THE PRACTICE OF ARGUMENTATION AS A METHOD OF TEACHING AND LEARNING MATHEMATICS

José Messildo Nunes
messildo@ufpa.br

Universidade Federal do Pará

Saddo Ag Almouloud
saddoag@gmail.com

Pontifícia Universidade Católica de São Paulo

ABSTRACT

In this article our goal is to reflect on the practice of argumentation as a method of teaching mathematics, thus acknowledging argumentation as a process that favors the appropriation of mathematical knowledge, evidencing the stages required for this fact to be established. The method consisted of a literature review in studies on mathematical argumentation, based on a qualitative research approach. To achieve our goal, we used some examples of plane Euclidean geometry as a reference, based on scientific studies on this topic. In this essay we consider the argument as a sociocultural construction, and highlight the importance of the scenario that enables its development, listing the stages that comprise the argumentative process motivated by Toulmin's ideas. We believe that modeling the argumentative process according to our proposal can promote the acquisition of argumentation competences in mathematics and therefore assist in the understanding of this discipline.

Keywords: Argumentation; Mathematics; Toulmin Model; Process Validation; Interaction.

RESUMO

Neste artigo temos como objetivo refletir sobre a prática da argumentação como um método de ensino da matemática, assim reconhecendo a argumentação como um processo que favorece a apropriação de conhecimentos matemáticos, evidenciando as fases necessárias para que tal fato se estabeleça. A metodologia consistiu em uma pesquisa bibliográfica de estudos que enfocaram a argumentação no ensino da matemática, com baseada em uma abordagem qualitativa de pesquisa. Para alcançar nosso objetivo realizamos uma reflexão utilizando alguns exemplos da Geometria Euclidiana plana como referência, nos baseando em estudiosos sobre o tema. Neste ensaio assumimos os argumentos como uma construção sociocultural, evidenciando a importância do cenário que possibilita seu desenvolvimento, elencando as fases que compõem o processo argumentativo, inspirados nas reflexões de Toulmin. Acreditamos que modelar o processo argumentativo de acordo com nossa proposta possa favorecer a aquisição de competências argumentativas em matemática e, assim, auxiliar na compreensão desta disciplina.

Palavras-chave: Argumentação; Matemática; Modelo de Toulmin; Processo de Validação; Interação.

1. Introduction

Douek and Pichat (2003) argue that the development of argumentative competences has become a matter of utmost concern for mathematics educators for different reasons, such as the need for an approach, from an early age, to competences that are relevant in the process of justification; exploiting the potential of social interaction in the development of mathematical knowledge; and the importance of argumentation competences focusing on increasing students' intellectual autonomy.

The Brazilian National Curricular Parameters - NCP (Brazil, 1997) emphasize that teachers should use methodologies that "prioritize the development of strategies, evidences, justification, argumentation, and critical thinking and encourage creativity, collective work, and personal initiative and autonomy that result from confidence in their ability to learn and face challenges "(Brazil, 1997, p. 31).

As a facilitator or mediator of learning, the teacher should encourage cooperation and discussions about hypotheses between him/herself and the students and between students, thus creating favorable conditions for the involvement of students in learning experiences that focus on the explanation and justification of reasoning in situations that include formulation, evaluation, and validation of conjectures.

The NCP also highlight that argumentation is a competence that should be developed in the elementary grades and that the argumentative competence is not an inherent human trait; students need to be placed in contexts that enable them to practice their argumentative skills (Brazil, 1997).

According to the principles and standards established by The National Council of Teachers of Mathematics (NCTM, 2000) regarding Mathematics Education, schools should create favorable opportunities for students to formulate and investigate mathematical conjectures and develop and evaluate mathematical arguments and proofs. Still according to this organization, mathematical reasoning is a habit of mind, and therefore it should be developed through the coherent use of multiple contexts.

Some studies on mathematical argumentation consider it as a facilitator of learning mathematical proofs and demonstrations (Boero et al., 1996; Douek, 1998, 1999, 2000, Mariotti 1997, 2002; Pedemonte 2002).

On the other hand, Boavida (2005) justifies the relevance of engaging students, especially in elementary school, in practices of argumentation since the argumentative competence includes the ability to talk, think, choose, and commit: "the ability to talk leads to an opening towards others that becomes effective by the desire to communicate and willingness to listen; the ability to think leads to a critical attitude and attention; the ability to choose and commit is related to individuals who seek to stand up for what they think making an effort to intervene. The role played by argumentation in a given context reflects how important freedom of thought and action are" (p. 67).

By creating conditions that enable students to develop their argumentative competence, we are contributing not only to objective thinking, but also and especially to individuals' character formation, i.e., for the development of their social relationships, their critical thinking, autonomy, etc.

Boavida (2005) adds that the involvement of students in argumentative activities is rarely seen in mathematics classrooms. According to the author, "there is still much to investigate, either about its potential or ways to make it a reality as a classroom practice showing that these practices pose significant challenges to the teacher" (p. 11).

Studies of Lampert (1990), Douek and Scali (2000), Boavida (2005) and Boero et al. (2008) showed that promoting and encouraging argumentation in mathematics classrooms are major challenges for teachers. The common difficulties are engaging students in situations that require argumentations, mediating conflicting situations, combining the argumentation activities with concepts' learning, and having sufficient knowledge to admit unexpected solutions ensuring they are consistent with the mathematical requirements, etc..

In order to play the role of a mediator, it is of fundamental importance that in addition to having comprehensive knowledge of the subject matter being taught, the teacher has to challenge students to face difficult situations that stimulate them to think independently by making decisions, relating theoretical and empirical activities, and respecting others' opinions, counter-arguing when necessary.

Another important fact in the development of argumentation is the context that enables its development. Accordingly, Goodwin (2009, p. 140) states that "the conversation[1] in which the arguments are developed is the primary means by which people organize a context for their interaction"; this is how the rules of argumentation for each context are set, according to the author in *"argumentative institutions*[2]*"*.

Goodwin (2009) argues that "the rules of argumentation include some requirements (standards, ideals, etc.) that the individual argumentation has to fulfill (to measure up to, to perform, etc.) in order to hold a strong conversation" (p. 140). The strength is provided by the normative criteria of each area, mathematics in this case, which should be set by the teacher to get responses that may be validated or refuted according to the argumentative process. "These quality requirements are included in the pragmatic context-dependent standards of the argumentation context" (Goodwin, 2009, p. 141).

One of the tasks of the teacher is to organize this conversation to challenge students to get involved in argumentative situations, *i.e.*, to create conditions that may favor and support the communication of ideas. There will certainly be difficulties engaging in conversations like these, but the teacher has to help students overcome difficulties that may arise by foreseeing that the students will initially use *abductive*[3] argumentation, but they will later use an argumentative organization according to the validation rules established by the argumentative institutions standardized by mathematics.

[1] Argumentative conversation that involve strategy acts, speeches, etc. (Goodwin, 2009).

[2] Social institutions (broadly speaking), where the argumentations can be found including academic types of discourse (such as an article for a conference), formally organized situations (criminal trial), sets of rules for conduct at meeting (Robert's Rules of Order), and undoubtedly many other somewhat lengthy patterns of expectation (including rules) about how argumentative conversation should occur (Goodwin, 2009).

