TEACHING FRACTIONS — VISITING THE PRIMARY SCHOOL CLASSROOM

ENSINO DE FRAÇÕES – VISITANDO A SALA DE AULA DO ENSINO BÁSICO

Paula Cardoso

paula.c.cardoso@outlook.com

Ema Mamede

emamede@ie.uminho.pt

CIEC – University of Minho

ABSTRACT

The most recent guidelines for mathematics in basic education (6-10-years-old) in Portugal preconize an in-depth contact with fractions. These guidelines suggest the approach to quotient, part-whole, measure and operator interpretations of fractions. Thus, it seems pertinent to investigate whether the current teaching practices reflect those guidelines. The research presented here sought to answer the following questions: 1) What role do teachers give to the interpretations of fractions to introduce the concept of fraction to children? 2) How do teachers approach and articulate the different interpretations of fractions? 3) What difficulties do teachers reveal about these issues? A collaborative working program with four teachers was conducted comprising the observation of lessons. The present paper presents the results concerning one of the cases — teacher Inês (fictitious name). A qualitative analysis of the collected data suggests difficulties in selecting examples and introducing the meaning of the fraction when quotient interpretation is involved; difficulties in approaching the ordering and equivalence of fractions; no exploration of the reference unit; unsuitable emphasis on algebraic procedures when operator interpretation is involved. In-service teacher training should be regularly promoted in order to ensure greater convergence between curriculum and teaching practices and improve the quality of the latter.

Key words: teaching of fractions, teachers' knowledge, curricular guidelines.

RESUMO

As orientações curriculares mais recentes para a Matemática no 1.º ciclo do Ensino Básico (6-10 anos de idade) em Portugal preconizam uma abordagem aprofundada ao conceito de fração, contemplando as interpretações quociente, parte-todo, medida e operador. Parece então pertinente investigar se as atuais práticas de ensino refletem tais orientações.

O estudo aqui apresentado procurou responder às seguintes questões: 1) Que papel atribuem os professores às interpretações de fração na introdução do conceito de fração aos seus alunos? 2) Como é que os professores exploram e articulam as diferentes interpretações de fração? 3) Que dificuldades manifestam os professores no desenvolvimento das suas aulas sobre frações? Foi desenvolvido um programa de trabalho colaborativo, com quatro professores, que incluiu a observação de aulas. O presente artigo apresenta os resultados relativos a um destes casos – a professora Inês (nome fictício). Uma análise qualitativa dos dados recolhidos sugere: dificuldades na seleção de exemplos e na introdução do significado de fração quando está envolvida a interpretação quociente; dificuldades na abordagem à ordenação e à equivalência de frações; inexploração da unidade de referência; ênfase inapropriada dos procedimentos algébricos quando a interpretação operador está envolvida. A formação contínua de professores deveria ser regularmente promovida, a fim de assegurar uma maior convergência entre currículo e práticas de ensino e de melhorar a qualidade destas.

Palavras-chave: ensino de frações, conhecimento do professor, orientações curriculares.

1. Teaching fractions in Primary School

The concept of fraction is considered to be among the most complex concepts that children learn in primary school. Several studies point out students' difficulties in learning this concept (Behr, Lesh, Post & Silver, 1983; Kerslake, 1986). On the other hand, it is known as a complex concept to teach (Behr *et al.*, 1983; Kerslake, 1986; Nunes *et al.*, 2004).

1.1 Interpretations of fractions

Several authors distinguished the interpretations that might offer a fruitful understanding of the concept of fraction. Behr, Lesh, Post and Silver (1983) distinguished part-whole, decimal, ratio, quotient, operator, and measure as subconstructs of the concept of rational number. Also, Kieren (1993) considered the subconstructs of measure, quotient, ratio and operator. Mack (2001) proposed a different classification, using the term 'partitioning' to cover both part-whole and quotient interpretations. More recently, Nunes, Bryant, Pretzlik, Wade, Evans and Bell (2004) presented a classification based on the meaning of values involved in the fractions, distinguishing quotient, part-whole, operator and intensive quantities situations.

The most recent curricular guidelines in Portugal include quotient, part-whole, measure and operator interpretations for primary school grades, thus they were selected for the present study. In quotient interpretations, the denominator designates the number of recipients and the numerator designates the number of items being shared. In this situation, a fraction may either indicate the relation between the number of items to share and the number of recipients or the amount of an item that each recipient gets. In part-whole interpretations, which involves the division of continuous quantities, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. In measure interpretation, the fraction $\frac{1}{b}$

(b≠0) is used repeatedly to determine a distance; it is often accompanied by a number line or an image of a measuring instrument, being expected that students measure the distance from one point to another in terms of $\frac{1}{b}$ unities. Finally, in an operator interpretation, which involves discrete quantities (see Nunes et al., 2004), the denominator designates the number of equal groups into which a set was divided and the numerator designates the number of groups taken.

1.2 Fractions in the Portuguese curriculum

In the classroom, the concept of fraction is often, and sometimes exclusively, approached in part-whole and operator interpretations (Behr, Harel, Post & Lesh, 1992; Kerslake, 1986; Monteiro & Pinto, 2005). The teacher traditionally presents a figure (a rectangle or a circle) divided into equal parts, one of these highlighted. The fraction then arises as a relation between the highlighted part and the whole of the figure. Yet, this type of teaching gives the students a narrow concept of fraction. It particularly limits the development of the idea that a fraction can be greater than 'one'. Effectively, the procedure of starting with a 'whole' divided into several equal parts does not easily fit into fractions larger than the unit (Kerslake, 1986).

In Portugal, the most recent curricular guidelines (MEC-DGE, 2012, 2013) anticipate a more in-depth approach to the concept of fraction on the primary school levels (6-10-years-old). According to such guidelines, students shall contact with different meanings or interpretations of fractions (quotient, part-whole, measure and operator). This curriculum implies significant changes if one takes into account that, previously (ME-DEB, 2004), only the operator interpretation of fraction was regarded, merely implying a few unit fractions (one-half, one-third, one-quarter, one-fifth and unit fractions with denominator 10, 100 and 1000). Therefore, the current guidelines demand the use of mathematical and pedagogical knowledge that can be considered innovative in Portuguese reality, when compared with the necessary knowledge for the implementation of previous ones. The teachers are supposed to be fully acquainted with the labelling, ordering and equivalence between fractions, as well as with their different interpretations.

Thus, it becomes important to analyze thoroughly how Portuguese primary school teachers explore the concept of fraction in the classroom, in order to identify possible ideas and practices that can be improved.

