in Pólya's course, and then to help with the course and to watch Pólya teach. I had many opportunities. Also I took a number theory course from him later.



Cover of Freshman Seminar on Problem solving (1966)
(From Jeremy Kilpatrick personal archive)

HG — Number theory.

JK — Number theory. I took a number theory course from him later. And there are a number of things that

[6] See footnote 1.

[7] James W. Wilson, professor of Mathematics Education —still a colleague of Jeremy Kilpatrick in the Department of Mathematics and Science Education at the University of Georgia — was also a former student of Pólya at Stanford University (from which he received his PhD in Mathematics Education) and, by that time, was attending the mentioned Pólya's problem solving seminar.

[8] Polya, & Kilpatrick, 1973.

[9] *The Stanford Mathematics Problem Book, with hints and solutions* (Pólya, G. & Kilpatrick, J., 1974). This book was more recently published by Dover (Pólya, G. & Kilpatrick, J., 2009).

were so interesting because … by that time I had been a teacher in a school for three years, and I knew something about teaching. So, when I was in his class, I was looking at him not only as my teacher but as somebody—. I was trying to understand his teaching. So, for example, one of the things he did a lot was to … to act, to pretend he did not know the solution to the problem. So he would say, "What should we do? What should we try here?", and he would ask the students for their ideas. And I was interested because of course he knew what the solution was, but he pretended he didn't know what to do next and [that] he needed some ideas, he needed some help. He always talked about "inside questions" and "outside questions". Inside questions are questions that could occur to you, outside questions are like "Why don't you set this equal to zero and than integrate both sides?", you know (laughing). But [an] inside question is something that could occur to you like "Could I think of a simpler case?" "Could I try something— what about an extreme case?".

So, those kinds of heuristic questions were things that he would try to get us to ask. And if we couldn't come up with an idea, he would try to ask an inside question that would move the whole process forward. So he was really ending up … posing a problem to us and solving the problem *with* us, but trying to push us. … [To] convince us to solve the problem ourselves.

I remember things that struck me. One was the pace. He never went faster than the slowest person in the class. All of us in the class were teachers, had been teachers, and there were several people in the class who really struggled with the mathematics. So it was interesting to me to watch him teach a group of people where some people were moving very slowly. And in the seminar, as I say, he never moved faster than the slowest person. Some teachers move at the rate of middle [ability] students; some teachers move at the rate of the fastest students, [and] leave everybody behind. But Pólya always moved at the rate of the people who were … the slowest. …

I remember that in the number theory course—. The number theory course was a different kind of course [than the problem-solving seminar] because in addition to some of us who were teachers in that course, I think most of the people were Stanford undergraduate math majors. Pólya taught that number theory course in a very basic way. We did very elementary things — on rules for divisibility, and proving basic theorems, and so forth — very slowly. And it was very upsetting to some of those students. They wanted to know why we didn't move faster. We didn't move as quickly through the ideas of number theory as a regular instructor might have done. But Pólya's whole approach was to make sure that everybody understood everything we were doing and to move no faster than the class could keep up. And it wasn't boring. I found it really interesting. […] We didn't take up as many topics in number theory as a typical course would do, because Pólya moved at a slower rate — an interesting rate, but a slow rate. And that I found very interesting because it was an experience that I had not had. I had math professors at Berkeley who went like crazy. All of us in the class would sit there, and try and take notes, and go home and try to figure out what had happened in the class. And we had not followed anything. But Pólya was not like that at all. You came out of Pólya's class understanding everything that had happened.

HG - Going back to the problem-solving seminar, what kind of problems were most used?

JK – Mostly the kinds of problems the one finds in the *Stanford Mathematics Problem Book*[10]. Some of the problems were geometry problems; some of them were algebra problems. There was a mixture of them. We would take up certain topics like … I don't know … … I can't think of—.

[10] See footnote 9.