[3]In mathematics education, it means to make use of rules that are not under complete control of the students, but which seem more appropriate to justify a conclusion drawn from facts or information (Pedemonte, 2002).

Boero et al. (2008) emphasize that the argumentative process requires language competence higher than that which is commonly demonstrated by students. Hence, argumentation seems an appropriate activity to promote improvement in language skills, specially the specific mathematical language, either in oral or written records thus, according to the authors, helping the development of mathematical reasoning.

The teacher is responsible for taking into account this argumentative process uniqueness - the combination of natural and mathematical languages to enable students to grasp the concepts being taught focusing on problem solutions.

We see argumentation as a practice that promotes the understanding of concepts, consistent with the ideas of Grácio (2009), by considering argumentation as a detailed interaction that admits several debaters discussing over a given topic in order to result in an argumentative competence that can help the teaching and learning process. "We seek to find a descriptive basis taking into account empirical data about the way social actors present an argument and how this influence their own argumentative practices, leading to the interpretation of the meaning of argumentative competences and a way to promote them from a didactic point of view" (Grácio, 2009, p. 103).

Therefore, according to this author, argumentative competences are acquired through interactions coordinated with a given topic. Although each individual presents his/her views on the subject under discussion, the arguments are based on interactions. In mathematics education, for example, when teaching area of plane figures, the communication of ideas will revolve around this issue, the argumentations will be mutually influenced by the ideas of the students and the teacher, who, as a mediator, should manage the process, requesting justifications for assertions and confronting divergent ideas.

This idea leads argumentation towards a particular subject considering it as part of a process that involves actions, justifications, decision making, conflicts due to divergent views, refutation, and validation seeing it as a methodological alternative. Therefore, the author states" the field of argumentation in terms of the relational mutuality of the subject under discussion, we see it in such a way to allow us to analyze the interventions developed in changing the shift of words, which characterizes the argumentation dynamics as *detailed discourse*" (Grácio, 2009, p. 103).

The circumstances surrounding the argumentative process enable us to consider it as an interactive resource for use in the classroom focusing on the content being taught. It is essential that such interaction be mediated by the teacher for a thorough understanding of the content under study by centralizing isolated, scattered, and personal discussions with those that agree with the concepts, definitions, and properties of mathematics.

The argumentative competence should also enable controversial situations under the teacher's guidance, who, in addition to mediate such situations, can make them emerge from activities. According to Grácio (2009), the deadlock resulting from conflicting opinions is what seems to make it clear that it is an argument.

As the teacher facilitates, mediates or brings about situations where there are conflicting opinions or counterexamples, the argumentative interactions change progressively from competitive to cooperative attitudes.

In the field of mathematics education, Krummheuer (1995) addresses *collective argumentation,* which results from the argumentative interaction that usually involves moments of agreement and understanding, but which also involves controversy and disagreement that should, with the teacher's intervention, be led to adjustments or changes in the ideas that underpin the justification for the assertions. According to this author, at the end of the process a consensus should be reached.

The author considers the argumentative practice as a social phenomenon that occurs when the subjects involved in the activities proposed by the teacher cooperate trying to adjust their interpretations in order to understand the concepts being discussed or to extend the discussions in an attempt to incorporate new concepts.

Despite the polyphony of the term argumentation, the authors that usually adopt it as a mathematics classroom practice, such as Krummheuer (1995), Pedemonte (2002), Douek (2000), Cabassut (2005), Boavida (2005), among others, assume that in the argumentative process the articulation between discursive verbal and non-verbal elements should be considered, for example, speech, text production, figures, and numeric or algebraic data, to which we add gestures. These elements that compose the argumentative process demonstrate the complexity of this term.

Therefore, we acknowledge that argumentations in mathematics are practices related to a given subject that, on the one hand, are constituted mainly of actions that enable evidence collection and representations, either numeric, algebraic, or figural, which contribute to persuade and make the ideas tested plausible and carry firm conviction, and, on the other hand, they are constituted of assertions, either written or spoken justifications, consonant or controversial dialogues, and gestures. This practice should provide students with argumentative competence.

In our view of practice of argumentation, it can be presented as a method to help understanding mathematical concepts, and the use of this method should facilitate the acquisition of argumentative competence, which enables students to appropriate strategies to solve problems and them develop the language for expressing mathematical ideas, report, hear, and discuss the purpose of their understanding of the concepts studied, in addition to favoring respect for the opinion of others and promoting conceptual understanding.

This paper is a contribution to a recent discussion about mathematics argumentation to understand the concept not only from a didactic point of view, but also understand its complexity as a method of teaching and learning, highlighting the importance of the scenario that enables its development. From the standpoint of validity, we assume the argumentation organization as a structural model, which is divided into valid reasoning arguments. Thus, in this article we aim to reflect on the practice of argumentation as a method of teaching mathematics. This is a bibliographical research it conducting to apprehension of objectives, observance of steps, reading, that procedure offers the researcher the possibility of seeking solutions to a research problem.

2. Argumentation in the Mathematics Classroom: evidences for its implementation

According to Douek and Scali (2000), Boavida (2005) and Boero et al. (2008), argumentation in the classroom has been addressed in the context of mathematics education since the 1980s. These authors believe that this fact is due to the attempt to tackle the problem of specificity of mathematical demonstration and its relation to argumentation. Studies on this topic address epistemological, cognitive, and/or educational perspectives.

Boero (1999) states that there are two issues in the process of learning mathematics that must be taken into consideration. On the one hand, there is the *nature of the arguments* considered by the students as credible arguments, either empirical or theoretical, for validation. On the other hand, *the nature of reasoning* produced by the students, analogies, examples, etc. We believe that some activities encourage students to present various types of arguments and, in order to characterize them, we can take into account the dual nature of arguments, and, from this characterization, they can be analyzed. Hence, argumentation can be seen as a process that promotes understanding of mathematical concepts such as the measurements that are usually performed in geometry activities that enhance the production of pragmatic arguments[4].

Therefore, the studies on argumentation in the classroom should be thorough and promote the development of strategies that can encourage students to develop their argumentative competence, which has its relevance in social life, but can also help understand the classroom concepts, thus requiring appropriate activities for this purpose.

Researchers (Nunes & Almouloud, 2013a; Nunes & Almouloud, 2013b; Douek, 1999, 2000; Douek & Pichat, 2003; Boavida, 2005) investigated possibilities of locating mathematical argumentation in the context of classroom. We postulate that the practice of argumentation may constitute itself as a methodological alternative to the process of teaching and learning mathematics. Hence, we must recognize that the argumentative process needs to include three stages that may favor the fulfillment of that purpose; each one of these stages will be discussed in the following sections.

3. Stages of the argumentative process

Toulmin (2006) compares an argument to a living thing that has a gross *anatomical* structure and a finer *physiological* structure, "When we explicitly present in details an argument [...] the major stages of an argument evolution from the initial statement of the unsolved problem until to the final presentation of a conclusion can be distinguished. Each one of these parts [...] represents the main anatomical units of the argument - its "organs", so to speak. The finer structure within each paragraph can also be recognized, at the level of individual sentences, which has been the major focus of logicians. At this *physiological* level, the idea is proposed in a logical way since this is where the validity of our arguments must be established or refuted "(p. 135).

