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Mathematics teachers' specialized knowledge regarding fraction division in the context of problem formulation

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Abstract: The aim of this paper is to present the specialized knowledge revealed by mathematics teachers through a training task related to the formulation of fraction division problems. It is a qualitative investigation, and the methodology is an instrumental case study, whose analysis was based on the teachers' productions and plenary discussions during the training. The results show that the teachers have knowledge of: the sense of sharing; solving problems using the invert-multiply algorithm; and continuous representation in rectangular form. However, the teachers have difficulties with the sense of measurement of division; continuous and discrete representation; and problem formulation, especially when the divisor and dividend are fractions. It can be seen that problem formulation contributes to learning fraction division, as it helps to understand mathematical concepts and solve problems.

Keywords: Problem Formulation. Teacher Specialized Knowledge. Fraction Division.

Conocimiento especializado del profesor de matemáticas sobre la división de fracciones en el contexto de la formulación de problemas

Resumen: Este trabajo tiene como objetivo presentar los conocimientos especializados revelados por profesores de matemáticas a través de una tarea formativa relativa a la formulación de problemas de división de fracciones. El estudio trata de una investigación cualitativa combinada con un estudio de caso. Los análisis se basaron en las producciones de los profesores y en discusiones plenarias durante la formación. Los resultados muestran que los docentes tienen conocimientos respecto al significado de compartir, resolución de problemas mediante el algoritmo inversión-multiplicación y representación continua en forma rectangular. Sin embargo, los docentes presentan dificultades en cuanto al sentido de la medición de la división, la representación continua y discreta y la formulación de problemas, especialmente cuando el divisor y el dividendo son fracciones. Está claro que formular problemas contribuye al aprendizaje de la división de fracciones, ya que ayuda a comprender conceptos matemáticos y resolver problemas.

Palabras clave: Formulación del Problema. Conocimientos Docentes Especializados. División de Fracciones.

Conhecimento especializado do professor de Matemática em relação à divisão de fração no contexto da formulação de problemas

Resumo: Este trabalho objetiva apresentar o conhecimento especializado revelado por professores de Matemática mediante uma tarefa para formação relativa à formulação de problemas sobre divisão de fração. Trata-se de uma investigação qualitativa, e a metodologia é o estudo de caso instrumental, cujas análises foram pautadas nas produções dos professores e nas discussões em plenária durante a formação. Os resultados evidenciam que os professores detêm um conhecimento em relação: ao sentido de partilha; à solução de problemas por meio do algoritmo inverte-multiplica; e à representação contínua na forma retangular. No entanto, os professores apresentam dificuldades referentes ao sentido de medida da divisão; à representação contínua e discreta; e à formulação de problemas, principalmente quando o divisor e o dividendo são frações. Percebe-se que a formulação de problemas contribui para a aprendizagem da divisão de fração, pois auxilia na compreensão de conceitos matemáticos e na resolução de problemas.

Palavras-chave: Formulação de Problemas. Conhecimento Especializado do Professor. Divisão de Fração.

1 Introduction

Problem formulation is an important practice in mathematical learning (Cai, Hwang, Jiang & Silber, 2015; Ellerton, 2013), as it is relevant to the development of problem-solving skills, as well as to the deepening of mathematical concepts (*National Council of Teachers of Mathematics* [NCTM], 2000; Toluk-Uçar, 2009; Isik & Kar, 2012). We know that teachers' practice is shaped by their knowledge (Ball & Bass, 2000), which plays a fundamental role in teacher training (Ball, Lubienski & Mewborn, 2001). It is therefore essential to work with problem formulation in order to develop and expand the mathematical knowledge of mathematics teachers (Lee, Lee & Park, 2016).

In this paper, we will address the formulation of problems on fraction division, as this is a challenging topic both for students, from a learning perspective, and for teachers, from a teaching perspective (Behr, Harel, Post & Lesh, 1992; Lamon, 2007). Research indicates that teachers have limitations in formulating mathematically correct and meaningful problems about fraction division (Ma, 1999; Kilic, 2015; Lo & Luo, 2012). These studies indicate that teachers have difficulties in relation to the concept of fraction and division (Toluk-Ucar, 2009) and that these challenges affect the process of formulating problems on fraction division (Ervin, 2017).

