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Validating the constitution of mathematical knowledge through exploratory and investigative tasks in the classroom

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Abstract: The discourse encouraging the validation of mathematical knowledge established in the classroom is recurrent in literature. From this premise, this work, aims to understand how it is expressed in teaching practices through exploratory and investigative tasks found in scientific productions published in journals. The research approach is qualitative, according to the phenomenological view, and the interpretation of the data is hermeneutic. On this occasion, the validation of mathematical knowledge constituted exploratory tasks, and investigative tasks proved to be absent, such as language, empirical practice, social practice, and demonstration. Although they may be related, how validation manifests shows us distinct epistemological dimensions, as it contemplates different techniques and levels of formality to communicate the legitimacy of the constituted knowledge.

Keywords: Philosophy of Mathematical Education. Math Teaching. Demonstration.

Validación de conocimientos matemáticos constituidos con tareas exploratorias y tareas investigativas en el aula

Resumen: El discurso que incentiva la validación de los conocimientos matemáticos establecidos en el aula es recurrente en la literatura sobre el tema. A partir de esta premisa, en este artículo el objetivo es comprender cómo ésta (la validación) se manifiesta en las prácticas docentes realizadas con tareas exploratorias y tareas investigativas, comunicadas en producciones científicas publicadas en revistas. El enfoque de investigación adoptado es cualitativo, según la visión fenomenológica; y la forma en que interpretamos los datos es de naturaleza hermenéutica. Con motivo de esto, la validación del conocimiento matemático constituida con tareas exploratorias y tareas investigativas resultó ser una ausencia, como lenguaje, como práctica empírica, como práctica social y como demostración. Las formas en que se manifestó la validación, si bien pueden estar relacionadas entre sí, muestran distintas dimensiones epistemológicas al contemplar diferentes técnicas y diferentes grados de formalidad para comunicar la legitimidad del conocimiento constituido.

Palabras clave: Filosofía de la Educación Matemática. Enseñanza de las Matemáticas. Demostración.

A validação do conhecimento matemático constituído com tarefas exploratórias e com tarefas investigativas em sala de aula

Resumo: O discurso de incentivo à validação do conhecimento matemático constituído em sala de aula é recorrente na literatura sobre o tema. Dessa premissa, neste artigo, o objetivo é compreender como ela (a validação) mostra-se em práticas de ensino efetivadas com tarefas exploratórias e tarefas investigativas, comunicadas em produções científicas publicadas em periódicos. A abordagem de pesquisa assumida é qualitativa, segundo a visão fenomenológica; e o modo como interpretamos os dados é de cunho hermenêutico. Por ocasião do realizado, a validação do conhecimento matemático constituído com tarefas exploratórias e com tarefas investigativas mostrou-se como ausência, como linguagem, como prática empírica, como prática social e como demonstração. Os modos pelos quais a validação manifestou-se, ainda que possam estar relacionados entre si, mostram dimensões epistemológicas distintas ao contemplarem diferentes técnicas e diferentes graus de formalidade para comunicar a legitimidade do conhecimento constituído.

Palavras-chave: Filosofia da Educação Matemática. Ensino de Matemática. Demonstração.

1 Introduction

Mathematical knowledge has long been crystallized in axiomatic statements, whose epistemic value is supported by the parameters of deductive logic and communicated with the rhetoric of demonstration. Although there is some disagreement about its ontology, mathematical scientists have a consensus that it (demonstration) is a fundamental element in constructing and validating mathematics.

Regarding the phenomenology of mathematical research, Wichnoski's study (2021) revealed that the scientific practice of producing mathematical knowledge inspires how we understand mathematical research in mathematics education and that demonstration is fundamental for validating conjectures. In other words, it revealed that from the perspective of Mathematical Investigation in Mathematics Education, the epistemic value of a conjecture is established through demonstrations. However, it was not possible to clarify how it is understood in the field of Mathematical Investigation nor the "rigor with which it should be done (or required) at different levels of schooling" (Wichnoski, 2021, p. 149).

In this research, we found only "indications that a demonstration is a process that advances towards the formalization of doing mathematics, through deductions and logical arguments" (Wichnoski, 2021, p. 147), and, on the other hand, in an idiosyncratic way, that it can be made more flexible, "removing the weight of rigor and mathematical formalization, without removing its importance as a mathematical communication capacity" (Wichnoski, 2021, pp. 147-148). This antagonism in understanding what demonstration is in the context of Mathematical Investigation provokes questions that open up possibilities for research, such as understanding how it manifests itself in teaching and learning mathematics in the classroom.