[4]Arguments that use successful actions such as the perception of equivalence between areas of plane figures using equidecomponibility as rules of validation.

The author characterizes the elements of the anatomical part that compose the argumentative process but leads his ideas to the physiological part, which enables analyzing the arguments sentence by sentence to see how its validity or non-validity is connected to the way we conceive and establish the relevance of this connection to the traditional notion of logical form.

3.1. Physiological part: structure of arguments

In order to present his proposal, the author makes a distinction between the claim/ conclusion (C), and the facts as the basis on which the argument is based - called the *data* (D); in our case we consider that they are collected from experimental or theoretical evidences, facts, and information given by the teacher or obtained from teaching materials, etc., *i.e.*, the interaction of the student with the proposed activities.

In the mathematics classes, the *data* can be identified through the following question: *How will the student present a solution to a given problem?*

The question can also be: *How has the student arrived at the solution?* The student may present a specific number of data as a basis for certain specific conclusion that will take him/her from the data to the claim, and the question now is about the nature and justification of such transition; "Our task is not any longer to strengthen the foundation on which we build our argument, but instead, it is now to show that, by taking those data as a starting point, it is appropriate and legitimate to shift from the data to the claim or to the conclusion presented (Toulmin, 2006, p. 141).

Indeed, we need general hypothetical statements that can serve as bridges and allow the steps to which we are committed in each one of our specific arguments.

According to the author, this type of proposition is called warrants or justifications (W) that allow shifting from the data to the claim, to distinguish them from the conclusions and from the data. These warrants correspond in mathematics classrooms to the justifications given by the students of their conclusions when they solve a particular problem by handling teaching materials or through the computer screen or observing facts. These techniques are used by the students to allow inferences and are based on abductive reasoning, which, according to Crespo Crespo (2007), correspond to data interpretation by collecting important information that can explain or justify the conclusion.

The warrant is the part of the argument that establishes the logical connection between the data and the claim, and at first it is the reason for acceptance or rebuttal of the argument. This allows us to also identify the argumentation function since it can be an explanation, communication, and /or discovery.

From this new element, Toulmin (2006) proposes an initial standard scheme to analyze arguments (Figure 1).

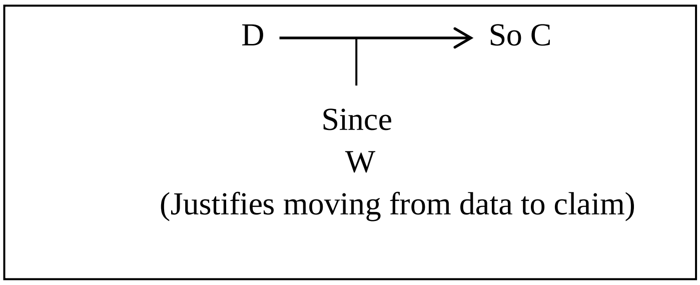

Figure 1. Model indicating warrant

According to this model, the explicit appeal to an argument emerges from the attempt to provide data that could support the claims, the warrant is, in a sense, incidental and explanatory, and its task is simply to register explicitly the legitimacy of the step involved. We sought to clarify this element, which can reveal its strength and allow the convergence of the argumentation into a specific solution based on definitions, properties, etc.

In certain situations, specifying the data, warrant and claim is not enough, and it may be necessary to add an explicit reference to the degree of strength of our data in relation to our claim because of our justification. Thus, "it may happen when a qualifier is inserted. It is also the case in the courts of justice, where, often, relying on a given statute or common law rights is not enough, but it is necessary to discuss explicitly the limit to which a specific law applies in a particular case; whether the law should be applied in that case, or if that case can be considered an exception to the rule, or if it is a case in which the law can only be applied if it is limited to certain qualifications" (Toulmin, 2006 , p. 145).

According to the author, some warrants allow us to qualify our conclusion using the adverb *necessarily,* but others can allow moving from data to claim temporarily, or allow it under certain conditions. Thus, we can introduce other more appropriate modal qualifiers, such as *probably, presumably, etc.* (Toulmin, 2006).

The qualifiers (Q) indicate the degree of force (warrant) conferred by the warrant on this step - true, probably true, probable. Q designates the truth qualifier, necessary or plausible, while the conditions of rebuttal (R) indicate the circumstances in which the general authority of the warrant would have to be set aside (Figure 2).

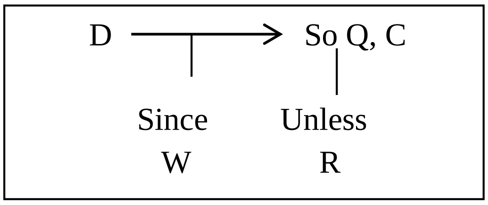

Figure 2. Model indicating the qualifier and rebuttal

According to Toulmin (2006), sometimes our warrant can be challenged; thus we need additional facts in order to legitimize and help warrant validation or rebuttal. Therefore, there is a new element in the model, the backing (B) for warrants, which does not need to be explained, "at least in the beginning, the warrants can be accepted without challenge, their backing may be implicit" (Toulmin, 2006, p. 152).

In fact, backing is what supports and allows our warrants, or as sometimes mentioned by the author, they are the knowledge base, without which, not even the warrants would have the authority or force (Figure 3).

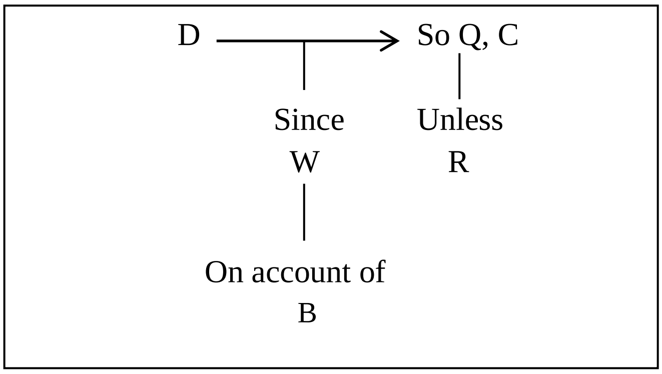

Figure 3. Complete Toulmin's Model

Warrant affirmations are hypotheses, but the backing for the warrants and the data used for direct support of our conclusions can be expressed as categorical statements of fact.

Backing in a mathematics classroom, most of the time, is not explained, but it is through this element of the model that the validation process of the argument is regulated. During the argumentative practice in the classroom, the teacher or sometimes the students seek backing to find counterexamples to refute argumentations that disagree with the properties, definitions, and mathematical axioms, as well as to validate arguments that comply with mathematical rules.

This model gives us an outline of argumentation and, at the same time, it allows us to view the links of arguments, "it is very useful to determine the type of reasoning (deductive, inductive, abductive, etc.) underlying argumentation" (Pedemonte, 2002 p. 40th).

In short, Toulmin (2006) argues that the argumentative process consists of two parts, the anatomical and the physiological. The former encompasses all stages of the process, from raising an issue to the proposal of a solution. The latter is formed by elements of the first (anatomical) part that favors the analysis for the acceptance or rebuttal of arguments; it includes all stages from the data to the conclusion and is presented in a model as shown in Figures 1, 2, and 3.