The aim of this study is to present the specialized knowledge revealed by mathematics teachers in a training task related to the formulation of fraction division problems. The research question is: What specialized mathematical knowledge do teachers reveal when they carry out a problem formulation task on fraction division? In this way, the teachers' knowledge of fraction division is investigated in relation to the representation of word problems for some predetermined division operations.

2 Formulating fraction division problems and teacher knowledge

Problem formulation plays a key role in establishing links between real-life cases and fractions, as well as operations involving fractions (Abu-Elwan, 2002; Iskenderoglu, 2018). To contribute to the goals of understanding abstract concepts and selecting the appropriate mathematical concepts to solve problems, word problems can be used (DeWolf, Grounds, Bassok, & Holyoak, 2014). In addition, teachers' knowledge of the division of fractions, acquired through the formulation of problems, contributes to developing and expanding this knowledge (Kilic, 2015; Xie & Masingila, 2017).

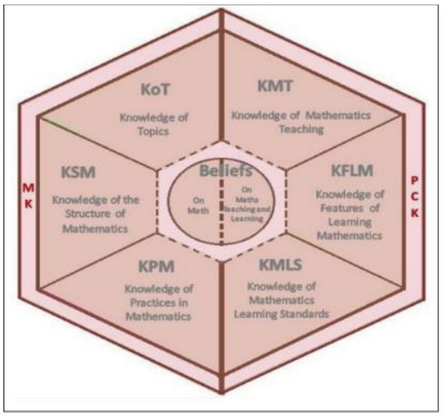
Studies point to the difficulties faced by teachers in relation to division and fractions. These challenges are related to various aspects, such as: the meanings of division, measurement and sharing (Simon, 1993; Lo & Luo, 2012); the conception that the result of division is always smaller than the dividend (Tirosh, 2000); the incorrect formulation of problems, such as elaborating multiplication instead of division (Ma, 1999; Zembat, 2004; Iskenderoglu, 2018); the lack of ability to correctly represent a problem in words (Utley & Redmond, 2008); and the ability to differentiate problems that refer to 6:2 and 6:1/2 (Ma, 1999). These difficulties occur due to teachers' lack of experience in formulating problems, as well as conceptual misunderstanding of fractions and division (Xie & Masingila, 2017).

Ma (1999) proposes the idea of a package of knowledge that teachers should have for fraction division. This model indicates that a deep understanding of fraction division is built on a network of prior knowledge. This includes the concept of unity, meanings of multiplication with fractions, concepts of division with whole numbers and the concept of the inverse operation of the operations (division and multiplication). These, however, are built on the concept of fractions, while the meaning of whole number multiplication is built on the meaning of whole number addition (Lo & Luo, 2012).

Therefore, teachers need to have specialized knowledge in relation to fraction division, in order to expand what they know about this subject. We understand specialized teacher knowledge in the sense of the *Mathematics Teacher's Specialized Knowledge*[1] [MTSK] model (Carrillo *et al.*, 2018), in our case relating to fraction division. We are therefore referring to specific professional knowledge used by mathematics teachers in their teaching work, built up from their initial training and throughout their career (Climent, 2002).

This specialization lies in the domain of Content Knowledge, in this case *Mathematical Knowledge* (MK), and *Pedagogical Content Knowledge* (PCK), including teachers' beliefs about mathematics, as well as about teaching and learning the subject.

Figure 1: Domains of Mathematics Teacher's Specialized Knowledge



Source: Carrillo et al. (2018, p. 241).

Here, we will focus on the subdomains of *Mathematical Knowledge* (MK), since we will be dealing with the specificities of the teacher's mathematical knowledge, which enables

[1] As this is a conceptualization of teacher knowledge that is disseminated and recognized internationally, we have kept the English nomenclature for all the terms in the model, as translating them could distort the understanding of the contents of each of the subdomains that make up the model that represents it (Figure 1).

the attribution of meaning regarding the topic of dividing fractions.

