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Building bridges between Mathematics Didactics and Inclusive Mathematics Education: the possibilities of T4TEL

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Abstract: This article aims to discuss access to knowledge by deaf and hearing students through tasks that legitimize their differences. To this end, it is based on two theories of Mathematics Didactics of Francophone influence: Vergnaud's Conceptual Fields Theory, for the deepening of studies regarding the mathematical knowledge studied, and Chevallard's Anthropological Theory of the Didactic, for the development of the investigation. It is conjectured that the T4TEL model is a possibility for deaf and hearing students to access mathematical knowledge in the same school space. The conclusions include that the T4TEL Model has proved to be efficient for research in the area of Inclusive Mathematics Education, since it considers the cognitive subject in the modeling of didactic variables. It also shows that the illustration variable has greater potential for deaf and hearing students to access knowledge.

Keywords: Access to Knowledge. Inclusive Mathematics Education. T4TEL Model. Deaf. Variables Legitimizing Differences.

Tendiendo puentes entre la Didáctica de la Matemática y la Educación Matemática Inclusiva: las posibilidades de T4TEL

Resumen: Este artículo pretende discutir el acceso al saber por parte de alumnos sordos y oyentes a través de tareas que legitiman sus diferencias. Para ello, se basa en dos teorías de Didáctica de las Matemáticas de influencia francesa: la Teoría de los Campos Conceptuales de Vergnaud, para profundizar en los saber matemáticos estudiados, y la Teoría Antropológica de la Didáctica de Chevallard, para desarrollar la investigación. Se conjetura que el modelo T4TEL es una posibilidad para que alumnos sordos y oyentes accedan al saber matemático en el mismo espacio escolar. Entre las conclusiones se encuentra que el Modelo T4TEL ha demostrado ser eficiente para la investigación en el área de Educación Matemática Inclusiva, ya que toma en cuenta al sujeto cognitivo al modelar las variables didácticas. También muestra que la variable ilustración tiene mayor potencial para que los alumnos sordos y oyentes accedan al conocimiento.

Palabras clave: Acceso al Saber. Educación Matemática Inclusiva. Modelo T4TEL. Personas Sordas. Variables Legitimadoras de las Diferencias.

Construindo pontes entre a Didática da Matemática e a Educação Matemática Inclusiva: as possibilidades do T4TEL[1]

Resumo: Esse artigo objetiva discutir o acesso ao saber por estudantes surdos e ouvintes por meio de tarefas legitimantes das diferenças dos estudantes. Para isso, apoia-se em duas teorias da Didática da Matemática de influência francófona: a Teoria dos Campos Conceituais de Vergnaud, para o aprofundamento dos estudos referentes ao saber matemático estudado e a Teoria Antropológica do Didático de Chevallard, para o desenvolvimento da investigação. Conjectura-se ser o modelo T4TEL uma possibilidade para a efetivação do acesso ao saber matemático por estudantes surdos e ouvintes, em um mesmo espaço escolar. Entre as conclusões, destaca-se que o Modelo T4TEL revelou-se eficiente para as pesquisas na área da Educação Matemática Inclusiva, visto que considera, na modelização das variáveis didáticas, o sujeito cognitivo. Evidencia-se, ainda, que a variável ilustração possui um maior potencial de acesso ao saber pelos estudantes surdos e ouvintes.

Palavras-chave: Acesso ao Saber. Educação Matemática Inclusiva. Modelo T4TEL. Surdos. Variáveis Legitimantes das Diferenças.

1 Introduction

The Genevan epistemologist Jean Piaget (1896-1980) in only two titles, published in Brazil, positioned himself in relation to school education: *Para onde vai a educação?*, written at the request of the United Nations Educational, Scientific and Cultural Organization (UNESCO) in 1948 and published for the first time in Brazil in 1973 and *Psicologia e Pedagogia* (1969), with the first Brazilian edition published in 1975.

In both works, Piaget takes away from the students the responsibility of the school failings in this discipline, when they still succeed in others.

In *Para onde vai a educação,* Piaget (1980), positioned himself:

> Our hypothesis therefore that the supposed different aptitudes of "good students" for Mathematics, Physics, etc., in equal levels of intelligence, consists mainly in their capacity to adapt to the kind of education they are given; whereas the "bad students" in those disciplines, who still succeed in other fields, are in reality perfectly apt to grasp subjects they don't seem to comprehend, as long as those reach them through other paths: it is therefore the "lessons" offered which escape their grasp, and not the discipline. (Piaget, 1980, p. 17). (Our translation.)

Furthermore, in the essay *Psicologia e Pedagogia* (1975), the Genevan master conjectures that the failure of children and teenagers, who display the same intelligence mechanism and, therefore, of learning, would be a consequence of the way in which it is taught, without considering the epistemological specifics of mathematical knowledge. In other words, Mathematics is taught in the same way as Biology or History, even though the nature of Biological knowledge is empirical and Historical knowledge, social.

From these reflections, allied to the failure of the New Math movement, which was started in the 1950s by North-American mathematicians who sought to solve the difficulties found in the processes of teaching and learning Mathematics by modifying curriculums in virtue of approximating mathematical knowledge presented to students to scientific knowledge,

[1] Este artigo se sustenta em parte de tese de doutorado da primeira autora e orientada pelos dois coautores.

emerged investigations of the phenomena that occurred in classrooms, particularly those related to teaching.

Mathematical education as a field of knowledge constitutes itself with the essential assumption that Mathematics isn't inaccessible to the students, rather the way they are presented with it. In that manner, studies and surveys seeking to comprehend how students learn Mathematics, how teaching should be realized, the transposition of mathematical knowledge, the different ways of presenting and promoting the (re)construction of mathematical knowledge, among many other aspects, were and are realized by mathematical educators.

Starting from the great boost that the development of Mathematical Education as a scientific field has received in the last decades, detailed and delimited fields of research were specified, giving origin to what Pais (2005) denotes theoretical tendencies, "[...] each one regarding different thematics of the Mathematical teaching" (Pais, 2005, p.10). (our translation).