2. Knowledge base for teaching

There is a set of specific knowledges for teaching (see Ball, Thames & Phelps, 2008; Shulman, 1986). Concerning the knowledge base for teaching, Shulman (1986) considers: a) curricular knowledge — knowledge of the curriculum; b) content knowledge — the theories, principles and concepts of a particular discipline; and c) pedagogical content knowledge — deals with the teaching process, including the most useful forms of representation and communication of content, and with the way students best learn the specific concepts and topics of a subject. Ball, Thames and Phelps (2008) further refined two of these categories. These authors suggested that Shulman's content knowledge could be subdivided into common content knowledge (CCK) and specialized content

knowledge (SCK). They provisionally placed Shulman's third category, knowledge of content and curriculum (KCC), within pedagogical content knowledge. The authors also suggested that pedagogical content knowledge could be subdivided into knowledge of content and students (KCS) and knowledge of content and teaching (KCT). The authors provisionally included as well a third category within subject matter knowledge: horizon content knowledge (HCK).

The domain common content knowledge (CCK) is the mathematical knowledge required for teaching but not exclusive to the work of teaching. It may include the ability to recognize errors, make correct calculations and pronounce terms correctly. The domain specialized content knowledge (SCK) is the mathematical knowledge exclusive to teaching. It involves a kind of unpacking of mathematics that makes a particular content visible to and learnable by students. The domain knowledge of content and curriculum (KCC) is the knowledge of the curriculum guidelines. The domain knowledge of content and students (KCS) combines knowledge of mathematics with knowledge of students, involving familiarity with students' usual thoughts and procedures. This includes anticipation of students' ideas and misconceptions. It also includes interpretation of students' understanding as it evolves. The domain knowledge of content and teaching (KCT) combines knowledge of mathematics with knowledge of teaching, involving familiarity with pedagogical principles for teaching a particular content. It includes the knowledge for planning instruction, for evaluation of advantages and disadvantages between different representations, and the ability to present examples able to create deeper understanding among students. The last domain, the horizon content knowledge (HCK), is the awareness of how mathematical topics are related. Teachers need to know how the mathematics they teach is related to the mathematics students will learn and, thus, they shall be able to set the mathematical foundation for future learning.

3. Teachers' knowledge on rational numbers

Studies on teachers' knowledge suggest that they are considerably less confident and successful in the domain of rational numbers than they are in the domain of whole numbers. As part of the Rational Number Project (RNP), Post, Harel, Behr and Lesh (1991) conducted a study involving 218 teachers (grades 4-6) to analyse their knowledge to teach rational numbers. The authors identified several difficulties, namely with the interpretations of fractions and with the ordering and equivalence between fractions. Post et al. (1991) emphasized that teachers have difficulties in presenting pedagogical explanations for computations with rational numbers performed by themselves.

Tirosh, Fischbein, Graeber and Wilson (1998), in the Conceptual Adjustments in Progress to Non-Negative Rational Numbers (CAPWN) project, carried out a diagnostic questionnaire to 147 prospective elementary teachers in order to examine formal, algorithmic and intuitive understanding of rational numbers. Prospective teachers' mathematical knowledge was found to be rigid and segmented. For most of them, mathematics was a mere collection of computational techniques not well mastered, formally unjustified. Their results also showed that the prospective teachers tended to over generalize their knowledge of whole numbers when working in the domain of rational numbers.

In Portugal, Pinto and Ribeiro (2013) carried out a questionnaire to 27 prospective teachers of elementary school (grades 1-4) revealing that they possess a limited

knowledge of rational numbers. The results particularly suggest difficulties with: the interpretations of fractions (quotient, part-whole and operator); the understanding of the role of the reference unit; the order and equivalence of fractions; and the density of rational numbers.

Mamede and Pinto (2015) carried out a questionnaire to 86 prospective teachers of the elementary school (grades 1-4). Their results suggest: difficulties of prospective teachers with the understanding of the reference unit; weak domain of the interpretations of fractions, mainly in the scope of problems involving the quotient interpretation and in the scope of problems involving the representation of rational numbers on the number line when numbers different than one are used as reference and when its necessary a redefinition of the scale; weak domain of the property of density of rational numbers; and difficulties with the ordering and equivalence between fractions.

In Portugal, little is known concerning the Portuguese teaching practices on fractions. As the recent Portuguese mathematics curriculum preconize an in-depth contact with fractions, this paper reports a research conducted with Portuguese primary school teachers focused on their teaching practices, when involved in a collaborative work program with a researcher — one of the authors of this paper. This research tried to address three questions: 1) What role do teachers give to the interpretations of fractions to introduce the concept of fraction to children? 2) How do teachers approach and articulate the different interpretations of fractions? 3) What difficulties do teachers reveal about these issues?

4. Methodology

This study followed a qualitative methodology, since it is intended to be a description and interpretation of educational phenomena in their natural environment (see Bogdan & Biklen, 1999; Merriam, 1998). A multiple case studies design was used, according to Yin (2010), such option is particularly appropriate, both to answer questions of the type "how?" and "why?", and to seek for a deep thorough understanding of the phenomena.

4.1 Participants

Participants were four primary school teachers of the district of Braga, in Portugal. This paper presents the results concerning only one of the cases — teacher Inês (fictitious name), with thirteen years of teaching practice, ten of them in elementary school. Her class had twenty-five students, aged 6 and 7 years. Inês informed that her students had not been formally introduced to the concept of fraction before. This teacher faced this participation as an opportunity to clarify eventual doubts.

4.2 Design

While introducing the concept of fraction to her 2nd-graders, Inês was involved in a collaborative working program with a researcher - one of the authors of this paper. This program was organized into cycles of activities, each consisting in the following sequence: working meeting, with all the participants, for reflection on the observed lessons and preparation of the next ones; observation of the lessons of each participant by

the researcher only; individual interview on the observed lesson occurring immediately after each lesson to assess teacher's critical view of his/her practices (Figure 1).
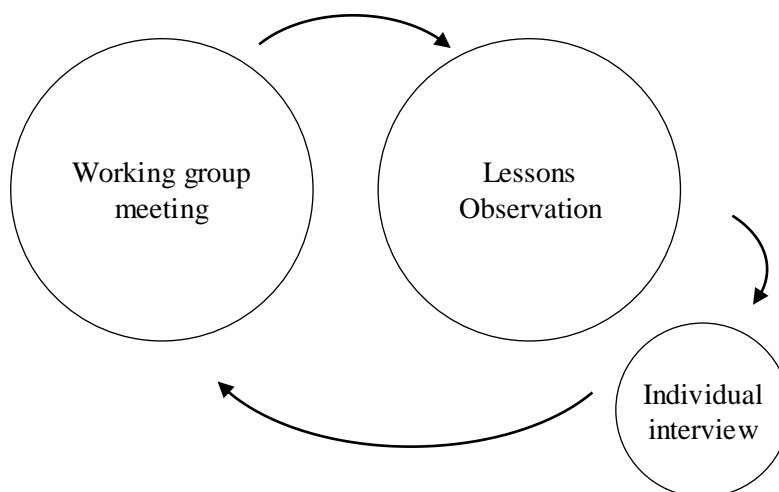

Figure 1– Standard cycle of the collaborative work program.