3.2. Anatomical Part

Toulmin (2006) presents the anatomical part divided into three stages: problem presentation, problem discussion, and the verdict. He believes that these stages are too complex and may require many pages if one wishes to transcribe and analyze them. Therefore, the author concentrates his efforts on the physiological part analyses, in which he extends the Aristotle idea of premises and conclusion by proposing a framework that includes, among other elements, justifications for shifting from the data to the conclusions and which can require backing, which give strength to the argumentation, and qualifiers that indicate the strength of justifications.

The most popular studies on mathematics education that used the Toulmin's model also focused on the physiological part, for example, the studies of Krummheuer (1995), Alcolea Banegas (1998), Yackel (2001), and Küchemann Hoyles (2002), Pedemonte (2002), Knipping (2003), and Houssart Evens (2004), Weber and Alcock (2005).

We agree about the importance of physiological analysis to qualify the arguments showing their validation or rejection. However, we believe that in a mathematics classroom it is very important to also focus on the anatomical part because it includes both the strategies and activities assigned by the teacher to engage students in the practice of argumentation. The study of the anatomical part can provide the required paths to implement this practice as a method of teaching.

Toulmin's proposal (2006) is not focused on discussing each stage of the anatomical part of the argumentative process, but rather the main points of argumentation, which, according to him, comprise the physiological part of this process. Our proposal includes the characterization of each stage in terms of teaching mathematics, from our studies on the concept of area of plane figures (Nunes, 2011) and relying on the studies conducted by Douek and Scali (2000), Pedemonte (2002), Cabassut (2005) and Perelman and Olbrechts-Tyteca (2005). Our inferences are made using the reference experience as a way to trigger a problem, identify the types of arguments and their respective validation functions during the communication of ideas, and analyze the strength of the arguments based on its convergence to a particular solution. Therefore, we believe that these stages can serve as guidelines so that the practice of argumentation can be developed in mathematics teaching, promoting the understanding of the concepts being taught.

The argumentation stages and our inferences, according to Toulmin (2006), are:

Presentation of the problem - the central issue concerning the grasping of a particular concept, such as the area and perimeter of plane figures, consists of peripheral problems that comprise particularities required for the conceptual understanding of the topic being studied, for example, understanding area as a region and area measurement as the measurement of this region in a certain unit.

Therefore, we believe that it is necessary to think of a problem that may mainstream a certain teaching sequence and thus engage students in a process of communicating ideas. In our view, this initial moment can occur in terms of experience of reference, as reported by Douek and Scali (2000), considering the necessary adaptations suggested here, by placing this experience at the beginning and, when necessary, to address new particularities of the concept under discussion.

Opinion about the problem - The reaction of the students to the activities proposed should enable them to collect evidence to support a specific solution. This stage can, in general, unfold in a series of stages that will be further discussed. According to Toulmin (2006), it will be always possible to dispute a specific assertion regardless of its nature; therefore, the foundation on which such assertion is based deserves full attention. Thus, we can analyze, classify, and evaluate the justification of the arguments based on their backings and structure within a field of study, such as mathematics.

Verdict - this stage is related to validation, *i.e.*, to one of the steps to which the argumentation process is subject. The verdict may refer to another judicial procedure, and the local validation may refer to a problem that requires global validation, for example, we can validate that the triangle can be used as the measurement unit to measure the area of plane figures; we can next show that several figures that have the characteristics required to the covering of a particular surface can be used the same way.

To understand each stage, we will focus on studies that addressed argumentation.

4. Reference Experience as an Element of the Argumentative Process Stages

Douek and Scali (2000) argue that argumentation affects the progressive development of basic mathematical concepts. Moreover, these authors believe that the attentive development of argumentation favors systematic connections between the concepts.

The major focus of Douek and Scali (2000) was to analyze the possible functions of argumentation in conceptualization. To achieve the conceptual understanding, the authors argue that it is necessary to have an activity that can be considered a *reference experience*. Such an activity is subject to an argumentative situation, in which the students need to explain, justify, or contrast an argument about that concept, either at the basic or advanced level. According to the authors, to construct a given concept from the reference experience, this experience must be connected to the concept's symbolic representations in a functional way. Therefore, an argumentation can be the way by which this connection is established.

The authors add that argumentation enables to reveal the implicit *operational invariants*[5] and can ensure its wise use. This function of argumentation relies heavily on the mediation of the teacher, and it is fulfilled when the students are asked to describe the procedure that led to the solution of a given task and the appropriate conditions for its use.

Douek and Scali (2000) highlighted that in order to fulfill some important functions of argumentation in the conceptualization, the teacher should: develop appropriate tasks focusing on the key points of conceptualization; use appropriate arguments in the first interactions with students in order to focus on the problem and transform a problem-situation into a reference experience; choose appropriate students' productions to be compared and discussed in the classroom; and manage the discussions in order to reveal significant aspects of the concepts, making them explicit.

Boero et al. (2008) and Douek and Scali (2000) support the construction of a *reference experience* characterized by raising arguments to explain, justify, or contrast a given concept.

The results found by Boero et al. (2008) led them to infer that deep knowledge of natural language in mathematical activities is required. They suggest the need to consider, on the one hand, the specificity of verbal mathematical language including representations and

[5] Propositions, propositional function and inferences that can be explained by arguments or by a chain of arguments (Vergnaud, 1996).

mathematical symbols and expressions, and, on the other hand, its role as a mediator between the flexibility of an ordinary language and the specific needs of mathematical activities.

Accordingly, Lampert (1990) argues that the dialogues between students and teachers resulting from the argumentative process should be functional not only in terms of communication but also in terms of reasoning. The teacher, as the representative of the mathematics culture outside the classroom, is responsible for providing students with conventional mathematical tools for its understanding including language and symbols, and he/she should discuss the meanings of these elements using a natural language that is familiar to the students.

The intentional or conscious use of representations of a given concept can be evidenced in the students' argumentations. Therefore, the arguments enable students to share knowledge that they normally keep to themselves when they are developing strategies to solve problems. This fact may ensure the intentional use of these representations. As previously mentioned, it depends on the mediation of the teacher, who should ask students to describe and discuss the procedures used in the activity. Knowledge and strategy sharing by the students when dealing with a particular problem can be related to appropriate symbolic representations, thus revealing important aspects of knowledge acquisition (Boero et al. 2008).

The reference experience may enable the combination of natural and mathematical languages, because the students need to rely on the former to communicate their ideas about the activity. At the same time, their previous knowledge and the mediation of the teacher will enable them to make use of and include representations, expressions, and symbols of a specific subject matter in their arguments.

We suggest that the *reference experience* should be explored in the *presentation of the problem*, *i.e.*, in one of the argumentative process stages; thus, we believe that this experience is presented to motivate and engage students in the practice of argumentation, that is, it should be considered as a strategy for the establishment of a *didactic contract*[6] that facilitates the communication of ideas in the classroom. On the other hand, this experience can be resumed whenever it leads to activities that detect new aspects of the concept involved.