Besides being research paths, these tendencies also characterize as educational approaches, or, as they're treated in the Parâmetros Curriculares Nacionais: Matemática (PCN) (or National Curriculum Parameters: Mathematics), "[...] ways to do Mathematics within the classroom" (Brazil, 2000, p. 42).

These ways, however, do not contemplate the different psychological subjects, that is, they consider only the generic subject, whether or not subjected to an institution, or still, a cognitive subject, but even then, still generic, which has led us to conjecture that the disregard of the differences between students could be the cause of the didactic gaps of proposals founded in different didactical theories, which end up not proportioning the access to knowledge to all students.

Our hypothesis is that it is necessary to have a didactical commitment to differences, which implies recognizing, respecting and valuing differences, that is, legitimize them. Thus, we corroborate with Perrenoud (2002), to whom differentiating education is to make accessible the action of teaching without, however, "[...] renunciate to instructing nor abdicate the essential objectives. Differentiate is therefore, fighting so that inequalities to the school become subdued, and simultaneously, so that teaching improves" (Perrenoud, 2002, p. 9). (our translation).

With these assumptions, some researchers of the work group 'Diferença, Inclusão e Educação Matemática,' or, 'Difference, Inclusion and Mathematical Education' in English, the GT-13, of the Sociedade Brasileira de Educação Matemática (SBEM), or 'Brazilian Society for Mathematical Education' in English, grounded on the idea of 'education for all', presented by the Salamanca Statement (1994), consider Special Education thought of in an inclusive perspective can fill in the didactical gaps left by the "ways of doing Mathematics in the classroom", by adding to them the didactical commitment to difference.

To reinforce this comprehension, we bring two examples of works that presented tasks developed to strengthen the idea of education for all. The first is the article *Cenários Multimodais para uma Matemática Escolar Inclusiva: dois exemplos da nossa pesquisa*, by Solange Hassan Ahmad Ali Fernandes and Lulu Healy (2015), which details an approach to elaborate tasks to be incorporated in inclusive scenarios to mathematical teaching. Such scenarios involve tools created to represent mathematical knowledge in an adequate manner for students with sensorial limitations and are developed to privilege multimodal experiences of objects, relationships and mathematical properties.

To illustrate the approach, the authors Fernandes and Healy (2015) present two examples of the work developed by the two with blind and deaf students. In the first, the tasks are mediated by material tools which explore tactile and visual resources; in the second they consider tasks directed at the concept of rational numbers mediated by a digital tool which offers visual and sonorous stimuli. According to the authors, the main worry when elaborating the learning scenarios, is prioritizing the emergence of a culture, in which the actors - teachers and students - feel ready for a pleasurable and satisfactory school experience for those who teach and for those who learn. The authors believe that, this way, it’s possible to reach all students through didactic actions which turn them into active subjects and capable of using not only their eyes and ears, but the entire potential of their perceptive body when establishing new relations as studied mathematical knowledge.

The second of these works is the *Ressignificação of the concept of diagonals of a convex polygon by deaf students under the light of compensatory mechanisms*, by Thamires Belo de Jesus and Edmar Reis Thiengo (2018). In this essay, the authors consider the difficulty of deaf students in attributing meaning to the formulae. They worked specifically with the formula d = n(n-3)/2, in which *d* represents the number of diagonals and *n* the number of sides, mainly because of the necessity for generalizing and abstracting which is involved in the utilization of letters and symbols without a real meaning to the students. The authors propose that the presentation of this content be realized through the construction of polygons in a plane, utilizing rubber bands. Once the polygon is built, still utilizing rubber bands, all possible diagonals are formed. A task to notice the existing regularity is done posteriorly, and the formula is deduced and presented in its final mathematical form. According to the authors, this task can also be developed with blind or low-vision students, in function of tactile possibilities, in addition to favoring the access of knowledge to hearing and seeing students. In the work of Jesus and Thiengo (2018, we see a possibility to provide a multimodal scenario, in which more than one sense is explored.

However, these works which consider the educational specificities of the students aided by Special Education, though destined to all students, are based theoretically on theories of psychological character, which don’t consider the nature of mathematical knowledge. From that finding emerged proposals for investigations which sought to contemplate both the educational specificities of students and of mathematical knowledge, and that way, bring the contributions of the Didactics of Mathematics, as the following examples will show.

The first example of investigations done with contributions of Mathematical Didactics of French influence and that considers the educational specificities of blind or low-vision students refers to the Master’s thesis of Pricila Basílio Marçal Lorencini, titled *Possibilidades inclusivas de uma sequência didática envolvendo representações gráficas da função afim*, defended in 2019. Lorencini’s research (2019) proved that tasks involving linear function’s graphs, in which the procedures and graphical representations are described in their natural languages (written or spoken) by pairs of students, constituted moments of learning to each of the classroom’s students. The main focus of the investigation was proving that this way of developing a sequence of tasks, thought with the explicit objective of favoring a student with low vision, contributed to the access of knowledge to all students. For that finding, the data was analyzed under the light of Gérard Vergnaud’s Conceptual Fields Theory.

The second example of the confluence between Mathematical Didactics and Special Education in an inclusive perspective, is the study related in the article *A influência da forma de apresentação dos enunciados no desempenho de alunos surdos na resolução de problemas de estruturas aditivas*, by Clélia Maria Ignatius Nogueira and Beatriz Ignatius Nogueira Soares (2019). In this research, the authors identified, through the resolution of tasks with additive structures for composition, transformation and comparison, the preference of deaf students to the means of presentation. The tasks differentiated in the written presentation, since some had diagrams and others illustrations. The research based itself on Gérard Vergnaud' Conceptual Fields Theory. The results pointed out that the visual aspect is determinant to the interpreting of the mathematical problem questions by deaf students. In Nogueira and Soares's investigation (2019), there was also a hearing student, who despite already being literate, also showed greater comprehension of the problems when those were accompanied by an illustration.