… … Well, for example, we would do all of a set of problems dealing with Pascal's triangle. We would begin to look at some of the different patterns. …

We would start out perhaps with, with … generating Pascal's triangle as a number of zigzag paths. I remember Pólya would start with the word ABRACADABRA and would show how we could make a zigzag path. If you wrote ABRACADABRA down as a diamond [see figure below], you could ask the question "How many ways can you spell ABRACADABRA by going from top to bottom?" And then he would say, "Well, we can have a street grid like Manhattan — with First Street, Second Street, Third Street, and so forth — and then you would count the number of the zigzag paths. This is taxicab, taxicab geometry. And [we] would generate Pascal's triangle. Then we would start proving. … We would observe certain regularities, like the sum of the terms in one row of Pascal's triangle, what is that? And how could we prove that for the general row of the triangle? And those were the kinds of things that we did.

HG – He mentioned his book How to Solve It *in the seminar? He used the book?*

JK – Yes, he used that book. … He didn't use it as a text[book]. … In fact, I don't remember if we had that book. But he would talk about, he would talk about … "Okay, now we need to develop a plan." He would always begin by saying, "What are we given? What are we to find?" … Always he used this metaphor — which is illustrated on the inside cover of the original first edition of *Mathematical Discovery* — of building a bridge: "Here is where you are. Here is where you are trying to go. Whether it is a problem to find or a problem to prove, you are *given* something, [and] you want *get* somewhere. You actually can work both ways. Sometimes you can work backwards; sometimes you work forward. The whole idea, though, is to develop a connection to go from here to there. Pólya says somewhere that the metaphor of building a bridge is something that came to him, is part of what came to him when he had trouble explaining something to this boy when he was in Zurich[11].

HG – He mentioned that in your [1978] interview[12].

[11] The first chapter of the 2nd volume of *Mathematical Discovery* begins with a section entitled "Metaphors". In this section, Pólya recalls the mentioned episode with a boy he is preparing for an examination, and says: "Discovering the solution is finding a *connection* between formerly separated things (the things we have and the thins we want, the data and the unknown, the hypothesis and the conclusion). […] We sometimes see the connection under the guise of a *bridge*: a great discovery strikes us as a bridging over a deep chasm between two widely separated things" (Polya, 1965, p. 2).

[12] In April 1978, Jeremy Kilpatrick interviewed George Pólya about his views on mathematical abilities. This interview, unpublished since then, has been recently appeared in a Portuguese translation ("Pólya e as capacidades

JK – When he was in graduate school — he talked about that a bit in the interview I had with him — [he said] that this boy had not understood, and Pólya gone home and realized that he hadn't explained [it] well. And he worked out this way of thinking about problems because he didn't want to get in that situation ever again — of not being able to … to explain how you go from what you are given to what want to prove or to find.

And he always asked that question. He would give us a problem and would say "What are we given?", "What is that we're given?", "And what are we looking for?", "And how do you get that kind of thing?". He would ask those general questions until we all understood this approach to problem solving: Lay out a plan, carry out the plan … *Always* check, checking over after you have a solution. Ask yourself, you know, "Does this make sense?", "Can I find this another way?", "What can I learn from this kind of problem?". All of the things he had written in *How to Solve It* he would bring out in his lectures.

HG —You mentioned a few minutes ago that Alan Schoenfeld in some way pointed that the approach of Pólya is too general (JK – Yes). *But Pólya would say "I like that," [that those] questions should be general.*

JK — Yes, yes. You see, Schoenfeld was much more interested, much more oriented to cognitive psychology, and I think that Schoenfeld thought that you needed to turn these heuristics into more teachable kinds of things. They are very difficult to teach, and Pólya taught them by giving example after example of how to use them. But, for example, Schoenfeld says, "Well, okay, Pólya says think of a simpler problem — that is thirty different heuristics, you know. That is not one heuristic 'Think of a simpler problem'. There are a lot of ways to think of a simpler problem.". But I have to say, 'Think of a simpler problem' is very good advice, "if you can't solve the problem in front of you, can you simplify it in some way [so] that you can solve it?". I find that so, so… helpful, you know. I don't understand why Schoenfeld would criticize it. But he did, and he felt that~~,~~ … basically he felt, I think, that Pólya's heuristics were not teachable. I would say that they are difficult to teach, and difficult to learn, but I saw Pólya teaching them in the seminar we had with him and in the other courses I saw him teach.