The texts on the subject, at least those with the greatest circulation in the area, refer to this moment as *demonstrating*, *proving* (Ponte, 2003; Mata-Pereira & Ponte, 2018), *showing* (Brunheira & Ponte, 2019), and *justifying* (Ponte, Brocardo & Oliveira, 2013). As a result, we realized that, over the years, different expressions have been used to designate the same element: demonstration, which allows us to consider it, in the context of published research, as similar to these adjectives.

Demonstrate. In the lexicon, this term means "to prove by conclusive reasoning; to prove [...] to show" (Ferreira, Anjos, Ferreira, Geiger & Barcellos, 2010, p. 225). Also, in the lexical sense, justifying is related to "demonstrating or proving [...] presenting the reason for (a procedure, way of thinking, etc.) or the explanation for (a fact, etc.)" (Ferreira et al., 2010, p. 450). It should be noted that the meanings of *demonstrate, prove, show,* and *justify* are intertwined in these definitions and express an ordinary meaning: the idea of making legitimate what is done. This thinking leads us to understand them as ways of *validating*, which implies focusing on demonstration from the point of view of validation.

Wichnoski's study (2021) was based on significant academic works published between 1996 and 2013, which considered exploratory tasks as possible ways of dealing with Mathematical Investigation in the classroom, especially in the first experiences with this type of work, as Brocardo (2001, p. 120) says: "a more structured task may be more suitable for students who are beginning to have their first experiences of investigation, without this meaning a lower quality of the task as a proposal for an investigative task".

However, in the most recent context of research in Mathematics Education, the nature of exploratory tasks has been the main argument for putting into vogue another teaching paradigm: Exploratory Teaching. Ponte (2020) tells us that

> the differences between the investigation and exploration tasks, let's say, they are a little bit, there in continuity with each other [...] in principle we only talked about investigation tasks, but at a certain point we thought it was better to distinguish between the tasks The simplest, the simplest investigation tasks we started to call exploration tasks [...] when we have an investigation or an exploration we formulate conjectures and generalizations... and therefore, let's say... the idea of the investigation work continues to be here (verbal information)[1].

Furthermore, in an interview, in an interview, Ponte (2022) tells us that he has been "working on the concept of Exploratory Teaching, within which new ideas are developed about how to structure mathematics lessons and which deepens ideas already introduced concerning Mathematical Investigation" (Ponte, 2022, p. 13). In view of this, and without intending to enter into this discussion, we understand that, although Exploratory Teaching grows in other directions and to some extent differs from Mathematical Investigation, it is connected with it in principle.

However, it is critical to note that the nature of the tasks, in itself, is not a sufficient condition to characterize a teaching perspective, since it requires, in addition to this, other conditions, such as ways of being a teacher, ways of being student, contexts and intentions. In this sense, we are not, here, taking positions regarding the (dis)junction between Mathematical Investigation and Exploratory Teaching, but only considering that, based on the research of Wichnoski (2021), exploratory tasks are possible ways of being with Mathematical Investigation in the classroom. Obviously, this type of pedagogical work cannot be reduced solely to the nature of the proposed task, which is a necessary condition, but not sufficient to characterize it, as we said.

Therefore, even though the research that gave rise to this work focused on Mathematical Investigation, and that, in it, the validation of conjectures is fundamental and important, we believe it is pertinent to consider as the primary material of this research the report of Mathematics teaching practices carried out with investigative tasks and with exploratory tasks, given the aforementioned theoretical transitions, whose boundaries still present some uncertainty, at least for us. In order to understand the validation of mathematical knowledge constituted with these tasks, with a phenomenological-hermeneutic stance we question: *how does the validation of the mathematical knowledge constituted with exploratory tasks and investigative tasks in the classroom appear?*

We can understand the research movement presented here as a conscious experience, exemplified by the cube metaphor:

[1] Lecture given by Professor João Pedro da Ponte in the 1st Cycle of Lectures of the Degree Course in Mathematics at the State University of Paraná – Campus de União da Vitória, online, on August 11, 2020.

> Consider how we perceive a material object, such as a cube. We see the cube from an angle, a perspective. We can't see the cube from all sides at once. It is essential for the experience of a cube that perception is partial, with only a part of the object being directly given at any time. However, it is not the case that we only experience the visible sides from our present point of view. [...] other sides are given, but given precisely as absent. They are also part of our experience (Sokolowski, 2012, p. 25).

In this respect, the sides that compound a cube are presented in perspective; they are given in different ways, called aspects. In turn, an aspect can be provided through a succession of temporally different appearances, called profiles. Therefore, all this multiplicity speaks of the same cube; in other words, "the sides, aspects, and profiles are presented to us, but in them, all the same cube is being presented" (Sokolowski, 2012, p. 28). Philosophically, this implies that an aspect of the cube that was absent at the time of the aforementioned doctoral research is now brought to light by the profile we focus on. Therefore, the research presented in this paper is an extension and reprise of Wichnoski's thesis (2021), as it focuses on the cube (Mathematical Research in Mathematics Education) from another aspect (validation) and under another profile (exploratory tasks and investigative tasks).