Whilst these works were developed with students aided by Special Education and with distinct differences, all of the research mentioned considered that education for all is a process, through which we have to learn how to live with differences, and recognize what we learn from them. For that, both the researchers and teachers have to consider human diversity as a factor which enhances the access to knowledge for all students. This way, the assumption is that teaching done based on Mathematical Didactics which considers the student's educational specificities may promote the access of mathematical knowledge in an inclusive perspective.

With that assumption, considering the surveys already done and with the comprehension that, the surveys thought of for deaf students can contribute in the access to knowledge for them and for hearing students in the same school space, we have constructed the investigative question of this work: how is the teacher able to think of tasks with inclusive potential in the classroom? We list as an objective: discussing the access to knowledge by deaf and hearing students by means of tasks which legitimate the differences of students.

The first step was choosing a mathematical topic taught in classrooms. We opted, therefore, considering the Additive Structures studied by Vergnaud (2014, seeing that the Conceptual Fields Theory brings contributions of what should be considered in this topic when forming the tasks.

After having decided what would be the basis for the mathematical topic, we worked on identifying what other theories of Mathematical Didactics would allow us to propose the tasks, considering the cognitive subjects in question. That way, we reached the Anthropological Theory of the Didactic, which provided institutional elements, that is, mathematical and didactic arrangements, through which we could detect didactic gaps in the teaching of a topic, and from that, elaborate proposals to fill those gaps.

Considering, however, the question of the deaf subject, we comprehended that the starting point should be their differences. Therefore, it was the differences of deaf students which led us to choosing the T4TEL model[22]. We conjecture it to be a possibility to actualize the access of mathematical knowledge to deaf and hearing students in the same classroom, given that, by considering didactic variables, it allows us to include in those, variables which legitimized differences.

We organized this work in six sections, in which we will: discourse about the Inclusive Mathematical Education of the deaf; bring considerations about the Anthropological Theory of the Didactic and the T4TEL model; describe the methodology for forming tasks; and end with analyses parting from the model in question. Lastly, we left some considerations.

2 Inclusive Mathematical Education

We considered as Inclusive Education any offered by a school "[...] centered in the

[2] The T4TEL model is part of the Anthropological Theory of the Didactical and was developed by Hamid Chaachoua as a formalization and extension of the praxeological model. The term T4 refers to the praxeological quartet (types of Tasks, Techniques, Technology and Theory). The T4TEL model has as an objective making possible the structuring of a set of specific tasks of a determined school subject, it is not a mold or a platform.

community, free of barriers (architectural or curricular), promoting collaboration and equity" (Rodrigues, 2006, p.302) (our translation). In other words, a school which has as an objective education for all, where all can learn equally.

Guiding ourselves through the assumption of Inclusive Education, that all students should have access to everything a school offers in any moment of their education, and the assumption of Mathematical Education, that all efforts should be put forth so that mathematical knowledge becomes accessible to all students, Nogueira (2020) highlights that talking about Inclusive Mathematical Education is redundant, given that, Mathematical Education itself is, or should be, naturally inclusive.

Nogueira (2020) considers that Inclusive Mathematical Education parts from the viewpoint that didactic actions should be practiced so that mathematical knowledge is accessible to each and every student, and that all of them are attended with the same quality. Even still, according to the same researcher, in Inclusive Mathematical Education, it is fundamental that differences are neither belittled or hidden. Au contraire, they should be legitimized (recognized, considered, valued) through the adoption of curriculums and different teaching and learning situations, which can coexist in the same classroom to support access of knowledge to all its students.

Nogueira (2020) affirms that the action of teaching, aiming to construct mathematical knowledge, is a long process, which requires dedication, and also "[...] the starting point should be marked by the previous knowledge of the student, and the finish line, by his potential and time spent for learning in a school context" (Nogueira, 2020, p.127) (our translation).

With our intention of legitimizing the differences within the classroom, we considered the T4TEL model of creating tasks as a possibility for all students to have the same opportunities in the classroom. That is because this model's task types generator allows us to create types of tasks which consider the educational specificities of mathematical disciplines, whilst also considering and valuing the differences between students (in this text's case, the deaf) in the classroom, given the adequate choice of didactic variables.

In this survey, to highlight the imbrications between the theories of Mathematical Didactics and Inclusive Mathematical Education, we based ourselves upon a doctorate research which had already been done to show the formation of a sequence of types of tasks made with legitimizing the differences of deaf students in mind, and we proposed that the teacher, upon learning the dynamic of this model, should be able to listen to their students and their respective guardians, to then form sequences of tasks which consider and value the differences of other students aided by Special Education.

The dialogue between the students, those responsible for them, and the teacher, contributes to the recognition of their differences, which have to be considered and valued in the elaboration of tasks, through the choice of pertinent didactic variables, as well as enriching the interpersonal relationships, favoring the mutual maturing and contributing to the empowering of students aided by Special Education.

Now that we've discussed it, we'll present the T4TEL model.

3 Anthropological Theory of the Didactical and the T4TEL model.

According to Bosch and Chevallard (1999), the Anthropological Theory of the Didactical considers any and all mathematical activities and the mathematical arrangement knowledge which stems from them. To these authors, a (educational) mathematical organization has its origin in the analyses, performed by teachers, of official educational documents[3], from which stem the mathematical contents which should be taught. Starting from this, the educator will begin to determine which kinds of tasks would hold up the teaching process of these contents, bringing with them the respective praxeological components (technique, technology and theory) (Bosch; Chevallard, 1999) (our translation).

In turn, a didactical organization appears from the intention of actualizing, or conducting, a determined mathematical organization, such as to enable its (re)construction or transposition. According to Bosch and Chevallard (1999), we cannot expect the (re)construction, in the span of a learning process, of a mathematical organization to arrange itself in only one way. However, to the researchers, no matter the way of teaching, certain situations would necessarily be present, both qualitatively and quantitatively, even if in heterogeneous ways.