I not only saw him teach number theory, I also saw him teach the so-called freshmen seminars, which were courses for small groups of freshman. And that was very interesting to me, too. That was later in my career at Stanford. Because … I lived in the same part [of Palo Alto], near campus, as Pólya, and I used to give him a ride in my car. The seminars were in the evening, and it was too dark to walk back. So I would give him a ride back to his home. And so he gave [me] a wonderful opportunity. After he had taught [a class] — it was interesting to me, [and] that's another feature of Pólya's teaching — he would go over it in his mind and think about questions that he could have asked [and] things that students had trouble with. He would replay — I've written about this too because this was astonishing to me. Pólya had no doubt taught the same thing for fifty years or more, and if anybody knew how to do it, it was Pólya. But still, after every lesson, he would replay the lesson taught through with me: Tell me what he thought had gone well; tell me what he thought it not gone so very well; talk about how he would do it differently if he was going to teach it again. … He was really analytical about his teaching. He really would replay his teaching and would think very hard about … "If I was going to teach this lesson again what I would do differently." He was doing that partly because he knew I was interested in what

matemáticas"), published in the Portuguese Mathematics Teachers Association research journal, *Quadrante* vol XVII (2), 103-120, issued in May 2011. An abridged version in English is about to appear in *The Mathematics Educator* journal (Department of Mathematics and Science Education at the University of Georgia).

he was doing. And that is part of why he did it, you know, it was a conversation that he and I had. But I thought it was really interesting.

HG — Let me highlight some aspects of what you said about the way you see Pólya as a teacher. You mentioned that he always went at the pace of the slowest [in the class] (JK — That's right); [that] he[used to] look back to his own teaching (JK — That's right), and could rewind, could recall (JK — Yes), and reflect on his on teaching; and that he was very able to ask good questions (JK — Yes), and to pretend, to act [before the students]… (JK — Acting, demonstrating…). These are four dimensions in his teaching…

JK — And another dimension — which I did not see, but I could tell was the case — [was] that I am sure that he prepared his lessons, his lectures, very carefully. I didn't see him prepare, but I could tell when he came in [that] he knew exactly what he wanted to do. He had everything in his mind. He had worked out all of the problems. He knew how he wanted to present it. So he was prepared. And [although] … I didn't see him prepare, I did see him rehearse, rewind. I saw the rewinding, but I never saw the preparation. Except that I was in the class, and I could tell that he had thought about all of these problems and the questions that would come up.

One of the interesting things would be to watch him lay out things on the board. If you have ever seen the [TIMSS] videos of Japanese teachers… This is very striking, I think, the way a Japanese teacher lays out the lesson on the board. By the end of the lesson, everything is on the board. An American teacher just goes and erases and puts something [else]. … Pólya always had that clear display. So he never erased things that you were still looking at. He had it organized. … So you had the feeling that just like the Japanese teacher, Pólya knew that he was going place this in this particular way, and the whole lesson—. Well, sometimes he would have to erase, of course. But before he erased, he would always ask if everybody … if it was okay to erase. But he was always very careful about laying out [the mathematics] on the board in such a way that you could see the development of the ideas. Not many teachers in the U.S. do that. … When I saw the Japanese teachers, I said, "They are doing what Pólya does."

HG – In [Polya's] heuristics—the so-called four phases in solving a problem—there are people saying that it is somewhat hierarchical, sequential, too sequential. What do you think about this?

JK – No, they are not meant to be sequential. I think that is a distortion of what he said. And the way he used those phases, he would talk about how we would go from one phase another but not necessarily in that sequence. You might start carrying out your plan and decide that you have to go back (HG – To understanding). Yeah, "What am I given?"… So he illustrated in his own problem solving, going back and forth, from one phase to another. And he never treated those as a sequence that you march through.

It is natural [to treat it as sequential]. He had to put it in some order in order to put it in the book. But he is very careful. He doesn't talk about stages or levels. They are *phases,* and in any phase you can go in and out of one phase, you [can] go back from one phase to another. They were never meant to be sequential.

HG – Another criticism… problem posing [in Pólya] is underestimated?

JK – I think that this is probably true. Other people worked with issues on problem posing. I think Pólya did not … … did not emphasize problem posing in the general sense. I think what he did do a lot was to talk about how by taking one problem you could transform it into another problem, or how you could create a problem that was a generalization, or a special case or—.