How the research question was elaborated and expressed guides a research movement focusing on how mathematical knowledge is validated through exploratory and investigative tasks in the classroom. Therefore, it signals the research's region of inquiry, whose exposition of some theoretical elements constitutes the content of section 2.

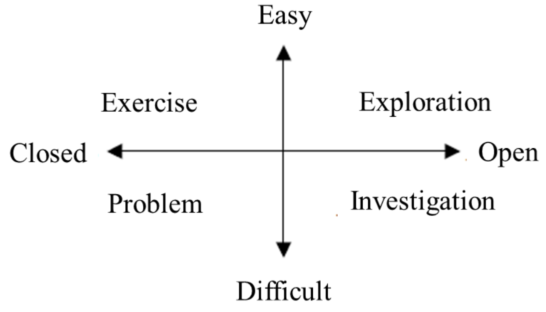
2 Theory notes

"Can the research work of mathematicians serve as inspiration for the work to be done by teachers and students in math classes?" (Ponte, Brocardo & Oliveira, 2013, p. 9). With this question, the authors start discussing what Mathematical Investigation activities are in the educational context, the consequences of which converge towards an affirmative answer. By bringing genuine mathematical task into the classroom, Mathematical Investigation facilitates mathematical thinking, which can be fostered with "activities that involve students in open-ended problems and mathematical explorations and investigations. These deal with fundamental mathematical task and thinking processes, such as formulating problems, making and demonstrating conjectures, or communicating discoveries" (Abrantes, 1999, p. 1).

Regarding exploratory tasks and investigative tasks, Ponte (2017) points out that they differ from other types of mathematical activities by the characteristics of openness in the enunciative structure and by the level of difficulty they contain, in such a way that investigative tasks have elements of vagueness in the enunciative structure and greater levels of difficulty than exploratory tasks. The levels of challenge refers to the perceived difficulty, and the level of structure refers to the openness of the tasks, varying between open and closed. Combined, they generate different types of task, namely exploratory and investigative, as shown in Figure 1.

Based on our interpretation of the diagram in Figure 1, investigative and exploratory tasks are open-ended and involve a specific "significant level of indeterminacy in what is given, what is asked, or both" (Ponte, 2017, p. 113). Therefore, from a pragmatic point of view, the level of openness of the enunciative structure of exploratory tasks is lower than that of investigative tasks. Comparing the level of openness and challenge, Ponte (2017, p. 114) recognizes that "not all open tasks carry a high level of challenge" but considers that exploratory tasks are easy and investigative tasks are complex, indicating a cause-and-effect relationship between openness and the level of challenge, which culminates in difficulties.

Figure 1: Relationship between different types of tasks, in terms of their level of challenge and openness



Source: Ponte (2017, p. 113)

It is relevant and critical to note that the degree of openness is not only guaranteed by the enunciative structure of the tasks, whether they are exploratory or investigative, because the texts of their enunciations unfold the activity, which, in turn, is always someone else's; another person about the author of the task. In this sense, although openness is present to a greater or lesser degree in the wording of tasks, it is not exclusive to them because it also depends on how the subjects engage with them, interpret them, and attribute meanings to them.

With these explanatory theoretical notes on exploratory tasks and investigative tasks, as they are conceived in the literature, we will now explain our stance on the research and the procedures derived from it. It is the content of section 3.

3 Research stance and procedures

It is with the phenomenological-hermeneutics stance that the intention of this research was pursued. It means not only how to proceed but also how researchers exist in the world and relate to their studies are constituted by a perceived reality. Thus, what is being questioned is not objectively given in the world but is formed in it through perceptual acts and, is seen as a *phenomenon*. In opposition to the Cartesian way, phenomenology states that the subject and what they question (the phenomenon) are united in the process of knowing, or "there is no separation between what is perceived and the perception of the one who perceives since a correlation of harmony is required, understood as giving, in the sense of exposure, between the two" (Bicudo, 2011a, p. 19).

Phenomenologically, in perceptual acts, what is perceived is, has been, and is no longer what it has just been. In other words, we can easily perceive that once the present moment has passed, we have the expression of what is seen through language, which requires analysis and interpretation procedures. Therefore, to proceed phenomenologically in research is to carry out "the very movement of working with senses and meanings that are not given in themselves, but are constituted and shown in the historical temporality of their durations and respective expressions mediated by language" (Bicudo, 2011b, p. 41).