In the first stage of the argumentative process, the communication of ideas is dispersed, i.e., at first they might not be focused on the subject being taught, but some evidences collected in the activity lead to argumentations about the mathematics subject being discussed. The *Reference experience* must then encourage the introduction of concepts related to the topic during the discussions.

This experience, according to a study of Douek and Scali (2000), should enable argumentations that may trigger ideas about the mathematics subject being discussed, besides engaging students in situations that encourage cooperation and communication of ideas necessary to trigger argumentation in mathematics. The use of some symbolic representations

[6] It is a relationship that determines explicitly, to a lesser extent, but implicitly, to a larger extent, the control responsibility of each partner, teacher and student, through which they will be, one way or another, responsible by each other. (Brousseau, 1986).

combined with the conscious[7] use of argumentations enable students to relate experience to the knowledge involved in it, thus favoring semantic interpretations of this knowledge.

The *reference experience* should involve students in a problem in which they can seek solutions interactively with their peers and teachers. It should also enable students to relate symbolic representations used in certain contexts to new situations, thereby making them once more responsible for conceptual acquisition. This experience, in our view, should take place when the activity is proposed and it should take place again other times transversely in order to facilitate the required relationship between the activities.

5. Argumentation Classification and the Validation Process: Selecting criteria for the analysis of the second and third stages of the argumentative process

One of the key points of our studies on argumentation, proofs, and demonstration is the classifications that are generally proposed by researchers as criteria for the analysis of students' assertions. No less important is the validation of these assertions. These points, argumentation, proofs, and demonstration will be further discussed since they were addressed and related in most of the studies on which our research proposal was based.

Balacheff (1987, 1988) was one of the first authors to highlight social interaction in terms of classroom proofs by considering argumentation as constitutive of validation processes. His types of evidence have been cited in many studies on this topic, even to assist the classification of mathematical arguments.

According to the author, certain situations require the application of argumentation solidly theoretically grounded. On the other hand, some other situations do not require validation, such as those when learning the rules of a game, operation of manipulative materials such as tangram pieces; or when choosing a unit of measure for covering a given region, etc. Therefore, according to the author, they are decision-making situation cases, since they do not require validation processes.

In situations that involve increasing familiarity with polygons, students may be asked to anticipate or predict, *i.e.*, guess whether a particular figure is a square or rectangle based on the shape of the figure only. Thus, they will face a decision-making situation, but by making decisions they would also be making validations, without, however, the need of an explicit proof.

Accordingly, Balacheff (1987) adds that even when students are faced with a decision-making situation, they may, for intrinsic or extrinsic reasons, seek justifications for their strategies. Thus, a decision-making situation can become a validation situation. In this case, the students will look for a justification that can be considered proof for their assertions. The environments that enable this process, ruled by a didactic contract, allow action rules that are regulatory sources of argumentative processes.

Turning a decision-making situation into a situation of validation can take place in cases in which students are asked, for example, to investigate whether two figures have the same area.

[7] According to Douek and Scali (2000), the comparison between alternative procedures to solve a given problem can be an important way to develop this conscious use.

Initially, the student is faced with a decision to make. At first, the student faces a decision-making situation but when he/she needs validation to convince his/her peers of his/her assertion, the changing of situations takes place, even if there is need of proof.

Hence, we believe that argumentation may be a common thread that implements a small mathematical society in the classroom. Therefore, the students can be inserted into an established culture to develop mathematical competence.

Moreover, the students' behavior towards the activities proposed can lead to decision-making, justifications of assertions, and debates in an attempt to defend or refute propositions, thus resulting in respective argumentation and validation processes. Therefore, the practice of argumentation can promote the understanding of the concepts involved.

5.1. Classification according to the nature of the arguments

As for the nature of the arguments, Cabassut (2005) listed three types of argumentation in mathematics: pragmatic, semantic, and formal.

Pragmatic argumentation uses successful actions as a validation rule, for example, the surface displacement reconfiguration action, moving an image on the computer screen, etc. The action can be effectively done or thought about. The author understands that choosing actions does not mean to follow formal rules.

On the other hand, semantic argumentations are those that use validation rules that are not formulated in a formal way, but which are rather based on the contents of mathematical objects being discussed. This type of argument uses rules that are not fully explained or formalized, for example, calculating the area of a rectangle using the product of two sides with the justification that this way is faster than counting one by one of its area measurement units.

Formal or syntactic argumentation is that with a clear explicit structure which is stated with well-defined data functions, validation rules, and conclusion. The application of validation rules is based on rules' terms and data, and it does not require interpretations based on the content or meaning of the terms or data. This is similar to the case of the logical sequence that enables the deduction of formulas for determining the area of plane figures using other formulas or mathematical relations (formula to determine the area of the triangle using the parallelogram formula).

The classification of arguments together with the functions of validation is used to analyze the quality of communication of ideas in the mathematics classroom. The analysis of the argumentative process goes through these classifications that contribute to the analysis of the effect of the argument on the practice of argumentation and indicate the taken for the acquisition of argumentative competence.

5.1.1. Validation functions

In response to the question: Why validate? Cabassut (2005) relies on two views about the functions of demonstration. On the one hand, considering the mathematical institutions, inspired by De Villiers (1990), and on the other hand, focusing on the didactic actions, based on the perspective of Hanna (2000). Therefore, the author proposes the following functions of validation of arguments in mathematics:

5.1.2. Verification functions

The main characteristic of this function is to validate the plausibility or need for truth of a proposition. At first, any validation uses this function, otherwise incorrect validations or other kinds of reasoning that do not require validation would be accepted. Accordingly, verifications can control the truth even when increasing the plausibility of an assertion, for example, by verifying the area conservation through the dissection method. This is a case of confirmation of the truth, but not an explanation of truth, as discussed below.

5.1.3. Explanation functions

Cabassut (2005) believes that the explanation of a validation is not always clear. The author argues, for example, that a method that leads to a certain formula "can explain" the relationship between actions and the use of this formula, for example, when choosing the use of the multiplicative method (using the formula) instead of the additive method (counting squares) to calculate the area of a rectangle. Thus, the explanatory issue becomes heuristic and exhibits a subjective dimension since, according to the author, students prefer intuitive and pragmatic arguments to explain their assertions due to the appeal to intuition and ease of understanding.

According to this author, this function is related to the understanding and not to the rigor and justification of a definition, theorem, or a more general property. It is developed and understood from specific cases that are studied collectively by the class, such as the case of determining the area of a figure by multiplying the length of its sides - as it is done for the rectangle –the same can be done for square and this strategy can be adapted to determine the area of other figures such as the rhombus and triangle.

5.1.4. Systematization function

According to De Villiers (1990), this function corresponds to the organization of the results into a deductive system. Cabassut (2005) argues that this idea can be seen as similar to the conception of Hanna (2000) to relate a known fact to a new idea and thus assign it a new perspective.

Cabassut (2005) points out that there are two different ways of this function included in the act of teaching:

a global level - methodical presentation in the field of mathematics. At this level, emphasis is placed on rigorous organization of knowledge.

a local level - a visual justification using a limited number of results and definitions is given, through which a local organization takes place.

The author states that, in teaching, this function is cultural, since a way to organize knowledge, specific to mathematicians, is taught. To analyze this function, it is necessary to understand the way it integrates the proposition validated in the organization of the two cases: local or global. The first attempts to the local validation can be seen in the initial series.