Five cycles of the collaborative program were carried out. Each cycle comprised one or two observed lessons. Each working meeting comprised: a) discussion on different interpretations of fractions referred in the official guidelines; b) discussion on teachers' suggestions for introduction of the concept of fraction in the classroom; and c) presentation of suggestions of the researcher on the topic. The selection and implementation of tasks in the classroom was teacher's responsibility. Tasks presented at the working meetings focused on the interpretations of fractions (quotient, part-whole, measure and operator) and on representation, equivalence and ordering of fractions in these interpretations.

In all moments, data collection comprised digital audio records and researcher's field notes. Photos were also taken but only during the lesson observation. A large and varied set of data was collected in order to guarantee validity. During the lessons, the researcher was a non-participant observer. The lessons were observed in locus only by the researcher (one of the authors of this paper).

The collaborative work aimed to help teachers to improve their practices in a reflective way, and in agreement with Saraiva and Ponte (2003), can help them to accomplish the desire to innovate and do better. The researcher and teachers acted as pairs, discussing mathematical and didactical doubts according to the rhythm, needs and teachers' interests when teaching in the natural context of the school.

4.3 Data analysis

Data analysis was based on the model about knowledge base for teaching presented by Ball *et al.* (2008). Thus, in order to interpret the data, a categorization of the analyzed aspects was made, according to the different parameters of the above-mentioned model: aspects of content knowledge and aspects of pedagogical content knowledge for teachers regarding the concept of fraction teaching.

5. Results

Concerning the observed lessons, the results suggest some difficulties of teacher Inês in the introduction of the concept of fraction, which are summarized below. The results presented here concern only to five consecutive observed lessons on fractions.

In the transcriptions of classroom dialogues presented here, the letter S represents the intervention of a student — numerated according to the order in which different students appear in each dialogue, T represents the intervention of the teacher, and Sv represents the simultaneous intervention of several students.

Lesson 1

In the working meeting that preceded the first observed lesson, the curriculum guidelines concerning the teaching of fractions were analyzed. Specifically, different interpretations of fractions (quotient, part-whole, measure and operator) were discussed. Inês agreed that it would be interesting to introduce fractions with the quotient interpretation and accepted the challenge to do so.

Fair sharing situations and division

The mathematical subjects approached in this first and in the following classes are preconized in the official curriculum guidelines, which suggests teacher's knowledge of content and curriculum. Also, teacher's carefulness in registering on the blackboard and discussing students' answers was a manifestation of knowledge of content and students.

Inês started her first observed lesson by approaching fair sharing activities, introducing the symbol "÷" to illustrate such a situation. For example, she asked students to distribute 6 buttons by 2 coats (Figure 2), then wrote on the blackboard "6÷2=3". She also introduced multiplication as the inverse operation of division (writing on the blackboard, for example, "6÷2=3   2×3=6").

Figure 2 — Distribution of buttons by coats.

Students were given time to answer to questions of the textbook related both to the division and to the multiplication as inverse operation of division (Figures 3 and 4). Students successfully divided items by recipients and correctly traduced such divisions using the symbol "÷". However, difficulties arose regarding the understanding of multiplication as the inverse operation of division.

Figure 3 — Student's answer to a problem of fair sharing of 12 items by 3 recipients.
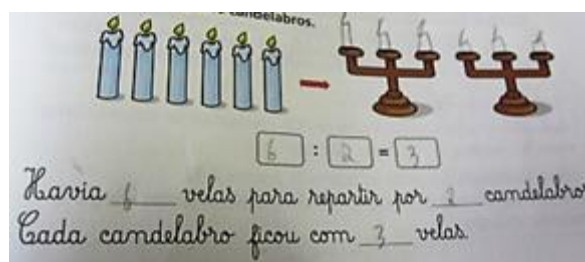
Figure 4 — Student's answer to a problem of fair sharing of 6 items by 2 recipients.

Then, the teacher presented on the blackboard tasks in which it was intended to find the dividend, with previous knowledge of the divisor and quotient (Transcription 1).

Transcription 1 – Fair sharing of 6 items by 3 recipients.

T —   ["__÷2=8" and "__÷3=7" were written on the blackboard and students had to fill in the blanks. Students remained in silence. The teacher did the correction on the blackboard:

$$16 \overset{\overset{x}{\frown}}{:2} = 8 \quad 21 \overset{\overset{x}{\frown}}{:3} = 7 \quad]$$

S1 —   I don't understand.
T —   What is it that you don't understand?
S1 —   I don't understand those calculations.
T —   Imagine you have 6 dolls and you have to divide them between you and your sister. How many dolls does each one get?
S1 —   Three. But I have more sisters.
T —   I know that you have more sisters, but this is an example. So, there were 3 dolls for each. You and your sister. That is 2 times 3, gives…?
S1 —   Six.
T —   Six was the number of dolls, that is, the inverse operation of the division is the multiplication. To find 21 multiply 3 times 7. Twenty-one divided by 3 is 7. Got it?
S1 —   Yes.

Before introducing students to fractions, Inês seemed to approach too many mathematical ideas in a short period of time. This lesson was thought to be centred on the concept of fraction, but the teacher's options revealed difficulties in focusing on fractions instead of division only. It is true that, according to the curricular guidance, the problems of fair sharing should precede the introduction of the concept of fraction. However, this introduction does not need to be anticipated by an approach to the symbolic writing of the operation of division neither by the multiplication as the inverse operation of division. Thus, the options of the teacher revealed difficulties concerning the knowledge of content and teaching.

Fair sharing activities: inattention to the variables

Subsequently, the teacher asked students to distribute "groups of straws" by 3 recipients. The fact that each "group of straws" was constituted by 10 straws led to some inconsistences on the approach of this situation by Inês. She suggested S1 to carry out the division of 3 "groups of straws" by 3 recipients, stressing that the total of straws was 30 (Transcription 2). When questioned about "how many groups did each recipient get?", some students answered "1" and other students answered "10" (Transcription 2). The teacher attended only to the number of "groups of straws" and not to the total number of straws, writing on the blackboard "3÷3=1". However, the teacher should have also registered the division of the total number of straws by the recipients, i.e., "30÷3=10", as that emerged in the class discussion.

<div align="center">Transcription 2 – Fair sharing of 3 groups of straws by 3 recipients.</div>

T —	Divide these 3 groups by the 3 recipients. Each recipient should have the same number of groups of straws. How many groups does each recipient get?
Sv —	One.
Sv —	Ten.
T —	We have 3 recipients and 3 groups of straws. We want to divide the groups of straws by the recipients so that each recipient has the same number of groups of straws. I want to know how many groups each recipient gets.
S2 —	One.
T —	Very good! So, 3 groups of straws divided by 3 recipients… how many groups of straws does each recipient get?
Sv —	One.
Sv —	[Writes on the blackboard: "3÷3=1 3×1=3"].