According to Bicudo (2011b), the interpretation of expressions mediated by language requires a hermeneutic graft to reveal meanings and significance. Given this, this research took the path of hermeneutic phenomenology, which allowed us to transcend modes of understanding attached to the objectivity of the word and achieve an experience in and through the historical and cultural context of those who, living, understand, and interpret.

Regarding that, understanding is a way of being and "becomes possible because man inhabits a world that is not the universe as seen by the scientist, nor the totality of all beings, but the totality of relationships in which man is immersed" (Hermann, 2002, p. 34). Thus, hermeneutic phenomenology takes us out of the naivety contained in the scientist's view and gives us the possibility of interpreting, understanding, and producing knowledge, not as a mental attribute or emanating purely from the phenomenon being questioned but as a corporeal attitude, always with the world.

Turning contemplatively to the research question and focusing on *what* it asks makes the ways to deal with it possible with math teaching practices performed with exploratory tasks and investigative tasks, reported and shared in academic articles. In the *Scientific Electronic Library Online* (SciELO)[2] and Periódicos Capes[3], a significant collection for the research was searched using the indexers: *exploratory task(s), investigative task(s), mathematical investigation task(s)*, and their combinations.

After reading the articles suggested by the search engines, we identified those that reported on math teaching practices carried out with exploratory and investigative tasks. We selected 11 articles for the research analysis collection, hereafter called primary material, shown in.

It is essential to mention that we do not question the conceptual legitimacy of the tasks analyzed because we understand that the respective authors of the scientific productions that include them do so.

Chart 1: Primary research material

Identification	Title	Authors
2008	The study of functional relations and the development of the concept of variable in 8th grade students.	Ana Matos João Pedro da Ponte
2012	Mathematical reasoning in elementary and higher education students.	João Pedro da Ponte Joana Mata-Pereira Ana Henriques
2014	Representations as support for students' mathematical reasoning when exploring investigation activities	Ana Henriques João Pedro da Ponte
2018	Promoting students' mathematical reasoning: a design-based investigation.	Joana Mata-Pereira João Pedro da Ponte
2019	Justifying geometric generalizations in the initial training of early years teachers.	Lina Brunheira João Pedro da Ponte
2020ª	Mathematical reasoning in the early years: two teachers' actions when discussing tasks with their students.	Eliane Maria de Oliveira Araman Maria de Lurdes Serrazina João Pedro da Ponte
2020B	Mathematical reasoning processes in solving exploratory tasks in the 3rd grade.	Eliane Maria de Oliveira Araman Maria de Lurdes Serrazina
2020C	Mathematical reasoning processes mobilized by 6th grade students when solving a geometry task	Luís Felipe Gonçalves Carneiro Eliane Maria de Oliveira Araman

[2] https://www.scielo.br
[3] https://www-periodicos-capes-gov-br.ezl.periodicos.capes.gov.br/index.php?

Identification	Title	Authors
		Maria de Lurdes Serrazina
2020D	Mathematical investigation: a possibility for teaching 1st degree functions.	Rosimiro Araujo do Nascimento Marli Teresinha Quartieri
2021ª	Mathematical reasoning processes mobilized by calculus students in tasks involving graphical representations.	André Luis Trevisan Eliane Maria de Oliveira Araman
2021B	Arguments presented by calculus students in an exploratory task.	André Luis Trevisan Eliane Maria de Oliveira Araman

Source: Own elaboration

The articles that formed the primary material were used as the basis for manifesting what we intend to do in this research because their scope includes accounts of the author's experiences with the phenomenon being questioned here. We must emphasize that we do not attribute any judgment to them since the specificities with which they were constructed are irrelevant in the light of our question, i.e., our interest lies in what is exposed about the moment of validation of mathematical knowledge constituted with exploratory tasks and with investigative tasks, and not in the articles themselves.

Once the search and selection of primary material was complete, we first read to understand the meanings in all the texts. In a second reading, we highlighted excerpts whose content contained essential aspects of the phenomenon being questioned and which were linked to the research question. With those excerpts, we built a text on the text of the excerpts, called a meta excerpt, to expose our understanding of what was said, making it clear and consistent with the region of inquiry of the research.

It is important to emphasize that the description with the meta excerpts does not describe what is perceived directly and immediately, as is assumed in supposedly objective observation with positivist roots, but instead describes it as a mode of expression that is always intertwined with the world (Bicudo, 2011b). With the meta-excerpts, we constructed units of meaning to bring together the meanings that were distinguishable and present in the previous description. Chart 2 exemplifies this movement.