A mathematical arrangement and a didactical arrangement can be implemented into an institution through the structure of the T4TEL mode, introduced by Chaachoua and Bessot (2018). The T4TEL model inserts itself into the Anthropological Theory of the Didactical by extending the praxeological approach through the introduction of the notions of variables and personal praxeology[4].

The goal of introducing variables in the structure of T4TEL is to form a set of specific situations of a discipline, characterized by a restricted set of relevant variables. To Chaachoua and Bessot (2018, p. 120) the notion of variables "[...] appears above all as a methodological tool in a shaping process, associated to the *a priori* analysis of a particular or fundamental situation". (our translation).

The first function of a variable is creating types and subtypes of tasks considering the values of the variables which depend on the subject, the mathematical concept and institution in question. In T4TEL, a type of task T is described by a verb and a complement, T = (verb, complement). The verb characterizes the types of tasks, such as: "calculate", "add", "subtract", among others. The complement is defined according to the level of granularity (particularities), from the specific to the generic (for example, "calculate the sum of two numbers" is more generic than the task type "calculate the sum of two natural numbers with value in the tens") (Chaachoua; Bessot, 2018) (our translation).

Considering the concept of granularity, Chaachoua and Bessot (2018) presented the notions of a task type generator and system of variables. A task type generator (TG) is defined by a type of task and a system of variables, and it can be described in the following manner: TG = [verb, fixed complement; system of variables]. The verb and the complement identify the type of task, and the system of variables accounts for the variables and the values which they can have within the realm of a discipline in a specific institution.

This way, to model the system of variables, we considered the epistemological, institutional and didactical perspectives. The epistemological perspective of variables comprehends that "[...] the division of the values of a variable is such that the altering of a value modifies the scope of possible techniques for a type of task" (Chaachoua; Bessot, 2018, p. 124-125) (our translation). To illustrate that perspective, we presented the task type T_1 = (Calculate the sum of two natural numbers with their first value in the tens and second value in the units). There is an economic technique to solve this type of task, overcounting, where the student takes the greater value, that is, the tens; overcounts with the second value, that is, the units; and

[3] Such as laws, decrees, curriculums, educational programs and manuals, among others.
[4] We utilize the notion of personal praxeology, developed by Chaachoua and Bessot (2018), as the difference between the personal and institutional relations of a student relative to the content studied.

represents the solution. This technique is less relevant, for example, for two numbers with values in the tens and hundreds, because it would require more effort and is prone to error.

In an institution, there will always exist conditions and restrictions which will not only restrict the type of task, but also the possible values of an epistemological variable for a type of institutional task. In the early years of Middle School, for example, for T_2 = (Calculate the sum of two numbers), the numbers involved, in the majority of cases, are within the realm of the Natural numbers (N), and the values are restricted to the units, tens and hundreds. A variable and its institutional values model explicit and implicit conditions and restrictions of the levels of codetermination under which a praxeology exists or may exist in a given institution. An example of institutional values are the values of numbers with which we work. In the 3rd year of Middle School, obeying the conditions of this institution, we work with natural numbers with values as high as the hundreds.

Regarding a didactic variable, it is that which is within an institution and potentially at the reach of a teacher. The teacher can enrich *a posteriori* the values of the didactic variables, considering the personal praxeologies of the students, that is, they can sculpt the values of these variables through a posterior analysis of knowledge already acquired by the students. With regards to the didactic variables, in this survey, we considered that, within the modeling of the values attributed to the variables, the praxeologies which the student already knows regarding studied mathematical concepts and the differences between each student present in the classroom. We considered then, that by attributing values to the variables, with the objective of legitimizing the differences between all students, in a social perspective of impairments, it is possible to promote equity of access to the studied concepts.

The second purpose of a variable consists in characterizing the scope of the techniques. Outside of its scope, the technique may fail; it may be applied, but it is prone to error. For example: the successive counting method may be applied for T = (calculate the sum of two whole numbers). If applied to larger numbers, it is highly likely it will fail. Therefore, the scope of a technique is the set of tasks in which it is reliable for performing these tasks with a low chance of failure or reasonable effort.

The third and last function of a variable is the notion of personal praxeology. This is an important notion to the diagnostic of the learning paths of students in a given institution, for the inclusion of the cognitive subject and the mistake as the study object in the Anthropological Theory of the Didactic. The researchers comprehend that the notion of personal praxeology encompasses the praxeological quartet, taking into account the description of errors in both the techniques and technologies of the student.

In this article, we approach the role of these variables related to the tasks and techniques, alongside its developments, considering the variables and their different values as tools which enable students to access mathematical knowledge ‘problems involving different meanings of addition and subtraction with natural numbers’.

4 Methodology

The *lócus* schools are bilingual schools for the deaf and a school which aims to be inclusive[5]. The first has as its language of instruction the Brazilian Sign Language - Libras - and Portuguese in written manners, as its second language; the second has as its instructive

[5] We consider an inclusive school as being the one which understands human diversity as an enriching factor of the educational process and that constitutes itself as a space for interaction, teaching and learning for all.
Therefore, in this investigation, we utilize the nomenclature of ‘aiming to be inclusive’ given that, we comprehend that the school is still in the process of becoming an inclusive school.

language Portuguese, both written and spoken. The institutions surveyed in this study were: a class from the 3rd grade of Middle School; a class from the second stage, phase 1, of the Education of the Youthful and Adults[6] of a bilingual school for the deaf; a 3rd grade of Middle School class from a school which aimed to be inclusive. In virtue of its age range and the year of schooling, the deaf students who participated in the survey are in the midst of the process of learning reading, and hearing, writing.