Having solved a problem, you could ask yourself, "What is another problem that is related to this?". So, generating one problem from another, he did all the time. But just generating a problem out of nowhere, he did not … he did not do that.

HG –That is "looking back"…

JK – Yes. I find that looking back—. First of all, I found when I was teaching that looking back was the hardest thing to get children to do. And it is the phase that people, I think, tend to avoid because we've solved the problem, so why should we look back? It is too difficult to look back. But Pólya always spent a *lot of time* on looking back, and he did all kinds of interesting … he had interesting ways of checking — "Is this a reasonable answer?" — he had interesting ways of saying, "How can I change this problem, and how would the answer change?". He had … he had many things to say about looking back because he was concerned that you should learn something from having solved this [complicated problem]. You spent a lot of time on solving this problem. You should learn something from that. And the only way to do it is to look back and see what you did, and say "Could I … could I use this in some other way? Could I … could I change this idea a little bit and get a slightly different problem and solve that?", and that is something that most people don't do.

HG –What other [kind of] work you did with Pólya?

JK – Well, I mentioned the freshmen seminars; I mentioned taking the number theory course. … I think that there was one thing that he did — and I've written about this, too — there was one thing he did at the end of his seminar in the spring, when the weather was good. We would go outside, and we would sit in the Stanford … Greek theater, actually. It is a kind of Greek theater, [an] open theater. And we would get into groups of four, usually four people. And he would give us problems of different types. Usually each person got a different kind of problem. It might be [that] one problem was number theory, and another problem was geometry, whatever. … The idea was [that] our first … our first assignment was solve the problem. And then we were to take turns teaching the problem to the other three people in the group. And to … essentially to follow the way with it [that] Pólya would introduce a problem, you know. You needed to say what the problem was, but you didn't want to give it away. But you might have to give some careful hints. If people were struggling with the problem, you gave some hints to solve it. And, and … but the group then would work on the problem, and after they finished one problem then they would move to the others. So, it was an exercise, it was an activity in teaching three other people to solve a problem, how to do it. And to see how they came up [with a solution] because sometimes they would get a solution that was different from the one that you did your way.

HG – [What was] Pólya's role … What was he doing… ?

JK – What was he doing? Well, he was mostly just watching. We were sitting in groups around the Greek theater, and at that point he was just observing us. He was just observing us. I think that this came from his own experience in Switzerland when he taught there in Zurich. And they would do some teacher education, and this was partly … his way of training teachers to present problems to other people. How to present a problem in such a way that you don't give away the solution, but if people are struggling, you try to find a way to help them, some suggestion [about their] work. Inside questions.

HG - Inside questions, again the questioning (JK – Yes). Non-obtrusive questions…

JK – Yes, non-obtrusive questions. If people are struggling, you try to ask the question that would take them one step further but that doesn't give everything away. It is an art because if you ask the wrong question or if you give the wrong hint, then there is nothing left to do. So you have to think very carefully how you… how you make a suggestion. You try a way for them to do it first…

HG – If you give too much…

JK – Yeah, you ruin the problem.

[…]

HG – You mentioned that when you found How To Solve It, *you were not [yet] thinking about teaching, but mainly about the learning (JK — The learning problem [solving], yeah, I think so), about the learning of how to do mathematics. Can you add something else about how did Pólya's ideas, or what you came to know from Pólya, match your interest concerning mathematics and doing mathematics.*

JK – Well, I can't really remember the lectures that he gave when he was… when I visited the class… … at Berkeley. So, I think actually it probably wasn't until I was at Stanford and was taking a class from him, a regular class […] that I started to think… And I had already being teaching at that point, so… I really didn't think… I was interested in problem solving, but I have to say that when I was teaching… in Berkeley, I was not thinking about the kinds of problems that Pólya was using in *How To Solve It* or in his teaching. I was thinking more about the kinds of problems that were in our textbooks, the textbooks of my students were using, which were mostly very routine word problems, and I was interested in how they learned to solve those problems, in trying to help them solve those problems, but I really had a very narrow view of what problems were, even though I had seen more interesting problems when I took the examination in school, more interesting problems were in the Stanford mathematics exam. Even though I had seen those, I hadn't think about teaching those to my students, and my master's thesis at Berkeley, for my master's of education, was about what makes problems difficult, what makes routine word problems difficult. It was not about the kinds of interesting problems, non routine problems that Pólya presented.