Subsequently, concerning the division of 9 "groups of straws" by 3 recipients, Inês asked "How many groups do we have?" and several students answered "90" (Transcription 3). This dialogue suggests students' difficulties in distinguishing "groups of straws" from the total number of straws. The teacher insisted: "We have 90 straws, but how many groups of straws do we have?", to which students correctly answered "9" (Transcription 3). In order to approach all the possibilities of a correct answer, the teacher should have written on the blackboard, not only "9÷3=3", but also "90÷3=30", i.e., teacher should have written, not only the division of the "groups of straws" by the recipients, but also the division of the total number of straws by the recipients (Transcription 3).

T —   [With the groups of straws and the recipients] How many groups do you have in your hand?
S1 —   Nine.
T —   How many straws are in total?
S1 —   Ninety.
T —   Very good! We are going to distribute by the 3 recipients… so that each recipient gets the same number of straws. How many groups of straws did each recipient get?
Sv —   Three.
T —   How many groups do we have?
Sv —   Ninety.
T —   We have 90 straws, but how many groups of straws do we have?
Sv—   Nine.
T —   How many groups did in each recipient get?
Sv —   Three.
T —   [Writes on the blackboard: "9÷3=3" "3×3=9"]

When straws, groups of straws and recipients were involved, it was not clear for students that the teacher was only referring to groups of straws and recipients, excluding the variable related to the total number of straws. Actually, in face of the chosen situation, Inês should have accepted all the answers — those involving groups of straws and recipients and those involving the total of straws and recipients — given that all of them were acceptable. Since this was an introductory lesson to division, it would have been even better that the teacher had chosen an activity that did not promote this kind of doubts, with items and recipients well defined, revealing a stronger knowledge of content and teaching.

Quotient interpretation and pictorial representation

Then, Inês introduced the quotient interpretation. This option revealed knowledge of content and students, since the quotient interpretation fits particularly well to students' informal knowledge (Nunes *et al.*, 2004; Mamede, 2008; Streefland, 1991). However, in order to approach the quotient interpretation, Inês selected a task with a pictorial support that created some doubts. She presented a fair sharing situation, in which a chocolate bar was equally shared by two children (Figure 5), and asked the students: "I want to know two things: How would you share the chocolate bar by two children? How much does each one eat?" In the picture, the chocolate bar was divided into 18 equal parts (Figure 5). Students' answers were indeed: "eats 9", "half of the chocolate bar", "2 parts", "each one eats 3 columns (6÷2=3)", "one part" (Figure 6). The approach to this task was pedagogically rich; the teacher registered on the blackboard several answers given by the students and asked for their explanations. Notwithstanding, Inês highlighted the answer "half of the chocolate bar" to the detriment of other answers (Transcription 4). When highlighting this answer, certainly the teacher had in mind the main theme of the lesson: to introduce the concept of fraction. However, what did the students who answered "eats 9" and "each one eats 3 columns (6÷2=3)" think? These were also correct answers to the

task presented. In order to avoid such situations, Inês could have selected a picture of a chocolate bar that was not previously divided into equal parts, or should have explained to the students that different unit references were being considered in their responses.

The selection of tasks and their pictorial support must be carefully thought according to the lesson's subject. This would reveal a stronger knowledge of content and teaching. Moreover, when planning the lesson, the teacher must consider all the possible answers to the selected tasks. This particular aspect of lesson preparation is especially important when it comes to introducing a new concept.

Transcription 4 – Introduction to the quotient interpretation.

T — All the answers are correct, excluding the answer "2 parts". But I wanted half [Figure 6]. That is, we shared the chocolate bar by how many children?
Sv — Two.
T — [Writes on the blackboard 1 and, below, 2 – Figure 6] But something is missing there...
Sv — The division.
T — [Makes a dash between 1 and 2 – Figure 6] We put a dash... has the same meaning of division. In the horizontal form we use the symbol of division. In the vertical form we use the dash. All of this is a fraction [pointing to the fraction $\frac{1}{2}$]. What is a fraction? A fraction is a division. Look to the picture [Figure 5] and look to the fraction. What does 1 represent?
Sv — The chocolate bar.
T — What does 2 represent?
Sv — The children.
T — It means that we have 1 chocolate bar that is going to be divided by…
Sv — Two children.
T — So, how much does each boy eat?
Sv — Nine.
Sv — Half.
T — Ok. How much does each boy eat? Read the fraction. Look to the image: 1 to 2, 1 divided by 2.
[…]
T — It means that each boy is going to eat one-half.

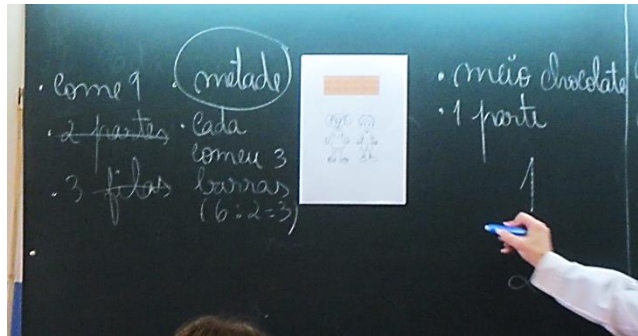Figure 5 – Fair sharing of 1 chocolate bar by 2 children.

Figure 6 – Students' answers on the fair sharing of 1 chocolate bar by 2 children.

Till the end of the first lesson students answered to three tasks more, involving the fractions $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{3}{2}$ in the quotient interpretation. These tasks included a pictorial representation both of the items and the recipients, and students were requested to indicate the fraction that represented the part of item that each recipient would get. In general, students answered correctly. Finally, the teacher clarified some doubts about the verbal representation of the fractions (for example, regarding the fraction $\frac{1}{2}$, some students said "one two" instead of "one-half").

After this first lesson, the teacher was asked to reflect critically on her class. Inês stressed the importance of the preceding collaborative work: "If it wasn't for this work, I couldn't speak so easily about fractions". She also stated that "students adapted well to this way of thinking on fractions…using quotient interpretation… more work needs to be done…they need to practice more".

Lesson 2

In the working meeting, Inês recalled the first observed lesson, stating that "without this [collaborative] work, I wouldn't have started by introducing the quotient interpretation, using instead the part-whole". She also affirmed that "students seemed to understand what was said". Once students reacted very well to the quotient interpretation, Inês declared that this was a good option to start approaching fractions. When questioned about the amount of mathematical subjects approached in one same lesson, the teacher argued that "the students had never spoken before about the division and I thought this could be a good moment for that…Too many things…? These things are going to be approached in other classes… maybe more time was needed for each issue, I guess…more time for fractions…" Indeed, the students just answered to three questions about the representation of fractions.