Chart 2: Constitutive movement of meaning units

Text excerpt	Meta excerpt	Meaning units
Students can also use deductive reasoning based on mathematical definitions and properties or by performing treatments within the algebraic representation system to formulate conjectures that acquire validity and a general nature. In this case, algebraic representation is used to explore and present conjectures and formal justifications.	When analyzing teaching practice, the authors conclude that conjectures acquire validity and a general nature because students can use deductive reasoning based on mathematical definitions and properties or perform treatments within the algebraic representation system to formulate conjectures that acquire validity and a general nature.	Conjectures acquire validity and a general nature through deductive reasoning. (2014.17) Algebraic representation is used as a tool for formal justification. (2014.18))

Source: Own elaboration

Searching for the most comprehensive meanings, based on the individual characteristics expressed in each unit of meaning, we crossed the units with confluent meanings, thus constituting the nuclear ideas representing the first invariants. However, concurrent meanings were still perceived, which called for another convergence, revealing the nuclei of ideas, which concluded the process of phenomenological reduction because they expressed the essential aspects of the phenomenon being questioned that, although manifested in different ways, did not change in meaning. These nuclei are *N.1 — validation as absence, N.2 — validation as language, N.3 — validation as empirical practice, N.4 — validation as social practice, and N.5 — validation as a demonstration*. The movement of phenomenological reduction described, from the units of meaning to the nuclei of ideas, is shown in Chart 3.

Chart 4: Movement of phenomenological reduction constituting the nuclei of ideas

Códigos das unidades de significados	Nuclear ideas	Nuclei of ideas
(2008.6) (2008.7) (2012.2) (2012.7) (2012.10) (2012.11) (2012.12) (2012.15) (2014.16) (2014.19) (2018.8) (2019.10) (2020B.8) (2020C.4) (2020D.4) (2021A.3) (2021A.5) (2021A.16) (2021A.17) (2021B.1) (2021B.2)	On the absence of validation	N.1 — Validation as absence
(2008.2) (2008.3) (2008.5) (2012.6) (2012.9) (2012.14) (2012.20) (2012.22) (2012.23) (2014.2) (2014.3) (2014.9) (2014.13) (2019.14) (2014.18) (2019.8) (2020A.1) (2020D.5) (2021A.8) (2021A.14)	The use of mathematical language	N.2 — Validation as language
(2008.1) (2012.21) (2014.1) (2014.6) (2014.8) (2014.12) (2014.15) (2019.5) (2019.8)	The use of natural language	
(2008.4) (2012.20) (2019.18) (2020B.2) (2020B.4) (2020B.6) (2020B.7) (2020B.9) (2020C.10) (2020D.1) (2021A.1) (2021A.2)	On the use of particular cases	N.3 — Validation as empirical practice
(2018.1) (2018.2) (2018.3) (2018.4) (2018.5) (2018.6) (2018.7) (2018.9) (2018.11) (2020A.1) (2020A.2) (2020A.5) (2020B.1) (2020B.3) (2020B.5) (2020B.6) (2020B.10) (2020B.11) (2020C.2) (2020C.3) (2020C.5) (2020C.6) (2020C.7) (2020D.3) (2021A.4) (2021A.7) (2021A.9) (2021A.10)	About peer validation	N.4 — Validation as a social practice
(2020A.4) (2021A.12) (2021A.13) (2020C.1)	On counter-evidence	N.5 — Validation as demonstration
(2012.4) (2012.13) (2012.18) (2012.19) (2014.10) (2014.5) (2014.17) (2019.16)	On the deductive process	
(2012.5) (2012.17) (2014.7) (2014.11) (2019.6) (2019.7) (2019.9) (2019.11) (2021A.15) (2021B.3) (2021B.4)	About mathematical proof	

Source: Own elaboration

Using the nuclei of ideas, we constructed a descriptive text to expose the aspects we felt and perceived about the phenomenon in question. We were guided by careful listening to what the primary material texts revealed and, as we have said, by a supposedly prudent stance, free of a priori judgments about what is said in them. That is explained in section 4.

4 Description and interpretation of the nuclei of ideas

The senses and meanings revealed by the units of meaning are now articulated with the description and interpretation of the nuclei of ideas, bringing to light the understanding of how the validation of mathematical knowledge is shown in math teaching and learning practices with exploratory and investigative tasks.

The first nucleus of ideas, N.1, shows validation as an absence in the constitution of mathematical knowledge with exploratory tasks and investigative tasks in the classroom. The units of meaning articulated in it expressed that there is greater value placed on the solution than on the inherent justifications (2008.7) (2021A.3); furthermore, they expressed that students do not feel the need to validate a rule (2012.12) or to justify conjectures (2012.11) (2014.16) (2020C.4), tending to generalize them without validating them (2012.15) (2014.19) (2018.8) (2020B.8) (2021B.1). When attempts at validation were made, they were incomplete, as we can see from the meaning units: *the justification was based on incomplete geometric structuring (2019.10); the rationale lacked mathematical support* (2021A.17) (2021B.2).