So I think… I really didn't appreciate Pólya's approach to teaching problem solving until I watched him teaching and then I understood better how important was to choose more complicated problems, problems that were not—. The word problems in the text books that I was teaching from were… simply opportunities to practice… writing an equation and solving an equation, they were very straightforward algebra word problems […]. Students had trouble with that but I didn't realize that those were not… Those problems are giving you practice in a very low level kind of problem solving, because there are all also solved more or less in the same way, and watch you solve one of them you can see how to solve all of them, more or less.

So, the idea that you would take a problem that is completely unfamiliar to you and try to solve that was not something I did in my teaching when I taught. But when I sat in the class that Pólya gave then I saw how a more interesting problem could really change you to think about mathematics in a different way. So… I think I really had a very narrow view of problem solving before I took the class from Pólya. But the way Pólya taught, the way he carefully choose the problems, set them up, help us solve them… pretending he didn't know the solution, trying to get us to help him — "What do we do now, how can we solve this?", "What were we given, what were we trying to find, how do we get this kind of thing?" — asking all of the questions that he

puts it in *How To Solve It*. I Think that it was completely new to me, nothing in my teaching experience, or in my experience as a student before, had really opened my eyes to that, and so that was a big important change in how I thought about teaching and learning mathematics.

HG — You are talking about the academic year?

JK – Yes, it was an academic year institute it went from September to June, I think.

HG — And that was the first time you saw Pólya teach?

JK – It was the first time I saw him teach a class, as I said, I think that I saw maybe two lectures or three at Berkeley, but those were guest lectures, it was just—. May be it was only one I can't even remember what the topic was, but I know he was a guest in somebody's else class.

[…]

HG — Do you want to highlight anything else about [Pólya's] teaching?

JK — Well, I roughly spoken considerably about it, it's the rate at he worked, the careful preparation, the reflection… … It's hard… it's hard to describe in any other way. He used a lot of humor… There is some humor in what is written in his books, but there was even more humor, I think, in his teaching. He would tell stories, and he had a lot of humorous expressions. I don't know if you've seen his film *Let us teach guessing*[13]?

HG — I saw a film [of him] at Budapest…

JK — It is the partition of space (HG — Yes, yes). Okay, then it is the same film, and it has a little bit the sense of his teaching is like. I wrote a tribute to him in a journal of research in mathematics education,[14] and I talked about how that film failed to capture his teaching because… he was basically in a studio. I think he was… It could be in a classroom at UCLA [University of California, Los Angeles]. It was in Los Angeles, I know that, but the people in the class, I think, were college students. But he didn't know them; they were not acquainted with him, and his accent was always a little bit hard for some people to understand. So I think there was a communication problem. You could see it in the film a little bit… It wasn't a realistic snapshot of his teaching, because… they didn't know each other. And I think that one of the things that was so great about him was how he got to know his students and could speak to them. He could tell jokes, and do things like that in a very open and friendly way. It doesn't really come across in that film, because it was a little bit artificial; he has a microphone on him and… the students are not typical.

HG – When you become his assistant, what did this experience add to your view of Pólya?

JK – Well, it gave me… The first time I've observed him I was a student in the class, so the second the time [was] when he taught the academic institute again […]. But then I assisted Pólya, so I could watch many of the same lectures, that he had given before being given again, and see

[13] The film *Let Us Teach Guessing* (1966) was produced by the Mathematical Association of America and presents George Pólya teaching a lesson using a partition of space problem. It was presented in 1968 in the Educational Film Library Association's Annual Film Festival, where it was distinguished with the *Top Honor Blue Ribbon*, in the category of Mathematics and Physics. The film is available from the Mathematical Association of America in video format.

[14] Kilpatrick, 1985, shortly after the death of George Pólya.

how he adjusted his presentation to the students, how he would… If they asked a question how he responded. He was not bothered by a question, and sometimes questions took him in a slightly different way, and that was fine. He knew where he was going; he knew that he had time to respond to questions. So I learned to better appreciate how interactive he was with the students, and if the students were confused, then he would slow down and sometimes go back and explain again something. If the students asked questions that indicated that they didn't understand what he was saying, then he would explain some more. So, he was very sensitive to whether the students were following his discussion, and as I say, if a student brought up a question that took the discussion in a different direction, he would go in that direction, for a while at least, until he thought that was far enough, and then he would bring it back. So, I saw that he didn't teach this class the same lesson that he taught my class. It was a lesson that was constructed differently and followed a different path, but attempted some of the same things.