However, we highlight that whilst the hearing had already mastered the Portuguese Language in the oral modality, the deaf students were still in the process of learning Libras. That is because, according to Gomes (2010, p. 35), more than 90% of deaf children are born to hearing parents, and therefore, do not naturally acquire the language in their family environments, coming to school with a homemade communication in signals, very close to mimicking, such that their first contact with formal Libras happens in school. In other words, deaf children acquire their first language at the same time as they learn the Portuguese Language in written form.

To deepen the studies referring to the studied mathematical concept, we based ourselves on the Conceptual Fields Theory. In this theory, Vergnaud (2014) identified in the study of additive structures, six forms of basis which spent all of the possibilities relating to this concept, and that parting from those it is possible to elaborate elementary arithmetical addition and subtraction tasks, which could mobilize for their solution, ternary (with three measures involved) or quaternary (four measures) schemes. In function of the conditions (mathematical concepts) of the surveyed institutions, we limited ourselves to the fundamental ternary schemes of the six following categories.

> First category: two measurements come together to form a third. Second category: a measurement is transformed to result in another measurement. Third category: a relation connects the two measurements. Fourth category: two transformations come together to form one transformation. Fifth category: a relative state (a ratio) is transformed to result in another relative state. Sixth category: two relatives (ratios) come together to form a relative state (Vergnaud, 2014, p. 200) (our translation).

The researcher establishes as the conceptual field of the additive structures, the set of situations where their solution implies one or multiple additions and subtractions, as well as the set of concepts and theorems which enable us to analyze such situations as mathematical tasks. The six categories of addition and subtraction situations are conceived from three ideas: composition, transformation and comparison.

We initially proposed discussing different tactics referring to the six categories displayed by Vergnaud (2014), concentrating our attention only on the natural numbers, given that was the focus of our investigation. However, upon consulting documents like the National Common Curricular Basis[7] (2018) and the Paranaense State Network Curriculum[8] (2020), we found that the fourth, fifth and sixth categories weren't considered in these documents for the surveyed institutions.

[6] Phase 1 comprehends from 1st to 5th grade of Middle School. More specifically, the second stage of phase 1 corresponds to the 3rd grade of Middle School.
[7] The National Common Curricular Basis is a document of normative character which defines the organic and progressive set of essential concepts to students of all schools in the country.
[8] The Paranaense State Network Curriculum establishes the essential concepts for Middle School and High School in the state of Paraná, in accordance with the National Common Curricular Basis (2018).

Considering then, the available options in the documents which orient the teaching of Mathematics in the state of Paraná, we restricted our study to the first three categories (which encompass the ideas of composition, transformation and comparison between measurements). A conjecture we made in regards to the absence of the other categories in this level of schooling is related to the ascending order of complexity of the situations: the higher the level of the category, the harder they may be given the situations.

To continue our research, considering the objective of the survey, based on the T4TEL model, we performed four studies:

- In the historical and epistemological span of the studied mathematical concepts, we emphasized the conditions and restrictions of the existence of mathematical and didactic arrangements with mathematical knowledge in an academic context which aims to be inclusive. We studied the evolutions of these organizations through time and the possible evolutions of didactical arrangements such that they may legitimize the differences of all students present in the classroom, contemplating in the problem questions and in the types of tasks, legitimizing variables.
- In our research through official educational documents (such as the National Common Curricular Basis, Paranaense State Network Curriculum, as well as school manuals and books, among others), which structured the surveyed institutions in relation of studied concepts, we identified the types of tasks which exist within the studied mathematical topic and the contemplated variables in the presentation of these types of tasks.
- In Vergnaud's study of the Conceptual Fields Theory we searched for the epistemology of the studied mathematical knowledge. We identified the existing kinds of tasks, the epistemological variables related to these topics and the didactic variables which contribute to the access of knowledge by hearing students.
- In our studies within the area of Inclusive Mathematical Education we researched the teaching of Mathematics to deaf students and identified legitimizing variables of the differences of deaf students.

Upon completing the studies on the studied mathematical concepts, considering the conditions and restrictions imposed by the investigated institutions, we identified 14 types of tasks (in the three first categories presented by Vergnaud (2014)). In the conditions imposed by the institutions, we highlight: the type of number – natural; the meanings of composition, transformation and comparison – between measurements.

- First category:

T_{11} = (Calculate the result of the composition of two or more measurements).

T_{12} = Calculate a measurement which is composed of another known measurement, knowing the resulting value of the composition).

- Second category:

T_{21} = (Calculate the final state (measurement) resulting from the transformation (positive) of a known initial state (measurement)).

T_{22} = (Calculate the final state (measurement) resulting from the transformation (negative) of a known initial state (measurement)).

T_{23} = (Calculate the transformation that occurs on an initial state (measurement) to result in a final state (measurement) with (final state > initial state)).

T_{24} = (Calculate the transformation that occurs on an initial state (measurement) to result in a final state (measurement) with (final state < initial state)).

T_{25} = (Calculate the initial state (measurement) which was transformed (positively) and resulted in a known final state (measurement)).

T_{26} = (Calculate the initial state (measurement) which was transformed (negatively) and resulted in a known final state (measurement)).

- Third category:

T_{31} = (Calculate the referred of a comparison between measurements with a positive relation).

T_{32} = (Calculate the referred of a comparison between measurements with a negative relation).

T_{33} = (Calculate the comparison relation between two measurements with (referred < referent)).

T_{34} = (Calculate the comparison relation between two measurements with (referred > referent)).

T_{35} = (Calculate the referent of a comparison between measurements (addition)).

T_{36} = (Calculate the referent of a comparison between measurements (subtraction)).

To identify the variables and constitute the variable system, we considered the notion of variable as presented by Chaachoua and Bessot (2018), which is a methodological tool in the modelization tool, which allows students to access studied mathematical concepts. This way, we formed the variables system, as we described in the following, with different values made explicit between parentheses:

- Variables and values attributed to the variables identified in the study of the book of the school which intended to be inclusive:

$V_{1/2}$[9] = Naturally spoken/written tongue (Portuguese in the oral manner[10], Portuguese in the written manner).