This organization is based on concepts, definitions, theorems, etc., enabling the review of knowledge and expertise. Therefore, it is possible to have access to the systematization of knowledge acquired and its application gives it value and relevance (Cabassut, 2005). For example, the various possibilities to cover a region can lead the student to say that the square, rectangle, triangle, etc. can be used as units to measure area.

5.1.5. Discovery or invention function

This function is related to the reaching of a conclusion and to the validation of the technique that led to that conclusion. According to Cabassut (2005), students should not be expected to use this function, but it can be taught in a propaedeutic way, for example, one can imagine this function during a demonstration, when the method being used allows demonstrating a more general result, or in the case of open problems, in which the demonstration leads to finding the solution.

The use of this function can be identified in the actions that lead students to discover new relationships that connect concepts, thus increasing their functionality.

With reference to a research conducted by Douady and Perrin-Glorian (1989) and Baltar (1996), it is clear that students have difficulties identifying the same figure when it its position is changed during rotation. Therefore, an activity that proposes the rotation of a square, which is built on the computer screen according to its properties, can ensure that the shape of the figure is not changed after rotation and translation. This discovery can lead the student to argue that, in general, any other figure has the same characteristic.

5.1.6. Communication function

In mathematics teaching, this function is characterized by the organization of different records, whether oral - towards a scientific debate, or written – text, drawings, or text reading -as well as the simultaneity of these records and gestural records to support an argumentation (Cabassut, 2005).

This discursive or communicative function corresponds to the fact that an argument or proof must be explained by those presenting it. It can be observed that this function is related to other functions due to the need to use them to communicate records, actions, drawings, etc.

These functions, ruled by the last one, control, in a certain way, the validation of communication and the dissemination of mathematical knowledge. Argumentation, included in this interaction process, also involves a subjective negotiation not only of the meanings of the concepts involved, but also, implicitly, of the criteria related to an argument considered acceptable. Therefore, the validation of the assumptions included in the process of communicating ideas reveal their strength and can support them or even refute them by counterexamples.

Validation functions generally occur simultaneously, but the prevalence of one or two of them in the argumentative process can be observed. Based on this assumption, these functions can be characterized by such preponderance.

According to Perelman and Olbrechts-Tyteca (2005), isolated arguments that focus on a particular study interact to get a consistent and meaningful conclusion within the context in which they occur. Therefore, the conditions in which the arguments develop determine to a large extent the type and the validation function of the arguments used to ensure the necessary strength to validate a particular conjecture.

5.2. Classification according to the nature of reasoning

With regards to the nature of the arguments produced by students, Pedemonte (2002) argues that the argumentations in mathematics can be abductive and inductive.

According to the author, "abduction was introduced by Peirce [...] as a model of inference used in the discovery process. It is expandable because its conclusion provides new knowledge. The search for the solution of a given problem is often constructed from the conclusion" (Pedemonte, 2002, p. 67).

According to Pedemonte (2002), abductive reasoning consists in finding the best or more plausible explanations from a set of facts or information given. The goal of this type of argument is to use incomplete, inaccurate, unreliable information to explain observed facts. When teaching mathematics, rules that are new and unknown to the students can be used, since they seem more appropriate to make a conclusion.

According to Pedemonte (2002), inductive reasoning is also expandable and like the abductive reasoning, it leads to new knowledge, but it uses observations of individual cases that are generalized to a broader set of cases. The goal is to infer a rule from specific facts or private data. The attempted application of observed facts in new cases highlights analogy, besides generalization.

As a result of the generalization and analogy processes, the author distinguishes three types of inductive argumentation: inductive argumentation by generalization, Passage to the limit and recurrence

In the first type, particular cases are used until the establishment of a general law: this enables to abstract properties from the analysis of several different cases.

The second type can be considered a special case of the first one; it consists in verifying whether a property is true for a given situation, and thinking that it may be true in a different situation that somehow resembles the first one. This characteristic makes this kind of argument similar to the crucial experience of Balacheff (1987).

In the third case, it is assumed that there exists $n_0 \in N$ such that $P(n_0)$ is true for all $n \geq n_0$ if $P(n) \rightarrow P(n+1)$ is true; thus for all $n \geq n_0$, hence $P(n)$ is true.

Although the studies of Pedemonte (2002, 2005, 2007) focused on the relationship between argumentation and demonstration, they reveal important points about the nature of reasoning used by students in the argumentative process.

5.3. Strength of an argument and the argumentative convergence process

In mathematics, effectiveness and validity are combined to ensure the validity of an argument, and they are subject to the standards set by this field of study. In this case, the strength of an argument is due to resistance to objections resulting from counterexamples. The practice of argumentation, necessary to the acquisition of argumentative competence, enables students to access the specificities of mathematics. Hence, they can communicate their ideas with the strength needed to validate their assertions.

> Any initiation to a rationally systematized domain not only provides knowledge of facts and truths of the subject being studied and of its own terminology and the way its tools should be used, but it also educates due to the strength of the arguments used in this field. (Perelman and Olbrechts-Tyteca, 2005, p. 528).

Thus, the strength of the arguments depends on the context in which they are inserted. In mathematics, for example, it is restricted to the habits, laws, methods, and techniques of this discipline, and in most cases, it is implicit and must be taken into account in the interactions with the specific issues addressed in the classroom.

According to Perelman and Olbrechts-Tyteca (2005), the interactions that ensure the strength of an argument are subject to the *convergence* process. This process occurs when several distinct arguments lead to a single conclusion, "be it general or partial, final or provisional, the value attributed to the conclusion and to each argument will therefore be strong enough to be validated in the context in which it is inserted" (Perelman and Olbrechts-Tyteca, 2005, p. 535).

Since the communications of students' ideas are composed of mathematical and non-mathematical argumentations, the teacher should manage this convergence ensuring that the arguments undergo this process during the collection of evidences through activities organized with this purpose.

In the case of opposite arguments, the convergence will be facilitated by the mediation of the teacher, who should lead the process towards the conflicting views, so that the argument that does not satisfy the requirements of this area loses strength and converges according to mathematical criteria.

These ideas allow us to propose and analyze the stages that comprise the argumentative process and thus to show that the practice of argumentation can promote the understanding of concepts in mathematics. These stages must include the reference experience and the types of arguments and their respective roles in the argumentative process, as well as the evaluation of the strength of each argument and the argumentative convergence, which is necessary to find a specific solution.

5.4. Characteristics of the elements of Second Stage

The communication of ideas intensifies during this stage. It is when thorough argumentation analysis (both functional and structural) should be performed classifying and leading the ideas to the last stage, validation. Toulmin (2006) subdivided this stage into three elements:

In the first stage, no matter the problem, one has to admit that a series of different suggestions deserves consideration; all of which, in the first stage, must be candidate *solutions.*

The second stage consists of the proposal of candidate solutions. The subjectivity of the arguments in this stage shows connections between the knowledge already acquired and that being developed. Intuition leads to gather this knowledge. The object becomes explicit articulating the representations, the gestural communication, and confrontation of contradictory ideas. This stage includes situations similar to those reported by Perelman and Olbrechts-Tyteca (2005), when there are isolate arguments that should articulate and interact with a given subject.