The researcher stressed that, Inês' students answered correctly to the questions about fractions. However, the selection of pictorial supports when quotient interpretation was involved arose some doubts among students. Inês concluded that, to introduce fractions in quotient interpretation, "it'd have been better to have chosen a picture that wasn't previously divided in equal parts". This didactical option would have revealed a stronger knowledge of content and teaching, since the teacher's selection of tasks would have fitted better her purposes for the lesson. This would avoid the need to highlight some students' answers in detriment of others (see observed lesson 1).

For lesson 2 Inês planned to give students more tasks about the representation of fractions, so that they could practice more in the quotient interpretation.

Quotient interpretation: inaccuracies on the meaning of the fraction

The second lesson observed was devoted to the representation of fractions in the quotient interpretation. This option revealed teacher's knowledge of content and teaching and knowledge of content and students, since students had the opportunity to recall and apply the knowledge about the quotient interpretation acquired in the first lesson.

Inês started the lesson by remembering the meanings of both the numerator and denominator of fractions, in quotient interpretation. The students correctly associated the numerator of the fraction to the number of items to be shared and the denominator to the number of recipients (Transcription 5, Figure 7). Transcription 5 illustrates the dialogue that the teacher promoted on this issue.

Transcription 5 – The numerator and the denominator of fractions in quotient interpretation.

T — What are fractions?

[…]

S1 — We have two numbers.

T — Yes, we have two numbers. One is above and the other below the dash. What represents the one that is above? [Pause] Let's think on the example of the other lesson 1 chocolate bar divided by…

Sv — A boy and a girl.

T — First the chocolate bar was divided by 2 children and then 1 chocolate bar was divided by 3 children [Draws 1 chocolate bar and 2 children on the blackboard] What do you put above the dash [pointing out to the numerator]?

Sv — One chocolate bar… what is going to be shared…

T — And below?

Sv — The children.

T — In this case, there are 2 children sharing 1 chocolate bar. [Writes on the blackboard:

$\frac{1}{2}$ $\longrightarrow$ Chocolate bar

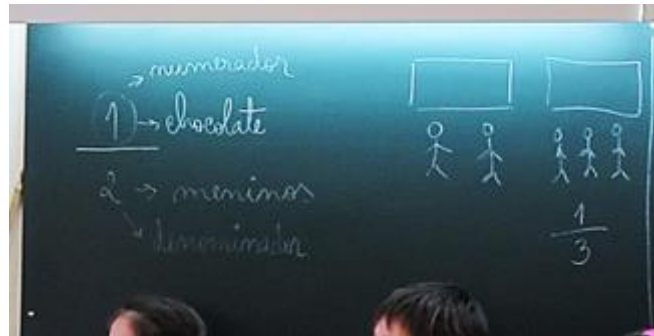$\longrightarrow$ Children   (Figure 7)]

Figure 7 – Fractions $\frac{1}{2}$ and $\frac{1}{3}$ in the quotient interpretation.

However, besides the meaning of the numerator and the denominator, the teacher should also have referred that the fraction represents the amount of item that each recipient receives. Indeed, in quotient situations, one usually starts with two quantities — a and b ($b \neq 0$) — treating a as the dividend and b as the divisor, and obtaining a single quantity — $\frac{a}{b}$ ($b \neq 0$). For example, when 1 chocolate bar (a) is shared among 2 children (b), the fraction $\frac{1}{2}$ represents both the division (1 divided by 2) and the quantity each child will receive ($\frac{1}{2}$). The above-mentioned inaccuracies on the meaning of the fraction can suggest fragilities in Inês' knowledge of content and teaching as the teacher did not approached the meaning of the fraction in the quotient interpretation in a complete way, not even in another moment of the lesson. Till the end of the lesson, students answered to tasks about the representation of fractions in quotient interpretation.

In the individual interview, Inês said to be fulfilled with her lesson plan and pleased with the students' performance and reaction to the subject of fractions, as well as pleasantly surprised by the ease of their learning on representing fractions in the quotient interpretation.

Lesson 3

The working meeting that preceded the third observed lesson fulfilled the goal of clarifying that, in quotient interpretation, the fraction represents not only a relation between items and recipients but also the amount that each recipient receives. Inês stated: "I guess I had these two meanings in mind, but during the class perhaps the concern about students presenting the right fraction for the items and recipients was stronger...". Inês informed that she would approach the comparison of fractions in quotient interpretation. The possibility of applying the proportional reasoning to compare fractions when quotient interpretation is involved (see Nunes *et al.*, 2004) was one of the main topics of the working meeting 3. It was also highlighted that the comparison of fractions in their symbolic form does not need to include the comparison between the correspondent decimal numbers — teachers seemed to use this strategy frequently.

Ordering and equivalence of fractions in quotient interpretation

The third lesson was devoted to the ordering and equivalence of fractions in quotient interpretation. Inês presented tasks to promote class discussion about the comparison of fractions in this interpretation. Firstly, Inês presented tasks involving ordering of fractions with the same numerators. Transcription 6 illustrates a dialogue about the correction of a task involving $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{1}{2}$. In general, the students answered correctly, reasoning on the proportional variation of the number of items.

Transcription 6 – Ordering of $\frac{1}{4}$, $\frac{1}{3}$ e $\frac{1}{2}$ in quotient interpretation.

T —	Why do you think that $\frac{1}{2}$ is greater than $\frac{1}{3}$?
S1 —	Because firstly there are only 2 children and then there are 3… more children to share the chocolate bar… each child eats less.
T —	The more boys share the chocolate bar, the less each eats. Which is the smallest fraction?
Sv —	One-fourth.
T —	And then?
Sv —	One-third.
T —	Which is the greater fraction?
Sv —	One-half.

The teacher concluded that "the more boys share the chocolate bar, the less each eats". This kind of reasoning promotes the understanding of the inverse relation between the fraction and the denominator, when the numerator is the same. The comprehension of this relation is essential to the understanding of the concept of fraction. Often students' mistakes arise from the failure in understanding that natural and rational numbers involve different ideas.

During the class, Inês also presented a task about the comparison of the fractions $\frac{1}{3}$ and $\frac{2}{3}$ in quotient interpretation. One chocolate bar was shared between 3 boys, and 2 chocolate bars were shared between 3 girls. The student that corrected the task on the blackboard wrote: "Each girl eats more chocolate because the girls have two chocolate bars and the boys only have one". The teacher accepted this answer without any reference to the fact that the number of boys and the number of girls are the same in both situations. This aspect is fundamental to conclude correctly about the ordering of the fractions. Apparently, the student only compared the integers 1 and 2, and did not take into account the magnitude of the fractions. Such reasoning does not promote students understanding of a major difference between fractions and whole numbers: in the first set of numbers, two numerical signs are used to represent a single quantity — it is the relation between the numbers, not their independent values, that represents the quantity.