The absence of validation was also shown to be a consequence of the difficulties encountered by the students in producing it. We see, in different units of meaning, that *the subject found it challenging to demonstrate the initial conjecture (2012.7), that the subject couldn't formulate a justification verbally* (2021A.5), *that the subject couldn't validate the conjecture* (2021A.16); *that the subject couldn't to explain why he was doing what he was doing* (2008.6); *that the subject couldn't prove that all triples of consecutive numbers are multiples of three* (2012.2); *and that the students couldn't justify precisely* (2020D.4).

The nucleus of ideas N.1 reveals the students' denial of the validation of mathematical knowledge built up through exploratory and investigative tasks. Epistemologically, the units of meaning of this core are expressed through two actions of the subjects: the failure to progress in the analysis of conjectures or the voluntary intention to deny the need to justify conjectures. These two epistemological positions are distinguished in that the failure comes from one or more attempts by the subjects to formulate a (conclusion) to what they were investigating and, therefore, provides the teacher with information about the epistemological obstacle preventing their progress, while the voluntary intention to deny the need for justification imposes, in the progress of the activity, suppression of the subjects' epistemological obstacles and prevents the progress of the activity on the part of the teacher.

For example, failure could be seen at different levels, in the meaning units (2008.6) (2012.2) and (2012.7), which illustrate cases in which the subjects are unable to produce a validation for the conjecture they are analyzing, or in the meaning units (2019.10), (2020D.4) and (2021A.17) which show an inability to be precise, complete or formal in the justification produced. On the other hand, the subjects' voluntary intention could be seen in meaning units such as (2008.7) (2012.10) (2012.11) (2012.12) (2012.15) and (2014.16) which, unlike the failure situations, provide no evidence of understanding, doubts or the subjects' ability to manipulate the conjectures analyzed to validate them mathematically.

It is in this sense that the nucleus of ideas N.1 expresses validation as an absence insofar as it consists of a break in the exploratory or investigative process about its final objective of constructing the validation of conjectures and, in the best of cases, provides the teacher with evidence of the origin of this break by the failure of subjects at different stages of the analysis of conjectures. In Brocardo's study, the voluntary intention to deny the need for validation was shown to be present in the student's experiences with research tasks, which they considered "the proof" of their conjectures as an unnecessary complication introduced by the teacher" (p. 544)" The study above also reveals that, during the activity, the students were more sensitive to the proof (validation) of their conjectures; however, they considered it to be external to the investigation itself.

With nucleus N.2, the validation of the mathematical knowledge built up with the exploratory and investigative tasks in the classroom is shown as *language*, using various forms. One of the forms of language used by the students was natural language to generalize (2008.1), as well as *to present and justify their reasoning* (2014.1) (2014.6) (2014.15) (2019.5) (2019.8). In addition, they used the language of mathematics to communicate the validation of their knowledge, i.e. *they used mathematical terminology correctly to present and justify their reasoning* (2014.2), using algebraic (2008.2) (2012.6) (2012.22) (2014.18), graphical (2008.5) (2012.9) (2021A.14) and tabular (2014.13), as well as the articulation between algebraic and graphical languages (2012.14), and between tabular and graphical languages (2020D.5). The non-algebraic languages were complemented with natural language for the justifications (2014.8) and, in some cases, *the generalization was expressed in an increasingly formal way (2008.3); in others, without much formalization* (2012.23) (2014.3) (2014.9) (2019.14).

Thus, the nucleus of ideas N.2 expresses validation as a language as it consists of a set of units of meaning that share an initial effort to validate conjectures that are not very formal, based on intuitive actions. Of course, these resources are also means by which investigative activity leads to demonstration; however, the intuition referred to here refers to the gap between the subject's perception and the generalization of conjectures.

In some excerpts, the meaning units expressed intuitions related to the particular characteristics of the objects analyzed, as follows: *the solution found is verified graphically (2008.5); the conjecture is justified with an algebraic expression* (2012.6); *the results are confirmed with relationships between algebraic and graphical representations (2012.14); the subject used the graphical representation as a verification tool (2014.13), and the justifications were supported with spreadsheets and graphs* (2020D.5). In these cases, the rationale are based on intuitions relating to singular objects, such as tables, charts, and algebraic expressions.