Finally, there are the candidate solutions that are proposed as singularly good solutions. Thus, some of them are rejected using qualifier modal verbs and specific terms to show possibility such as *cannot* and *it is impossible* among others in order to eliminate the solutions that do not fit the problem. However, a solution can be found using more appropriate words such as *probably*, *possibly*, *likely* indicating more plausible answers, although they are not put forward this way. These terms can be explicit by related statements that fit them in a process of possibility or exclusion.

We believe that the first two stages occur simultaneously and naturally in classroom settings; therefore, they are considered as a single stage.

Rejections of candidate solutions can result from new information that may lead us to discard them. The observations, discussions, and actions within a specific context can be useful in the search for specific solutions. At this point, *provisional* solutions are important to identify the changes of plausible and transitory solutions into categorical and irrefutable solutions.

Therefore, the activity actions should enable the collection of evidences required to strengthen or invalidate ideas underlying specific argumentations. Therefore, the transitory truths are put to test before the facts (the data) that result from activities and can strengthen the beliefs, *i.e.*, can add more or reduce the strength to the argument until there is evidence of validation or refutation of the argument.

This situation is in accordance with the ideas of Perelman and Olbrechts-Tyteca (2005), who argued that one of the functions of the argument is to gather around a particular subject. Several ideas that at first are scattered can become specific solutions according to the specific area of study due to the teacher's mediation. Hence, the strength of an argument becomes clear, once it conforms to the norms of the field of study in which it is inserted.

The ideas of Cabassut (2005) must be taken into consideration in the argumentative convergence process, who emphasized that the argumentation transformations related to mathematics education result from the fact that they are composed of a double *transposition*[8] of mathematical and non-mathematical arguments.

During this process, which Perelman and Olbrechts-Tyteca (2005) call convergence, we believe that characterization of the arguments is necessary. They can be pragmatic, semantic, or formal. As for the nature of the reasoning used by students, they can be deductive, inductive, or abductive. This characterization describes the arguments and can help either lead to validation, in accordance with its conjecture, definition, theorem, etc., or highlight the establishment of argumentative competences (cf. Pedemonte 2002, cf. Cabassut 2005, cf. Grácio 2009).

5.5. Characteristics of the third Stage

As previously mentioned, Cabassut (2005) argues that, in order to understand the validation process of argumentation in a mathematics classroom, it is necessary to consider that this process results from the articulation between mathematical and non-mathematical arguments.

Mathematical validation, according to the author, is related to mathematical knowledge, such as definitions, theorems, and properties that make up the demonstration process - organized by mathematical institutions[9] - where it is the object of knowledge. The appropriation of knowledge enables the understanding of this validation process. "Mathematical demonstration is the validation procedure used in mathematical institutions, where it is the object of knowledge. The organization of this knowledge may vary depending on the institutions being considered" (Cabassut, 2005, p. 77).

On the other hand, according to Perelman and Olbrechts-Tyteca (2005), there are the non-mathematical argumentations such as the pragmatic argumentations that are present in the classroom, for example, the use of an action, either an observation, manipulation etc., which could justify a conclusion. In this case, the proof will be sustained by the action that provides the data needed to reach a conclusion. Such arguments, according to Cabassut (2005), refer to the non-mathematical validations that provide knowledge, such as those used in other areas of knowledge and *non-scientific knowledge* of daily life resulting from institutions such as family, group of friends, etc.

The author believes that mathematics educational institutions use mathematical and non-mathematical validations. Therefore, the pragmatic argumentations, such as those resulting from computer screen visualization can help the understanding of geometric properties.

It is worth highlighting that the non-mathematical arguments and their respective validations can contribute to the understanding of properties, definitions, proofs, etc., thus helping validation in mathematics teaching (Cabassut, 2005).

To highlight that composition of arguments in the classroom, Cabassut (2005) classifies the arguments as pragmatic, semantic, and formal, as previously announced. Such classification,

[8] A knowledge content defined as knowledge for teaching, from which a number of adaptive transformations will make it fit as an object of education. The work that transforms an object of knowledge for teaching into an object of education is called didactic transposition (Chevallard, 1991).
[9] Institutions where new knowledge about a certain fields is developed, for example, universities (Cabassut, 2005).

together with the categorization suggested by Pedemonte (2002), contributes to the identification and analysis of the nature and reasoning of the arguments considered by students as credible arguments for validation.

To help the analysis and classification of the arguments produced by students, the argumentation validation functions related to the last stage of the argumentative process must be taken into account.

In order to understand the validation process, according to Toulmin (2006), we should be aware that a solidly built good cause, a well-founded allegation or strongly supported by the criteria of the field in which it is inserted, will withstand criticism, and it will be the cause that corresponds to the required standard and to which one can expect a positive verdict.

6. Final Considerations

We have to admit that in the first stage the problem takes the form of a *reference experience* according to Douek and Scali (2000). In the second stage, we propose that the first and second elements are deemed concomitantly, since the argumentative process in the classroom consists of multiple speakers, and the candidate solutions undergo the convergence argument process postulated by Perelman and Olbrechts-Tyteca (2005). Therefore, the candidate solutions that are not in accordance with the rules of mathematics may be either rejected or adjusted in order to find a specific solution. We suggest that the arguments are classified in these stages in order to understand, on the one hand, the nature of the arguments used by students, which can be pragmatic, semantic, or formal, and, on the other hand, the nature of reasoning produced by them, abductive, inductive, or deductive, using the classifications of Pedemonte (2002) and Cabassut (2005). The last stage, which Toulmin (2006) calls the verdict, was considered as the validation time using the functions of validation reported by Cabassut (2005) to qualify the validations.

We suggest that a process that is driven by these moments can encourage students to acquire argumentative competence, necessary not only for the development of mathematical knowledge, but also for social life requirements.

The studies reviewed here that addressed the theoretical perspective of Toulmin, focused on the physiological process of argumentation. In the present study, in addition to the physiological part, we studied the anatomical part, which enabled us to model the argumentative process in terms of stages, and thus introduce the practice of argumentation as a teaching method.

When proposing the anatomical part stages, Toulmin (2006) did not focus on the details of its complexity, mainly because the author's proposal is to universalize the argumentative process analysis. However, we realized that it would be more feasible focusing on a specific field such as mathematics, since we believe that the general characteristics of the stages proposed by the author can be modeled in terms of *reference experience,* types of arguments, argumentative convergence, and validation functions.

As for the second phase, Toulmin (2006) proposes that it be divided into three stages: the first would include a series of candidate solutions to the problem and the second would propose the solutions; however, we believe that in terms of argumentative interaction, these two stages are overlapped so that they cannot be identified separately in the process. In the last stage, the author states that one of the solutions will have enough strength since it satisfies the requirements of the field in which it is inserted, leading to the rejection of the other solutions proposed. Nevertheless, we believe that not only rejection, but rather the composition of candidate solutions can lead to a specific solution.

In the last stage, the verdict will take place, *i.e.,* validation or refutation of the argumentation. The functions of validation proposed in this stage were based on the theoretical reflections of Cabassut (2005).