The inattention to such common incorrect reasoning suggests Inês' fragilities on the domain of the knowledge of content and students, on one hand, and on the knowledge of content and teaching, on the other. Indeed, the teacher did not anticipate eventual incorrect type of reasoning of the students when comparing fractions with equal denominators, and she did not have in mind that such comparisons could create an illusory successful performance. Perhaps Inês should have reinforced the selection of problems

that promote a deeper understanding of the inverse relation between the denominator and the magnitude of the fraction, when the numerator is the same.

The equivalence of fractions is also crucial for the understanding of the concept of fraction. This might not be considered simple by the students, given that fractions that have different labels may represent the same quantity.

Inês approached very sparingly the issue of equivalence between fractions in quotient interpretation. Only the equivalence between $\frac{1}{2}$ and $\frac{2}{4}$ was explored. Inês mentioned that 2 girls were sharing 1 pizza fairly and 4 boys were sharing 2 pizzas fairly. She wrote on the blackboard two types of answers "$\frac{1}{2}$ $\frac{2}{4}$" and "$\frac{1}{2}$ $\frac{1}{2}$" – each fraction represented the amount of chocolate that each boy and each girl would eat. Then, the students presented their ideas on the given fractions. Actually, the fact that the students had the opportunity to justify their choices was a very fruitful moment to share and exchange ideas (Transcription 7). However, only very few tasks were explored, more time could have been spent on equivalence of fractions. These didactical options suggest fragilities on teacher's knowledge of content and teaching.

Transcription 7 – Equivalence between $\frac{1}{2}$ and $\frac{2}{4}$ in quotient interpretation.

S1 — It is the first.

T — Why can't it be the second?

Sv — It can.

S2 — It can be both.

T — Why is that, S2?

S2 — In the first they divide the pizzas in half and in the second they do not... the first is how it is in the drawing.

T — Very good! In the first, as S2 said, we do as usual. We look at the picture [draws on the blackboard 2 pizzas and 4 boys] and find that we have 2 pizzas divided fairly by 4 boys. We conclude that each boy eats $\frac{2}{4}$. But the second answer is also correct. As S2 said, we can also divide the pizzas in half [divides the pizzas into 2 equal parts] and we conclude that each girl eats $\frac{1}{2}$.

[…]

T — The boys and the girls eat the same amount… but here [referring to $\frac{1}{2}$] the fraction is more simplified.

In the individual interview, Inês highlighted the students' performance and stressed that they may be too young to learn the comparison of fractions: "The students are very young… many of them understood, but this is hard. They are too young... yet, I think they did well". These comments reflect some teacher's resistance to approach these matters in early grades. It is important to remember that the most recent Portuguese curriculum suggests a much thorough approach to fractions in primary school, when compared to the former curriculum.

Lesson 4

The working meeting 4 focused on the comparison of fractions. Inês recognised that, regarding the ordering of fractions, it is fundamental to approach the inverse relation between the denominator and the magnitude of the fraction, when the numerator remains the same. Inês also agreed that it would be important to stress that this relation is applicable only when the numerator is the same.

In spite of the students' reaction, Inês still felt that the ordering and equivalence of fractions was "for students a difficult matter that needed more time to be consolidated, in forthcoming years".

Part-whole interpretation

Inês' fourth lesson was centered in introducing students to the part-whole interpretation. A circle divided into 4 equal parts in which $\frac{1}{4}$ was written in one part was presented. Then, Inês introduced students to the meaning of the numerator and denominator, in part-whole interpretation (Transcription 8). Moreover, the teacher diversified the reference unit, which was always a circle, a rectangle, a square, etc. (see Figure 8).

Transcription 8 – Introduction to the part-whole interpretation.

T — Here we have a circle that was divided into how many parts?

Sv — Four.

[…]

T — What is written on the painted part?

S1 — One-fourth.

T — Exactly. It has a fraction. What does the number 1 mean?

S3 — That is one part.

T — It's one part of how many equal parts?

Sv — Four.

T — That is, in the numerator we will count the number of parts that are painted and in the denominator we will count the number of parts in which the figure is divided.
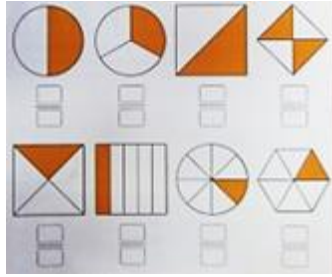
Figure 8 – Fractions $\frac{1}{8}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ e $\frac{2}{4}$ in part-whole interpretation.

Equivalence and ordering of fractions in part-whole interpretation

In one task about the representation of fractions in part-whole interpretation, it was presented a square divided into 4 equal parts, 2 of them were painted. In this context, the teacher approached the equivalence between the fractions $\frac{2}{4}$ and $\frac{1}{2}$. Inês asked students if $\frac{2}{4}$ and $\frac{1}{2}$ could represent the part of the square that was painted (Transcription 9).

Transcription 9 – Equivalence of fractions in pat-whole interpretation.

T — I'm going to ask a question: in the fourth figure [a square divided into 4 equal parts, 2 of them painted] you put $\frac{2}{4}$. Can't it be $\frac{1}{2}$?

Sv — It can.

S1 — Because $\frac{2}{4}$ is $\frac{1}{2}$.

T — Precisely! Because $\frac{2}{4}$ is $\frac{1}{2}$. We have 2 parts painted and 2 parts that are not painted. Half of the square is painted and half is not. It can be $\frac{2}{4}$ or $\frac{1}{2}$, because $\frac{2}{4}$ is $\frac{1}{2}$. [Writes on the blackboard: "$\frac{2}{4}$ or $\frac{1}{2}$"].

In order to consolidate knowledge on equivalence of fractions, Inês could have asked more questions of involving other equivalent fractions such as $\frac{1}{3}$ and $\frac{2}{6}$.

Figure 9 illustrates a task presented to students about the ordering of fractions, in part-whole interpretation. The first bar represented the unit and it was intended that students constitute this initial bar with two equal bars, each of these represented by $\frac{1}{2}$; with 3 equal bars, each of these represented by $\frac{1}{3}$; and so on till 10 equal bars – excepting the cases $\frac{1}{7}$ and $\frac{1}{9}$. The teacher approached the ordering of fractions by referring that for a same numerator, the greater the denominator the smaller the quantity represented by the fraction. However, this was done very briefly and students did not have the opportunity to conclude by themselves about the ordering of the fractions.

Figure 9 – Ordering of $\frac{1}{10}, \frac{1}{8}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}$ and $\frac{1}{2}$ in part-whole interpretation.

Being this a task full of potentialities, it would be interesting, for example, to approach the equivalence of fractions. For example, it would be interesting to challenge the whole class to notice that two bars of $\frac{1}{4}$ have the same size of one bar of $\frac{1}{2}$, and conclude that $\frac{1}{2}$ is equivalent to $\frac{2}{4}$. However, what could be a powerful task in the math class ended poorly explored.