For other units of meaning, *the generalization was described in ordinary language* (2008.1); *progressively, the generalization was expressed using symbolic language* (2008.2); *the generalization was expressed in an increasingly formal way* (2008.3); *the subjects used natural language to present and justify their reasoning* (2014.1); *the student justifies his algorithm using natural language* (2014.6*), and the subject justifies based on written natural language* (2019.5).

Based on this *status quaestionis*, there is a nucleus of ideas that shows the absence of validations and another that shows validation as intuitions mediated by language, that the nucleus of ideas N.3 emerges and distinguishes itself from the previous ones. Thus, validation as *empirical practice* brings together the units of meaning that express concrete but particular attempts to validate conjectures. In these cases, in general, students tend to take conjectures as conclusions, an aspect also revealed by Brocardo's study (2001, p. 544), which reports: "If a conjecture had withstood successive tests, it seemed true to them, so they felt no need to prove it".

On this basis, *the subjects tended to argue based on numerical regularities* (2019.18), as well as with data extracted from the task itself (2020C.10), such as the following units of meaning*: the subject elaborated the justification based on the diameters of the water surface, depending on the shape of the bottle* (2021A.1); *the arguments to validate the sketch were based on the 'way' in which the height of the water varied* (2021A.2). In addition, *the correct result was verified by making calculations* (2008.4) *with specific values* (2020B.4) (2020B.7) (2020B.9) (2020D.1), which indicate, for the subjects, a validation process (2020B.2). It should be noted that these units of meaning show a validation process based on solving contextualized problems with pseudo-real situations (2021A.1) (2021A.2) and (2020D.1), numerical regularities (2019.18) and (2020B.4) and particular issues (2020B.2), (2020B.6) and (2020B.7).

Validation as a *social practice*, with core N.4, is another way of validating the mathematical knowledge constructed through exploratory and investigative tasks in the classroom. It is based on the interaction between peers (teacher and classmates), so *justification is built through dialog* (2020A.1). In this interaction, the teacher is responsible for guiding, requesting, and encouraging validation and sometimes validating the students' knowledge themselves, as shown by the following units of meaning: *the teacher guided Marisa to justify her answer* (2018. 1); *asked for an explanation of 'why'* (2018.2); *encouraged students to present justifications* (2018.4); *challenged the student to present a rationale* (2018.9); *pointed out the justification as valid* (2018.3); *validated the student's answer* (2018.6) *and validated the strategy used by the student* (2018.7). In addition, the teacher's actions served as a basis for the students to realize the non-validity of a conjecture (2020A.5).

Colleagues are also seen as important peers when discussing validation processes. The units explain this: *validation happened after Bento corrected Monica's calculation* (2020B.3); *Agnaldo validated the strategy used by Marta (*2020B.10); *Monica validated Bento's resolution* (2020B.11); *Beatriz's speech was a validation for Lucas's conjecture* (2020C.6); *colleagues' validations pointed to the falsity of José's conjecture* (2020C.7).

Similarly, classmates are essential peers for sharing the knowledge they have built up (2020B.5) and for agreeing to the validations presented (2021A.7) (2021A.9) (2021A.10) which generally *occurred by comparing the relevance of the result found with that required by the task* (2020B.1) and *by sharing the same results (2020C.5), so that different results invalidated some conjecture*s (2020C.2). In some cases, the justifications were accepted almost naturally (2021A.4). In other cases, *the students disagreed with the justification given* (2020D.3).

The units of meaning in nucleus N.4 express the presence and influence of the teacher and classmates in validating the mathematical knowledge produced through exploratory and investigative tasks, validating the status of social practice in the classroom. Given this, the nucleus of ideas N.3 differs from N.4 since N.3 is composed of the units of meaning that relate to the knowledge produced by the student from non-immediate ways of investigating mathematically and, in N.4, this knowledge produced is evaluated, reinforced or put to the test by both the teacher and the students who have been involved in exploratory tasks and investigative tasks in the classroom.

Thus, when performed by the teacher, validation as a *social practice* is characterized by reflections on what has been produced, with a view to the teaching objectives (2018.2) (2018.9) (2020A.5) (2018.7) and (2020C.3). On the other hand, when performed by peers, validation as a *social practice* is presented as a technique of opposition, as in the meaning units (2020C.2) (2020D.3) and (2020B.3), or reinforcement of the epistemic practices adopted, such as in (2020C.6) (2020B.10) and (2021A.9).