Therefore, it is possible that the practice of argumentation can foster the acquisition of argumentative competences in mathematics. They will favor the mobilization of necessary reasoning that will enable the use of strategies needed for solving problems and the appropriation of symbols and specific mathematical language, but without limiting to these functions since there is an essential stage of personal relationships that includes personal intentions, beliefs, decision making, persuasion, respect for the others' opinions, etc.

7. References

Balacheff, N. (1987) Processus de preuves et situations de validation. *Educational Studies in Mathematics*, v.18, n.2, p. 147-176.

Balacheff, N. (1988)*Une étude des processs de prevue en mathématiques chez des eleves de college*. 1998. 591 f. Tese (Doutorado em Ciências e Didática das Matemáticas) – Université Joseph Fourier, Grenoble.

Boavida, A. M. R. *A argumentação em Matemática Investigando o trabalho de duas professoras em contexto de colaboração.***(**2005). 975f. Tese (Doutorado em Educação) - Faculdade de Ciências, Universidade de Lisboa.

Boero, P.; Douek, N.; Ferrari, P. L. (2008). *Developing mastery of natural language:*Approaches to Approaches to some theoretical aspects of mathamatics. Virtual Books, 2008. Disponível em:
<http://books.google.com.br/books?id=63voiv2afgMC&pg=PA262&lpg=PA262&dq=Developing+mastery+of+natural+language:+Approaches+to+Approaches+to+some+theoretical+aspects+of+mathamatics&source=bl&ots=7fHcShVTdt&sig=BKE4bi31C6iu-20TrObbjlNNoCI&hl=pt-BR&ei=7JLzTIjeFYiq8Aa0w4DMDA&sa=X&oi=book_result&ct=result&resnum=1&ved=0CBYQ6AEwAA#v=onepage&q&f=false> Acesso em 13 ago. 2009.

Boero, P.*Argumentation et démonstration: Une relation complexe, productive et inévitable en mathématique et dans l'enseignement des mathématiques.*(1999). Disponível em:<http://www.lettredelapreuve.it/OldPreuve/Newsletter/990708Theme/990708ThemeFR.html> 1999. Acesso em: 17 set. 2008.

Boero, P.; Garuti, R.; & Mariotti, M.A. (1996). *Some dynamic mental processes underlying producing and proving conjectures*, Proceedings of PME-XX, Valencia, v. 2, p. 121-128.

Brasil, Secretaria de Educação Fundamental. (1997). *Parâmetros Curriculares Nacionais:*introdução aos parâmetros curriculares nacionais/Secretaria de Educação Fundamental. Brasília: MEC/SEF, 126p.

Brousseau, G. Fondements et méthodes de la Didactique des Mathématiques. (1986) *Recherches en Didactique des Mathématiques*. Grenoble: La Pensée Sauvage-Éditions, v.7.2, p.33-115.

Cabassut, R. *Démonstration, raisonnement et validation dans l'enseignement secondaire des mathématiques en France et en Allemagne*. (2005). 424 f. Tese (Doutorado em Didática da Matemática) - Ecole doctorale Savoir scientifique: épistémologie, histoire des sciences, didactique des disciplines. Université Paris 7, Paris.

Chevallard, Y. *La transposition didactique*. Grenoble: La Pensée Sauvage Editions, 1991.

CrespoCrespo C. *Las Argumentaciones Matemáticas desde la Visión de la Socioepistemología.*(2007). 300f. Tese (Doutorado em Ciências e Educação Matemática. Instituto Politécnico Nacional. Centro de Investigación en Ciencia Aplicada y Tecnología Avanzada. México, 2007.

De Villiers, M. The role and the function of proof in mathematics, *Pythagoras.*n. 24, p.17-24, 1990.

Douady, R., & Perrin-Glorian, M. J. Un Processus D'Apprentissage du Concept D'Aire de Surface Plane. (1989) *Educational Studies in Mathematics*, 20(4), 387-424.

Douek, N. (1998). Analysis of a long term construction of the angle concept in the field of experience of sunshadows, *PME-XXII*. 2, 264–271, Stellenbosch.

Douek, N. Argumentative aspects of proving of some undergraduate mathematics students' performances. (1999). *PME XXIII*. 2, 273-280, Haifa, Israel.

Douek, N. (2000) Importance des Aspects Argumentatifs dans la Production et Demonstration de Conjectures. *Compte Rendu de l'Atelier*.

Douek, N. & Scali,E. (2000). About argumentation and conceptualisation, *Proceedings of PME-XXIV***,** 2, 249-256, Hiroshima.

Douek, N., & Pichat, M. (2003). From oral to written texts in grade I and the approach to mathematical argumentation. *Proceedings of PME-XXVII*, 2, 341-348, Honolulu.

Goodwin, J. (2009). A argumentação não tem função. *Comunicação e Sociedade*, 16, 123-144.

Grácio, R. (2009). Com que é que se parece uma argumentação? Representações sociais do argumentar. *Comunicação e Sociedade*, 16, 101-122.

Hanna G. (2000). Proof, explanation and exploration: an overview, *Educational Studies in Mathematics*, 44(1-2), 5-23.

Krummheuer, G. (1995). The ethnography of argumentation. In: P. Cobb; H. Bauersfeld (Eds.), *The emergence of mathematical meaning*: Interaction in classroom cultures (pp. 229-269). Hillsdale, NY: Erlbaum.

Lampert, M. (1990).When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27(*1), 29-63.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCM.

Nunes, J. M. V., & Almouloud, S. A. (2013a). O modelo de Toulmin e a análise da prática da argumentação em matemática. *Educação Matemática Pesquisa* (Online), 15(2), p. 467-512.

Nunes, J. M. V., & Almouloud, S. A. (2013b). Argumentação no Ensino da Matemática: perspectiva metodológica. REMATEC. *Revista de Matemática,* Ensino e Cultura, 8(13), p. 143-167.

Nunes, J. M. V. (2011). *A prática da argumentação como método de ensino: o caso dos conceitos de área e perímetro de figuras planas*. Unpublished doctoral dissertation. São Paulo: Pontifícia Universidade Católica de São Paulo-PUCSP. Brazil.

Pedemonte, B. (2002). *Etude didactique et cognitive des rapports de l'argumentation et de la démonstration dans l'apprentissage dês mathématiques*. Unpublished doctoral dissertation, Université Joseph Fourrier, Itália.

Pedemonte, B. (2005). Quelques outils pour I'analyse cognitive du rapport entre argumentation et démonstration. *Recherches en Didactique des Mathématiques. 25*, 313–348.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66(1), 23–41.

Perelman, C., & Olbrechts-Tyteca, L. (2005). *Tratado da argumentação: a nova retórica.* Trad. Maria Ermantina Galvão G. Pereira. 2 Ed. São Paulo: Martins Fontes.

Toulmin, S.E. (2006). *Os Usos do Argumento*. Trad. Reinaldo Guarany e Marcelo Brandão Cipolla. 2 Ed. São Paulo: Martins Fontes.

Vergnaud, G. (1996). A Teoria dos Campos Conceituais. In: BRUN, J. (Dir.) *Didáctica das Matemáticas*. Instituto Piaget, Lisboa: Horizontes Pedagógicos, p. 155-191.