HG — How long had you been his assistant?

JK – That's a good question, I don't remember, it was either one or two years. And then, let's see… I was in the academic year institute in sixty… … sixty and… sixty-one; [in] sixty-one and sixty-two, I was his assistant, and there was… I don't have that count very well in my mind. He took a half a year off, and that's when the *Mathematical Methods in Science*[15] was done… and I assisted in both. And then, for two… I know I assisted him for two years in the freshman seminars; those must be in sixty-four, sixty-five, something like that, I can't remember exactly. I know that there were two sets of freshmen seminars, and I think that it were two sets of academic years institutes, but maybe it was only a year and a half academic year institute, I'm not sure.

HG – You mentioned the Mathematical Methods in Science. I found, I don't know where, that this book was first written for SM…

JK – Yes, it was published first by the SMSG [School Mathematics Study Group] and then later it was incorporated into the MAA [Mathematical Association of America] series, so it exists in two versions[16].

HG — How do you deal with this… SMSG is very related to the New Math, Pólya…

JK – Pólya… Pólya is related to the New Math, too, but in a different way. He did not… he was very critical of SMSG on the one hand, but on the other hand, Ed Begle, who was also at Stanford, was very fond of Pólya. Pólya was critical of SMSG, but Ed Begle appreciated Pólya, and Pólya did some work on one of the New Math projects, not the SMSG, but [on] one I think it was called the Greater Cleveland Mathematics Project, and he was a consultant of that or something like that. But Pólya didn't like the formality and abstractness of SMSG. However, Begle had a lot of money because of the textbooks, and he had a number of series that he produced of materials for teachers. […] Pólya's book was in the SMSG series, but it was so popular [that] they decided to put it in the MAA series. But the point is, Pólya didn't had a problem with having SMSG produce his book because it was a way to get the book produced at a very inexpensive cost. They had people who would do that… who would type up the material

[15] Pólya, 1963. This book was published by the SMSG, (ed. Leon Bowden), as vol. XI of the Studies in Mathematics series. It was later published by the Mathematical Association of America, vol. 26 of the New Mathematical Library series (Pólya, 1977).
[16] See footnote 15.

from the lesson notes that [Leon] Bowden put together[17]. I don't know how different the two are, I have never looked at, I have both versions but… I they are basically the same. … … But it is a little bit curious.

HG — When I found this, I became a little bit troubled…

JK – Oh, no, there are… The SMSG series had Russian authors, it had… various mathematicians writing books for teachers, there were problem books, there were all kinds of books that were produced, and that was all done as a kind of service to teachers.

HG – Using one phrase [can you say] what had you best learned from Pólya?

JK — Oh… The thing is… For me he was such a gentleman and such a human… such a wonderful human person. I think I learned how you are to treat other people because he… he was so kind to me, and when I visited him with my wife and my children, he was so wonderful to us, he was so openhearted. And, despite all of the various things that did happened to him in his life, he was always so positive. So positive, so full of humor, such a gentleman, so kind, so nice. And he taught me how I should treat my students.

HG – Let me quote you and listen to your reply. 'Great teachers do not simply teach us to do, they teach us to be'[18].

JK — That's exactly right. And I meant that when I said it. I learned more about being a person from him; he was a tremendous… I never had a teacher who was as influential on my way of thinking about teaching and being a human being than Pólya.

HG – And I also find that you, talking about teaching [said], 'We have the better of it – we knew him'[19]. It's kind of very strong…

JK – It's very difficult to explain to anybody else, anybody who didn't know him, what a wonderful person he really was, really a fine person, so kind, always patient. I always remember him telling me… I used to go to his house; he lived only a few blocks away from me, not very far, just a few blocks away. And so I would visit him very often, and he was always open to my coming. And I remember once that I was complaining about something was not going well, maybe it was on my dissertation, I don't know, but he… he essentially gave me a lecture and told me not to feel sorry for myself, that I should not complain, I should be more positive. I always remember that.