The *Knowledge of Topics* (KoT) includes the teacher's mathematical knowledge associated with fraction division, underpinning an understanding of what is done, how it is done and why it is done in a certain way. It also encompasses knowledge of different types of representation registers and the multiple possible definitions for the same concept. In the context of dividing fractions, it includes, for example, knowing: different meanings attributed to this operation, such as measurement and partitioning (Simon, 1993); different procedures (algorithms) - traditional or unusual -; properties, such as the relationship between the divisor and the unit of measurement, as well as between the dividend and the whole to be measured; the equivalence of fractions, linked to the concept of rational number as a representative of a class of equivalent numbers; the different types of fractions and the different forms of representation associated with calculating the division of fractions - pictorial and algebraic.

The *Knowledge of the Structure of Mathematics* (KSM) corresponds to the teacher's mathematical knowledge of each of the topics, in a broad and deep way, which supports the connections between them. In relation to fraction division, this includes, for example, knowledge relating to comparison with division of whole numbers and the multiplicative inverse in rational, real and complex numbers.

On the other hand, the *Knowledge of Practices in Mathematics* (KPM) includes the teacher's knowledge associated with ways of doing mathematics. This includes knowledge of demonstration and the different ways of presenting mathematical results, as well as the criteria for establishing a valid generalization and the various problem-solving strategies. In the area of dividing fractions, this knowledge includes, for example, knowing the justification for generalizing different procedures.

The dimensions of the teacher's specialized knowledge, including beliefs, underpin their actions in developing mathematical knowledge and skills in students without limiting themselves to a single way of doing things, avoiding the teaching of rules as a starting point and, at most, using them as a finishing point. To this end, the teacher's actions must initially take on board the students' knowledge, which requires interpretative knowledge (Ribeiro, Almeida & Mellone, 2021). Therefore, in order to promote an improvement in the practice of mathematical learning, it is essential to focus on the teacher's knowledge, considering their specificities.

Thus, there is a need to broaden teachers' knowledge of the division of fractions in the context of problem formulation. We should therefore think about questions such as: *If I have the operation 5 ÷ 2, what problems can I formulate? How many mathematically distinct problems can I formulate?*

To do this, we must have knowledge of the meanings of division. Among the meanings of division (Simon, 1993), we assume partitive and measurement as a starting point. The concept of measure is associated with comparing quantities that express quantities of the same nature and with quantification. The number of items in each group is known. The partitive sense has meaning in contexts where, given a quantity of elements in a set (dividend), it is desired to share (distribute) a quantity equally between a certain number of sets (divisor).

Therefore, based on a problem, we can identify which of the two meanings of division is being evoked (Fischbein, Deri, Nello & Marino, 1985), just as the verbalization linked to the act of dividing indicates the meaning attributed to it. In this way, it is essential that the teacher takes a significant role in understanding the concepts related to this operation (Fazio & Siegler, 2011), so that division does not lose its meaning and teaching is not focused on the algorithm procedure.

There is research that has classified fraction division problems typically seen in curricula (Fischbein et al.,1985; Greer, 1992) in a general way, identifying five main problem structures (Table 1*): equal group measurement division and equal group partition division* refer to problems that deal with a certain number of groups, all of equal size. Comparison measurement *division and comparison partition division* deal with multiplicative comparison situations, i.e. one set involves several copies of the other. *Rectangular* division occurs when the product of the multiplication consists of a two-dimensional unit, such as the length and width unit for the area product.