- Variables and values attributed to the variables identified in the studies in regards to problems which involve different meanings of addition and subtraction:

$V_{1/2}$ = Naturally spoken/written tongue (Portuguese in the oral manner, Portuguese in the written manner).

V_3 = The size of the first measurement m_1 ($m_1 \in N \mid 0 < m_1 > 100$).

V_4 = The size of the second measurement m_2 ($m_2 \in N \mid 0 < m_2 > 100$).

V_5 = Presentation of the information (information in the temporal order of the presented facts, information presented in a disorderly fashion, inverse fashion).

[9] The numbers of the variables were picked at random, they do not correspond to the order in which the variables were identified. Because the variables system was constructed simultaneously in studies 2, 3 and 4, in some moments, variables 1 and 2 appear together.
[10] We consider 'Portuguese in the oral manner' as a value attributed to the 'natural tongue' variable, given that it's the language in which instructions are given in the school which intends to be inclusive.

V_6 = Type of theme (common to the student's day-to-day life, uncommon in the student's day-to-day life).

V_7 = Visual aid (schemes to establish a relationship between the solution and the numerical data, a scheme to establish the relation between the solution and type of task).

- Variables and values attributed to the variables encountered in the studies within the field of Inclusive Mathematical Education of the deaf:

$V_{1/2}$ = Naturally spoken/written tongue (Portuguese in the written language, interlanguage[11], Portuguese in the written manner (with one sentence in each line), Libras).

V_7 = Visual aid (schemes, illustrations).

After identifying the existent types of tasks with the studied mathematical concept and the constitution of the variables system, we formed 14 sets of tasks, that is, 70 tasks that composed our didactic device.

For example, the first set, formed with the T_{11} = (Calculate the result of the composition of two measurements), was constituted in the following manner: the first type of task was generated with variables V3, V4, V5 and V6[12], contemplated in the task's question, the variable of Portuguese in the written manner (presenting one sentence per line), and was printed on yellow paper; the second, with the variables V3, V4, V5 and V6, contemplated in its presentation the variables: Portuguese in the written manner (presenting one sentence per line) to be signed in Libras by the teacher in the case of the bilingual deaf school, or by the Libras interpreter in the case of inclusive schools, and was printed on green paper; the third, with variables V3, V4, V5 and V6, contemplated in its presentation the variables of Portuguese in the written manner (one sentence per line), and interlanguage, and was printed on pink paper; the fourth, with the variables V3, V4, V5 and V6, contemplated in its presentation the variables of interlanguage and schemes, and was printed on blue paper; and the fifth, with the variables V3, V4, V5 and V6, contemplated in its presentation the variables of interlanguage and illustration, and was printed on white paper. The objective of the different colors was to highlight that the presentation of problem was distinct and the students would choose only the problem regarding to the same type of task, in other words, for problem 1, for example, the student would choose only one sheet, with the question in such a manner that they judged most favorable to their understanding. For problem 2, the choice could land on a sheet of the same color, should they wish to continue solving problems with questions presented in the same manner. An example of these can be verified in Figure 1.

Each set of tasks is composed by the same relational calculus, but with differently presented statements. This device was implemented at first, in the institutions of the '2 Step of Phase I Education to Teenagers and Adults', with five students, and the 'third year of middle school', with five students, who were part of a bilingual school for the deaf. Posteriorly, it was also included into the institution 'third year of middle school', with thirty-one students, which are part of a school which intends to be inclusive.

[11] This nomenclature is used in the field of linguistics to characterize a system of transition which a student of a new language creates throughout the assimilation process of the same. The value of 'interlanguage' is characterized, in this investigation, through short and clear sentences; through sentences which utilize the name of subjects to re-introduce them, avoiding the use of pronouns; through sentences without articles, prepositions and conjunctions; through sentences without unnecessary informations to understand the task; and through sentences which avoid the possibility of ambiguous interpretations.'

[12] We consider 'Portuguese in the oral modality' as a value attributed to the 'natural language' variable, since it is the language of instruction in a school that aims to be inclusive.

The implementation of the didactic device happened in a similar manner within the three investigated institutions[13]. The student-collaborators, in duos, received sets constituted by five sheets, each sheet with a different color and task. Between the five tasks showcased by sets, they chose three and solved them.

Figure 1: Task sets

T = Andriérilim has some bananas.
Tahyna has 14 bananas.
Tahyna has 5 fewer bananas than Andriérilim.
Andriérilim have how many bananas does?
R =

T = In the basket there are some fruits in the basket.
In the basket there are 15 strawberries.
In the basket there are 7 fewer strawberries than oranges in the basket.
In the basket how many oranges are?

Reading in Libras by the teacher in charge of the class (bilingual school for the deaf) or Reading aloud by the teacher in charge of the class (school that wants to be inclusive).

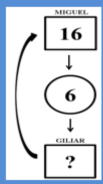
R =

T = Miguel has some bananas.
Rafael has 12 bananas.
Rafael has 2 fewer bananas than Miguel.
Michael have how many bananas does?

Miguel has bananas.
Rafael has 12 bananas.
Rafael has 2 less bananas than Miguel.
Michael bananas have?

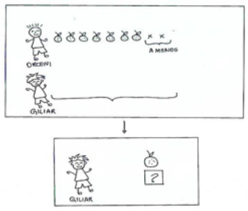
R =

T = Giliar has some pears.
Miguel has 16 pears.
Michael has 6 fewer pears than Giliar.
Giliar have how many pears does?



R =

T = Giliar has some oranges.
Orceni has 7 oranges.
Orceni has 2 fewer oranges than Giliar.
Giliar have how many oranges does?