The nucleus N.5 expresses the validation of mathematical knowledge constituted with exploratory tasks and investigative tasks in the classroom as a *demonstration*. Based on that, validation was supported by the mathematician's scientific practice, which makes use of procedures, properties, theorems, and mathematical concepts previously accepted as valid (2012.17) (2014.7) (2021B.3) so that *conjectures acquired validity through deductive reasoning* (2014.17) (2012.4). From this, it follows that validation as *demonstration* consisted of processes of contraposition, as in (2020A.4) and (2021A.12); of logical-deductive arguments, as in (2012.4) (2012.13) (2012.19) and (2014.7); and with definitions, as in

(2014.11). In addition, there were cases in which the epistemic value of a conjecture appeared to change from probable to false (2020C.1) so that the falsity of the conjecture (2020A.4) was validated with *a counterexample* (2021A.12).

In light of the above, we understand that the validation of constituted knowledge is a multifaceted phenomenon in mathematics teaching and learning practices with exploratory and investigative tasks. It consists of particular cases, informal justifications, the absence of justifications, and logical-deductive processes that reproduce the mathematician's scientific practice. In addition, each of the nuclei of ideas contains, as a way of giving materiality to the validation, different epistemological techniques that led the students to justify their conjectures.

5 Final Considerations

Returning to what the hermeneutic exercise exposed in the light of the question, *how does the validation of mathematical knowledge through exploratory and investigative tasks in the classroom appear?* We realize it occurs in language, empirical practice, social practice, and demonstration. It follows that the validation of mathematical knowledge created through exploratory tasks and investigative tasks in the classroom can be constructed of deductive processes, as mathematical scientists do, but is not limited to them. This result is to be expected from the pedagogical point of view of Mathematics Education since "the Mathematics of Mathematics Education [...] in its state of truth, is another Mathematics, radically different from that seen from the perspective of the professional practice of mathematicians" (Garnica, 2002, p. 98).

Fiorentini and Lorenzato (2006) emphasize that "if, during the activity, questions or conjectures are formulated which trigger a process of testing and attempts to demonstrate or prove these conjectures, then we have a mathematical investigation situation" (p. 29); otherwise, the activity may be restricted to the exploration and problematization phase. In a sense, discourses that converge with those mentioned by Fiorentini and Lorenzato (2006) are recurrent in the literature. Together with what we have seen in this work, they highlight a tension: on the one hand, discourses encouraging demonstration - as the mathematical scientist does - as a *modus operandi* for validating mathematical knowledge in the classroom with investigative tasks; on the other, teaching practices that indicate other possibilities, including the absence of justification with tasks of this exact nature. In this sense, we have raised the following question: in pedagogical work with exploratory tasks and investigative tasks in the classroom, "should the teacher be satisfied with informal justifications or ask students for mathematical proof of their statements?" (Ponte, 2003, p. 57).

In the Brazilian educational scenario, the Common National Curriculum Base (BNCC) proposes the presence of demonstration in the teaching of mathematics for the final years of elementary school as an essential contribution to the formation of hypothetical-deductive reasoning (Brasil, 2018). For secondary education, the document explains that students' mathematical education presupposes the development of "a set of skills aimed at investigating and formulating explanations and arguments, which can emerge from empirical experiences [...], but should also include more 'formal' arguments, including the demonstration of some propositions" (Brasil, 2018, p. 541).

Oliveira (2002) points out that "the idea of demonstration underlying many educational studies is strictly deductive" (p. 179) and identifies it in the mathematics classroom as a type of inferential reasoning. In this sense, demonstration is seen not only as legitimizing but also as justifying mathematical knowledge. This aspect, together with what has been shown in this study, corroborates the study by Ponte, Ferreira, Varandas, Brunheira, and Oliveira (1999), which extends the stage related to demonstration in the research process "to include now all aspects related to 'justifying a conjecture'" (p. 62).

According to Ponte, Quaresma, and Mata-Pereira (2020), justifying can take forms such as logical coherence, generic examples, counterexamples, exhaustion, and absurdity. From this perspective, justifying is related to understanding and validating the results found and functions as a mechanism for communicating their legitimacy. For Brunheira and Ponte (2019), these aspects are associated with demonstration, which they understand to be an argument or a sequence of interconnected statements that establish the truth for a person or a community.

Therefore, the validation of mathematical knowledge constituted exploratory tasks and investigative tasks in the classroom, characterized by logical-deductive processes, the use of natural language, empirical and social practices, and the absence of justification. It is also very likely that in other contexts, other modes of validation may emerge, given that the classroom is made up of subjective ways of being a teacher and being a student, together with the methodological perspective used, which places those presented here in a condition of possibilities and not univocities.

In this context, the understandings exposed relate mathematical knowledge constituted through exploratory and investigative tasks in the classroom to what has been reported in the literature, giving them a meta-comprehensive character. In view of this, we leave the proposition of studies that interrogate, in situ, the validation of mathematical knowledge constituted strictly with Mathematical Investigation in the classroom.

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