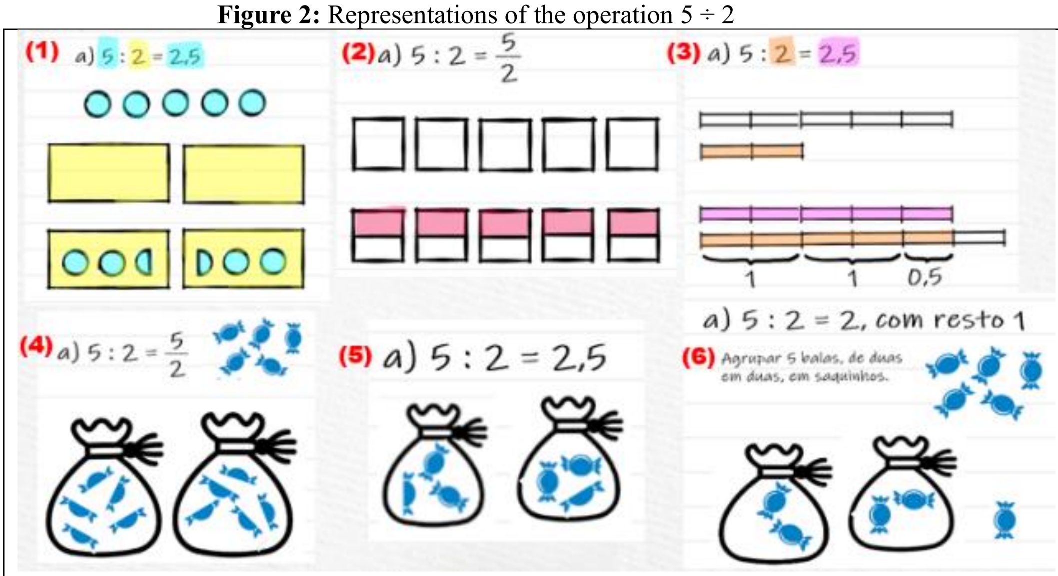
Table 1: Classification scheme for fraction division word problems

Fraction division	
1) Equal group measurement division	You need 1 1/3 cups of flour to make a batch of cookies. Marie has 11 2/3 cups of flour. How many batches of cookies can she make?
2) Equal group split division	Marie has 11 2/3 cups of flour. That's enough to make 1 ½ batches of cookies. How many cups of flour are needed for one batch of cookies?
3) Measurement division by comparison	This week, Mark exercised for 8 ¼ hours. Last week, he exercised for 4 1/2 hours. How many times more did he exercise this week than last week?
4) Partition division by comparison	This week, Mark exercised 1 ½ times more than last week. If he exercised 8 ¼ h this week, how many hours did he exercise last week?
5) Rectangular area division	If the area of a rectangle is 13 ½ m2 and the length is 3 1/3, how wide is it?

Source: Lo & Luo (2012).

Another very important element in interpreting division in the context of solving a problem is the representation associated with the meaning in the problem. We should therefore think about how many different and distinct representations we can make. Figure 2 shows some possibilities for the operation 5 ÷ 2.

Figure 2: Representations of the operation 5 ÷ 2



6) $5 \div 2 = 2, \textit{with remainder } 1$
Group 5 candies, two by two, in plastic bags.

Source: Prepared by the authors.

Different registers of representation play an important role in the process of teaching and learning mathematics, since they help in the understanding of mathematical concepts (Izsák, 2003). It is essential for the teacher to have knowledge of different fraction representation registers, such as: common language (one third); arithmetic (1/3, 50%, 0.5), algebraic ($x \in R / 2x - 1 = 0$); or figural (area, set or length) (Rojas & Moriel, 2016).

Three types of figural representation are often adopted to represent fraction concepts: area, length and set (Lo & Luo, 2012; Ervin, 2017). In the area model, fractions are based on parts of an area or region; in the length models, lengths are considered; and in the set model, the whole is understood as a set of objects, and the subsets are fractional parts. In relation to continuous and discrete representation, continuity and discontinuity play an important role in numerical development (Dehaene, 1997; DeWolf *et al.*, 2014).

Whole numbers indicate discrete or continuous quantities. However, to represent parts of such quantities, it is necessary to use rational numbers, such as fractions or decimals (1/2 of a marble, 0.5 l of water). In this way, continuous problems involve quantities of mass, such as weight, volume and length, while discrete problems encompass discrete (number of bicycles) or discretized entities and sets of individual objects that cannot be divided into equal units, such as balloons and grapes (DeWolf *et al.*, 2014).