R =

Source: Research data

After the implementation of the tasks, interviews were done with the student-collaborators of the school which intended to be inclusive, with the intention of establishing dialogue with them and comprehending their opinions of the multiple variables contemplated in the presentations of the enunciations. The interviews were done in duos, the same were done as tasks in the classroom. We recorded them in audio and, in the deaf student's case we recorded it in audio and video[14]. In the moment of the interview, in the room were only the student-collaborators and the teacher-researcher. The desks and seats were close, and on the desks was a set of tasks, just like the ones they had received in the device's implementation.

5 Analyses

Whilst performing our studies of both the didactic books and the works of Vergnaud (2014), we identified the types of tasks and the variables related to the studied mathematical concept. We identified as well, in the didactical book, a gap related to the way the problem questions were introduced, which may make the access to the mathematical knowledge for the students difficult. In the works of Vergnaud (2014), we encountered elements to question and

[13] The implementation of the didactic device and the interviews were done with all 41 student-collaborators. However in the bilingual school for the deaf, both the implementation of the device and the interviews were conducted individually, whereas in the school which intended to be inclusive, they were performed in duos.
[14] In regards to this deaf student, we respected the restriction imposed by the dad, and didn't use sign language. However, we recorded a video to contemplate in the description, both by the answers given verbally by the student as well as possible signs and body movements done, given he doesn't have a good mastery of the spoken Portuguese language.

fulfill this gap, when talking about hearing students. Through performing studies in the field of Inclusive Mathematical Education for the deaf, we identified difference-legitimizing variables which could contribute to the interpretation of the relational calculus involved in the problem questions of the tasks for both hearing and deaf students.

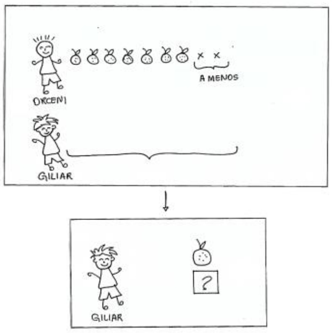
After identifying the types of tasks and the constitution of the system of variables, with the task type generator of the T3TEL model, we created a sequence of tasks which made up our didactic device.

The implementation of the device in the bilingual school for the deaf had as an objective to identify which variables legitimize the difference and which ones have a higher potential for accessing the studied mathematical concept. Before presenting some remarks made by this implementation, we highlight that ‘legitimize’, to our understanding, means recognizing, considering and valuing the differences; whereas ‘legitimizing variables for inclusive differences’ refers to those which not only meet the conditions imposed by the researched institutions, but also legitimize the differences and contribute so that deaf and hearing students have access, simultaneously, to the studied mathematical concept.

We illustrated, in figure 2, one of the tasks which contemplates, in the problem question, legitimizing variables for the inclusive differences of the deaf:

Figure 2 - Task which contemplates in it’s question presenting legitimizing variables of ‘inclusive differences of the deaf’

T = Giliar has some oranges.
Orceni has 7 oranges.
Orceni has 2 fewer oranges than Giliar.
Giliar have how many oranges does?



Legitimating variables of inclusive differences contemplated:

- First and second measure sizes smaller than 100;
- Presentation of information in the temporal order of the reported facts;
- Theme common to the student’s day-to-day life;
- Portuguese in written mode (presenting one sentence per line);
- Illustration to establish the relationship between the solution and the type of task (relational calculation).

Source: Authors (2021).

In this task we can identify the values of the variables which meet the conditions imposed by the researched institutions, in regards to the studied mathematical concept: the size of the first and second measurements; the presentation of information in the temporal order; and the theme being in common with the student’s day-to-day life. We also identified the variables which legitimized the differences of deaf students: Portuguese in the written manner (one sentence per line) and the illustration to establish a relation between the solution and type of task (relational calculus).

Upon analyzing the data formed with the implementation of 70 tasks, we saw that from the 210 tasks done, 160 of them were solved correctly (76,1%). In regards to the number of correct answers, according to the variables contemplated in the presentation of the questions, we verified that tasks with illustrations had 55 tasks out of 63 done correctly (87,3%); 45 out of the 57 tasks with schemes (78,9%); 39 out of 54 with Portuguese in the written manner (72,2%); 8 out of 13 with interlanguage (61,5%) and 13 out of 23 done with Portuguese in the written manner (56,5%) We observed, upon analyzing the data, that values attributed to the variables with highest potential for accessing the studied mathematical concepts were illustrations, schemes, Libras, interlanguage and Portuguese in the written manner (with one sentence per line).

To continue with the construction and analysis of the data, we performed the implementation in the school which intended to be inclusive with the objective of identifying, through the constructed didactic device, the possibilities of access to the studied mathematical concepts to the deaf and hearing students of this institution.

While forming the tasks that compose the device we considered all elements related to the notion of institution and to the mathematical concept recommended by the Anthropological Theory of the Didactical. In other words, we considered the praxeological model to model the mathematical and didactical organizations of the studied concept, in the researched institutions and in the actions by the students which were expected by the institutions.

While preparing ourselves to experiment with the device in the school which intended to be inclusive, which has in its institution of the Third Year of Middle School a deaf student, we were faced with two unexpected situations. The deaf student in this school, unlike other deaf students in the bilingual school for the deaf, doesn't know Libras, and by option of the family, seeks inclusion through speech and audio therapy. The option of the family is based in a clinical perspective of deafness, unlike the students in the bilingual school for the deaf whose model is based on a social perspective of deafness. Furthermore, there were between the students, one with low vision.

This way, it was possible to identify the gap brought on by the created device, given we considered initially only the subjects within the third year of highschool for the hearing and deaf, that is, in our modeling, we only considered institutional and epistemological elements. Upon constituting the variable system only with those related to these dimensions - epistemological and institutional - it wasn't sufficient to establish an effective modelization of the students present within the classroom. That way, it is necessary to identify the variables related to the didactical dimension, which consider the different forms of establishing new relations with mathematical knowledge, not encompassed in the institutional sphere.