In sum, the equivalence and ordering of fractions in part-whole interpretation was poorly explored by Inês, which suggests fragilities on teachers' knowledge of content and teaching. Indeed, the comparison of fractions is essential to promote a complete construction of the concept of fraction among the students.

In the individual interview, Inês felt to have approached everything that she planned and mentioned that students' reaction was better than she expected: "at beginning, there was the quotient interpretation, and now we spoke about part-whole, and they all did very well…they were comfortable speaking about fractions…"

Lesson 5

In the working meeting 5, Inês mentioned that students naturally integrated the part-whole interpretation and they spoke with familiarity about fractions. For the upcoming lesson, the teacher informed that the operator interpretation would be the main subject and that measure interpretation would not be approached, in spite of being in the curriculum guidelines. This may suggest teacher's insecurities with this interpretation of fraction, traditionally absent during the first grades.

Operator interpretation: emphasis on algebraic procedures

In the fifth lesson, Inês approached the operator interpretation. The teacher presented an example of fraction where the operator situation was involved writing on the blackboard "$\frac{1}{2} \times 6 = 6 \div 2 = 3$" arguing that, "in fractions, we have $\frac{1}{2}$ times 6, we do 1 times 6, that is 6, and 6 divided by 2… 2 times 1, 2; 2 times 2, 4; 2 times 3, 6". Students did not seem to follow these explanations. Instead, the teacher could have started by presenting the meaning of the numerator and denominator in quotient interpretation, followed by the pictorial representation of the situation. And, only afterwards, the approaching to an

expression such as $\frac{1}{2} \times 6 = 6 \div 2 = 3$. Thus, this approach suggests fragilities on teacher' knowledge of content and teaching.

Later, Inês proposed the following task: "John has 8 sweets and ate $\frac{1}{4}$ of them. Highlight the sweets that John ate" (a picture of 8 sweets was presented). Students also had to fill in the blanks "$\frac{1}{4} \times 8 = \underline{\quad}$ or $8 \div 4 = \underline{\quad}$". The teacher suggested: "Firstly, do the computations and afterwards highlight the sweets". Inês' suggestion immediately guided students in order to apply algebraic procedures. Despite doing the correct computations, some students revealed difficulties in highlighting the sweets, suggesting that they did not master the meaning of the numerator and denominator of fractions, in operator interpretation.

Afterwards, Inês approached the pictorial representation of fractions in operator interpretation referring the meaning of the numerator and denominator in this interpretation (Figure 10).
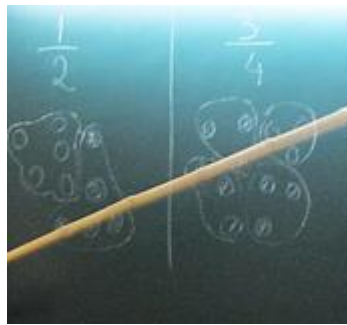

Figure 10 – Representation of fractions in operator interpretation.

Articulation between interpretations of fractions

In the lessons observed, the approach to the subjects taught was too segmented; the teacher rarely interpolated tasks involving different interpretations – Inês began by working in quotient interpretation, then moved to part-whole, and finally to the operator interpretation. The articulation of these interpretations of fractions would have promoted a consolidation and integration of knowledge, and would have revealed stronger knowledge of the teacher on the domain of pedagogical content knowledge regarding the teaching of fractions. This poor articulation of different interpretations of fractions suggests that the teacher either did not recognize its importance when building on the concept of fractions, or felt uncomfortable on doing this articulation.

In the individual interview, Inês stated that "not all the students succeeded in doing the computations… I chose the tasks from the textbook because these seemed very clear… but some of the students had some difficulties…" Indeed, students seemed to apply to the computations procedures without understanding. The algebraic procedures should be preceded by an approach to the meanings of the numerator and denominator in operator interpretation.

In the final working meeting, concerning operator interpretation, Inês agreed that students' difficulties resulted from an insufficient approach to the meanings of the

numerator and denominator. Notwithstanding, teachers highlighted the positive impact of the collaborative work program on her lessons and on students' learning.

6. Discussion and conclusions

To approach the work with fractions, Inês relied on fair sharing situations, as preconized in the Portuguese curriculum guidelines (see MEC-DGE, 2013). The teacher´s knowledge of content and teaching was well evinced in several moments, namely when using tasks that motivate students for the subject, when selecting situations that were close to students' everyday life, and when selected the quotient interpretation to introduce fractions, as this interpretation can be very helpful for children to establish the connection between their informal ideas about quantities represented by fractions and the formal representation of fractions (Nunes *et al.*, 2004; Mamede, 2008; Streefland, 1991).

Nevertheless, the results suggest some fragilities regarding the pedagogical content knowledge, which is considered one of the bases for teaching (see Shulman, 1986, Ball *et al.*, 2008). More specifically, this teacher presented difficulties concerning the subdomains of knowledge of content and teaching, and knowledge of content and students (Ball *et al.*, 2008), regarding the teaching of fractions. In order to introduce the concept of fraction, the teacher considered necessary to introduce formally the operation of division by using the symbol ÷, and also the multiplication as an inverse operation of the division. However, these are too many mathematical ideas to approach in only one lesson, and are not essential to understand fractions in the context of fair sharing situations. Inês seemed to ignore these issues, suggesting fragilities in the subdomain of Ball´s *et al.* (2008) knowledge of content and teaching.

In the task selection, some inconsistencies occurred, due to discrepancies between what is asked and the kind of answers that are considered correct, revealing fragilities in Inês' knowledge of content and teaching. In quotient interpretation, when a pictorial support includes items previously divided into equal parts, the students tend to give answers that consider different unit references. For example, when an item is shared between two recipients and previously divided into 18 equal parts, students tend to answer that each recipient will get 9 parts of it. In such situations, the teacher easily emphasized the students' answers that met the idealized purpose for that task — one-half — instead of considering all the correct answers. Alternatively, the teacher could have wisely selected a pictorial support not previously divided into equal parts.

Concerning the quotient interpretation, the results suggest some inaccuracies on the explanation of the meaning of fraction. Indeed, a fraction represents, not only a relation between items and recipients, but also the amount of item that each recipient receives (see Nunes *et al.*, 2004; Mamede, 2008). This last and important aspect tends not to be stressed properly by the teacher, which suggests, once again, fragilities in teacher's knowledge of content and teaching.

Regarding the operator interpretation, the results indicate that there is a tendency to an excessive emphasis on algebraic procedures, through stress on the rule "multiplies by the numerator and divides by the denominator" to calculate $a \times \frac{b}{c}$, being a, b and c whole numbers ($c \neq 0$). However, it is important to approach the meaning of both the numerator and denominator in order to explain and justify those computations. This difficulty with fractions in operator interpretation was also reported by Pinto and Ribeiro (2013), who

revealed that prospective teachers use algebra, completely dissociated of pictorial representations made by themselves.