HG – Well, talking about mathematics. How would you characterize Pólya's view of mathematics?

JK — Hmm…

HG — What most distinguished?…

JK — Most distinguishing was his great effort… to… … … try to understand how it came about. What he says in the interview with me[20] about how Ernest Mach[21] had this idea that you could…

[17] See footnote 15.
[18] See footnote 14.
[19] Kilpatrick, 1987, p. 300.
[20] See footnote 12.

[that] you learn mechanics or physics best by following the path that the people took to created it. And Pólya always put a lot of that into his… into the math—. He always went back to first principles and tried to find out "How did people think about this?" So, that helped him to make his own progress in mathematics because he would say "Well, how did we come up with this?", and tried to develop principles. That's why *How To Solve It* and *Mathematical Methods* [*in Science*], all these books are trying essentially to put into words his philosophy of doing mathematics, not just of teaching, but of doing mathematics, to try to find out what are the basic principles and how do we get from here to some place else. He tells a story about 'up-monkeys' and 'down-monkeys' and says that he was a 'down-monkey', which means that he would start with a particular case and try to say "How can I generalize this, how can I understand a particular case and then go beyond the particular to the general?" And that is the way he would approach problems. If he ran across a problem, he would start simplifying it and taking a simpler problem and asking, "If I solve this particular case, what can that tell me about the general case?" And that led him to a lot of discoveries in various parts of mathematics. He was very eclectic, he did a lot… mathematics in a lot of different fields, but he was always, I think, driven by "Let me start with the problem, let me take a simple case, let me understand the simple case, and them I can maybe move up to something more general." And that was his approach to mathematics, and to some degree his approach to teaching, too.

HG – Okay. And concerning mathematics teaching, how would you describe his major contribution?

JK – I think it is giving us the [idea of]… reviving the study of heuristics, so that, by studying heuristics, teachers can be introducing heuristics explicitly into their teaching of problem solving. Without Pólya… Of course, the four phases are important; of course, the ideas in *How To Solve It* are important. But the basic idea is the one of heuristics… In order to solve a problem you need to think 'How can I do this?", "What's a rule that I could try to use here?", "What's an approach?…". It may not work. Maybe working backwards would help in this case; maybe a simpler problem would help in this case. He gave us a lot of specific, maybe general, ideas about how to approach problems, ideas that can be brought into the classroom and that we can use to help children solve problems. So, I think that's what he really did: transform the way in which we approach problems. In a way, it's the same thing that happened to me when I went from being a teacher to be a student in Pólya's class. I saw problem solving in a different way, and I think that's what Pólya did for teachers.

References

Kilpatrick, J. (1985). Editorial. *JRME, vol. 16* (5), 323.

Kilpatrick, J. (1987). George Polya's influence on mathematics education. *Mathematics Magazine, 60*, 299-300

Pólya, G. (1945). *How to Solve It: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press.

[21] Ernest Mach (1838-1916), was an Austrian physicist and philosopher, author of *The science of mechanics*, cited by Pólya in the interview Kilpatrick refers.

Pólya, G. (1962 & 1965). *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*, vol. I & vol. 2. New York: J. Willey. & Sons. (Combined edition, New York: J. Willey. & Sons, 1981).

Pólya, G. (1963). *Mathematical Methods in Science*, ed. Leon Bowden (vol. XI of the School Mathematics Study Group series Studies in Mathematics).

Pólya, G. (1977). *Mathematical Methods in Science* (vol. 26 of the New Mathematical Library series). Washington, DC: MAA.

Pólya, G. (1977). *A arte de resolver problemas*. Rio de Janeiro: Interciência.

Pólya, G. (2003). *Como resolver problemas*. Lisboa: Gradiva

Pólya, G., & Kilpatrick, J. (1973). The Stanford University Competitive Examination in Mathematics. *American Mathematical Monthly, 80*, 627-640.

Pólya, G. & Kilpatrick, J. (1974). *The Stanford Mathematics Problem Book, with hints and solutions* . New York: Teacher's College Press.

Pólya, G., & Kilpatrick, J. (2009). *The Stanford Mathematics Problem Book*: W*ith hints and solutions*. New York: Dover.