As previously established, the didactic variable is a variable within an institution, potentially in the disposition of the teacher/researcher. According to Chaachoua and Bessot (2018), we can enrich the values of didactic variables considering the personal praxeologies of students. In this investigation, the notion of personal praxeology allows us to identify the difference between students in the three surveyed institutions and, consequently, contemplate the diversity of students present in the classroom.

This way, the gap identified in the planning of tasks considering only hearing and deaf students, upon noting the presence of a student with low vision, was fulfilled with contemplation of the T4TEL model and other variables, which allowed the inclusion of this student not adequately subjected to the institution initially considered: a school which attends the hearing and the deaf in an inclusive perspective. The contemplation, for example, of the value of 'Portuguese in the written manner (with amplified letters)' in the variable of 'essay', allowed us to attend to the specificity of a hearing student with low vision. The possibility of attributing new values to the variables reveals the efficiency of the T4TEL model, upon attending the educational specificities of the students and promoting the access to knowledge of different cognitive profiles of the subjects of a same institution.

5.1 Dialogues with the collaborator-students

During our investigation, upon the implementation of the didactic device and the interviews, we found that the cognitive variables, which consider the educational specificities of bilingual deaf students, contributed to the access of knowledge to hearing students and to the deaf non-signing student. An example of this was the attribution of value ‘illustration’ to the ‘visual aspect’ variable, given that when we attend to their differences in learning through interacting with the visual mean, we can understand the contributions of this variable to the other students in the classroom, including the student with low vision, who could access it through assisting tools like a magnifying glass, or in the case of blind student, materials in braille.

Upon asking the questions to the duos of collaborator-students in regards to the illustration, we got the following answers:

> Researcher: The illustration, I’d like you to comment a little about it. In the task: “In teacher Marisa’s classroom there are 4 boys and 3 girls. How many kids are in teacher Marisa’s classroom?”.
>
> Student Laisla (hearing student): Seven.
>
> Researcher: I’d like to ask what you understood from the illustration.
>
> Student Endrio (hearing student): I thought it was cool, because I could count the boys and girls.
>
> Researcher: Here, after the key-word, there’s a drawing of a boy, the word ‘and’ and the drawing of a girl, and below it, an interrogation mark.
>
> Were you able to understand that it was talking about addition?
>
> Student Laisla (hearing student): Yes.
>
> Researcher: Did the word ‘and’ give you the idea of adding?
>
> Student Laisla (hearing student): It gave me an idea in my head, then I just did the sum in my head, then four plus three, seven (Duo 2).
>
> Researcher: This one, with the illustration. You read the question and saw the image, right? Let me read it: “In teacher Marisa’s classroom, there are four boys and three girls. How many children are in teacher Marisa’s classroom?
>
> Student Gabriel (deaf student): Seven.
>
> Researcher: Did you look at the illustration? How many boys are there?
>
> Student Gabriel (deaf student): Four boys.
>
> Researcher: Four, and how many girls?
>
> Student Gabriel (deaf student): Three.
>
> Researcher: What about boys and girls? [pointing at the illustration]
>
> Student Gabriel (deaf student): Seven.
>
> Researcher: Perfect! Did the illustration help you understand?

Student Gabriel (deaf student): Yes.

Researcher: Which one was your favorite?

Student Amanda (student with low vision): This one, the white.

Researcher: Why?

Student Amanda (student with low vision): Because the illustration represents Math.

These answers led us to conclude that the illustration contributed to the interpretation of the task's question for all present students and, consequently, promoted the access to the studied mathematical knowledge.

6 Final considerations

In this investigation, we found that considering only variables related to institutional subjects, that is, epistemics, limits the access to knowledge for all students in the classroom. Considering what one given student should know in a determined level of schooling in function only of what institution they are subject to, which implies assuming all students have the same starting point and they establish relationships with the concept studied in the same manner, which isn't sufficient to promote didactical actions which contribute to the access of knowledge to all, given it doesn't contemplate the cognitive subject with their differences.

In this survey's case, the epistemological and institutional studies realized in mathematical and didactical organizations in the investigated institutions, which are based in the Anthropological Theory of the Didactical, from Chevallard (1992), allowed them to know *a priori* the epistemic subjects, deaf and hearing people.

However, the consideration of epistemic subjects, signing, bilingual deaf and hearing people, we couldn't contemplate the diversity of students aided by Special Education present in the classroom in which our device was implemented, given that, the deaf student wasn't bilingual in the time of the survey, having had a cochlear implant and educated in the oralist perspective. Furthermore, the specificities of the student with low vision, also weren't considered, in a way that some of the didactic variables we considered as legitimizing differences, such as Libras, didn't attend to our deaf student and the font of the questions, the schemes and illustrations, weren't adequate for our low sight student, and consequently, their differences, within the didactic actions from a inclusive school context.

Upon adding the these cognitive subject's difference-legitimizing variables to the task generator, we obtained a more detailed modeling of the personal praxeologies of the student-collaborators of the survey, based off of T4TEL model, including types of embryos of techniques and techniques which are not always visible to the institution. It's what happened with the deaf, non-signing student who needed the questions for the tasks to be read out loud so he could perform lip-reading; and the hearing student with low vision, who needed tasks to be presented with bigger fonts and other visual aids amplified, or that the questions were read to her.

This way, in regards to a research done in the field of Inclusive Mathematical Education based in Mathematical Didactics, we highlight the flexibility of the T4TEL model, which allowed us to model variables related to the differences of the cognitive subject and posteriorly, perform adequations on our didactic device, which contribute to the access to knowledge of each one of our students.

For future studies, we suggest exploring strategies to promote a more holistic approach in modeling the student's personal praxeologies, including the consideration of embryos of techniques and non-traditional techniques which may not be immediately evident to the institution. This proposition can offer valuable orientations to future research in the area of Inclusive Mathematical Education, with the goal of improving the pedagogical practices and guaranteeing the access of knowledge to each one of the students, independently of their individual differences.

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