The understanding of the ordering and equivalence of fractions is essential to understand fully the concept of fraction. Regarding the quotient interpretation, it was observed that the inverse relation between the denominator and the magnitude of the fraction, when the numerator is constant, tends to be achieved. Yet, it was also observed the use of tasks involving ordering of fractions with the same denominator, which seems to evince some fragilities in teachers' knowledge of content and teaching, and knowledge of content and students. This type of tasks promotes an illusory successful performance of the students with fractions as they can accomplish the tasks successfully only by reasoning on whole numbers. Similar results are presented by Nunes, Bryant, Pretzlik and Hurry (2006) when reporting 4th and 5th graders high levels of success when comparing fractions with the same denominator (3/7 and 5/7), but saw these success levels reduced to its fourth part when the numerators were the same (3/5; 3/4). Nunes *et al.* (2006) argue that, when the numerator is the same and the denominator varies, the students have to consider the value of the fractions in a way that is not in agreement with the ordering of natural numbers. Regarding part-whole interpretation, a very brief incursion was made by the teacher on the ordering and equivalence of fractions. Concerning the operator interpretation, these issues were not mentioned at all.

It is also important to provide the students opportunities to establish connections between the several forms of representation of fractions (see Mamede, 2008). However, this seems to be promoted poorly in the lessons observed. Generally, the tasks tend to be implemented in a segmented way, i.e., when an interpretation of fraction is approached, only tasks on that interpretation are selected. Bright, Behr, Post and Wachsmuth (1988) argue that teachers easily articulate different forms of representation but often are unaware of such articulation. Moreover, the authors stress that students need explicit help in learning to perform these articulations.

In spite of not being able to generalize, this study pointed out some of the teachers' fragilities that can be common to many other Portuguese elementary school teachers. Inês was enthusiastic about teaching fractions, but soon realized that this is a very challenging and difficult topic to teach, both for its novelty in the curriculum and its complexity. Teacher's fragilities identified here suggest that modifications on the curriculum guidelines are not always accompanied by modifications on teaching practices. In-service teacher training, based on collaborative working programs or other models, should regularly be available for teachers.

7. References

Ball, D. L., Thames, M. H., & Phelps, G. C. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education, 59*(5), 389-407.

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational Number, Ratio, and Proportion. In Douglas A. Grouws (Ed.), *Textbook of Research on Mathematics Teaching and Learning* (pp.296-333). New York: MacMillan Publishing Company.

Behr, M., Lesh, R., Post, T. & Silver, E. (1983). Rational-Number Concepts. In R. Lesh and M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 92-127). New York: Academic Press.

Bogdan, R., & Biklen, S. (1999). *Investigação qualitativa em educação: Uma introdução à teoria e aos métodos*. Porto: Porto Editora.

Bright, G., Behr, M., Post, T., & Wachsmuth, I. (1988). Identifying fractions on number lines. *Journal for Research in Mathematics* Education, *19*(3), 215-232.

Kerslake, D. (1986). *Fractions: Children's Strategies and Errors – A Report of the Strategies and Errors in Secondary Mathematics Project*. Berkshire: NFER-NELSON.

Kieren, T. (1993). Fractional numbers: from quotient fields to recursive understanding. In T. P. Carpenter & E. Fennema & T. Romberg (Eds.), *Rational Numbers: An Integration of Research* (pp. 49-84). Hillsdale, NJ: Erlbaum.

Mack, N. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education*, 32, 267-295.

Mamede, E. (2008). Um pouco mais sobre Frações. In Ema Mamede (Ed.), *Matemática — ao encontro das práticas — 2.º ciclo*. Braga: Universidade do Minho – Instituto de Estudos da Criança.

Mamede, E., & Pinto, H. (2015). O ensino de frações – conhecimento de futuros professores do ensino básico. *Caderno de resumos Conferência Internacional do Espaço Matemático em Língua Portuguesa*. Coimbra: Universidade de Coimbra.

ME-DEB (2004). *Organização Curricular e Programas Ensino Básico – 1.º Ciclo*. Lisboa: Departamento da Educação Básica. Ministério da Educação. (4.ª edição).

MEC-DGE (2012). *Metas Curriculares do Ensino Básico – Matemática*. Lisboa: Direção Geral de Educação. Ministério da Educação e Ciência. Acedido em http://www.dge.mec.pt/sites/default/files/Basico/Metas/Matematica/programa_ma tematica_basico.pdf

MEC-DGE (2013) *Programa de Matemática do Ensino Básico*. Lisboa: Direção Geral de Educação. Ministério da Educação e Ciência. Acedido em http://dge.mec.pt/metascurriculares/index.php?s=directorio&pid=17

Merriam, S. B. (1988). *Case study research in education: A qualitative approach*. San Francisco, CA: Jossey-Bass.

Monteiro, C., & Pinto, H. (2005). A Aprendizagem dos números racionais. *Quadrante*, *14*(1), 89-104.

Nunes, T., Bryant, P., Pretzlik, U., Evans, D., Wade. J. & Bell, D. (2004). Vergnaud's definition of concepts as a framework for research and teaching. *Annual Meeting for the Association pour la Recherche sur le Développement des Compétences*, Paper presented in Paris: 28-31, January.

Nunes, T., Bryant, P., Pretzlik, U., & Hurry, J., (2006). *Fractions: Difficult but Crucial in Mathematics Learning,* London: ESRC - Teaching and Learning Research Programme. Acedido em (http://www.tlrp.org/pub/documents/no13_nunes.pdf)

Pinto, H., & Ribeiro, C. M. (2013). Conhecimento e formação de futuros professores dos primeiros anos – o sentido de número racional. *Da Investigação às Práticas, 3*(1), 80-98.

Post, T., Harel, G., Behr, M. & Lesh, R. (1991). Intermediate Teachers' Knowledge of Rational Number Concepts. In E. Fennema, T. Carpenter, S. Lamon (Eds.),

Integrating research on teaching and learning mathematics (pp. 177-198). NY: State University of NY Press.

Saraiva, M., & Ponte, J. P. (2003). O trabalho colaborativo e o desenvolvimento profissional do professor de Matemática. *Quadrante, 12*(2), 25-52.

Shulman, L. S. (1986) Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), pp. 4-14.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research*. Norwell, MA: Kluwer Academic Publishers.

Tirosh, D., Fischbein, E., Graeber, A. O., & Wilson, J. W. (1998). *Prospective elementary teachers' conceptions of rational numbers.* Retrieved from: http://jwilson.coe.uga.edu/Texts.Folder/tirosh/Pros.El.Tchrs.html.

Yin, R. (2010). *Estudo de caso. Planejamento e métodos* (4.ª ed.). Porto Alegre: Bookman.