Thus, representations help to make sense of and model mental processes (Janvier, 1987), helping us to understand the concepts involved in the problem and associate them with the representation (Lesser & Tchoshanov, 2005). Teaching and learning mathematics means that cognitive activities - such as conceptualization, reasoning, problem solving - require the use of different registers of representation and expression, in addition to natural language or images.

However, in order to represent fraction division using pictorial models, teachers need to have knowledge of the flexibility of the unit of reference, which is the ability to keep track of the unit to which a fraction refers and to change their understanding of the quantity as the reference changes (Lee, Brown & Orrill, 2011). Therefore, in order to represent the division of fractions, teachers need to understand the units to which the numbers refer in their representations.

In this way, teachers need to be able to formulate problems, analyze and solve fraction division problems, and model solution strategies corresponding to pictorial models and different representations (Ball, 1990; Silva, Vidal, & Carvalho Filho, 2023).

3 Method and context

This paper focuses on the knowledge revealed by primary school teachers (who work with students aged 7 to 14) in relation to problem formulation in the context of fraction division, using a teacher training task developed by the authors. The sources of information for this study were produced during the teacher training session, which took place online and lasted six hours over two days. This research is qualitative and uses instrumental case study methodology (Stake, 1995). The focus of interest is not on the case itself, but on knowing that this instrument allows us to know and understand a specific element - teacher knowledge - in order to generate theories.

In this context, we asked the participants to formulate problems individually, according to the proposed operations, with the aim of revealing their knowledge of fraction division. The information was collected by means of a questionnaire, observations during the online training, the educator's notes, teacher productions and recordings. The teachers' productions were collected before the plenary session held after the task had been solved, making it possible to carry out a qualitative analysis based on criteria that would show the problems the teachers had formulated, as well as their conceptions of them.

The comments and production of each teacher are associated with their pseudonyms, indicated by their names: Bruno, Ana, Célia, Eva, Dina and Carlos. The group of participants was made up of two teachers with a degree in Mathematics and more than five years' teaching experience (Bruno and Ana); two future teachers who were in the final year of their Mathematics degree (Dina and Carlos); and two teachers with more than three years' experience (Célia and Eva).

The participants responded to a training task (Ribeiro *et al.*, 2021), which has its own structure, with two parts constructed in a mathematically meaningful way, taking into account the weaknesses of the teacher's knowledge. The first part has as its starting point a proposal that students at the level where the teachers teach can solve, hoping that the teachers can implement it with their classes. Here we will present one of the questions, which is related to the problem formulation in Part I, which is part of the student's task.

In the proposed question, we addressed different division operations, for example: division with whole numbers; dividend being a fraction and whole divisor; dividend being whole and divisor a fraction; and, finally, dividend and divisor being fractions; in order to identify what knowledge teachers reveal about the topic. The task asked teachers to formulate problems using the expressions provided, as illustrated in Figure 3:

Figure 3: Part of the training task

Task: Working with Fractions -

(You should always explain your reasoning by describing the process you use to answer the question. You can do this using diagrams, words, calculations, drawings...)

Consider the following expressions:

a) $5 \div 2$

b) $\frac{2}{5} \div 4$

c) $7 \div \frac{1}{2}$

d) $\frac{12}{15} \div \frac{3}{5}$

Formulate a problem for each of the above expressions and solve it, considering that the resolution implies the specific expression.

Source: Prepared by the authors.

Based on the teachers' production, we analyzed the structure of the problems created, the pictorial representations and the type of solution presented, as well as the correspondence between the problem and the operation requested. We analyzed the knowledge revealed by the teachers about fraction division through the problems formulated, based on the MTSK model - the teacher's expert knowledge.

To this end, we considered the knowledge revealed by the teachers regarding the meanings of division (KoT - phenomenology); types of word problems for dividing fractions (KoT - phenomenology); knowledge of situations that generate decimal numbers (KoT - phenomenology); properties of fraction division (KoT - properties, definitions and foundations); knowledge about discrete and continuous representations and different models for representing fractions applied to dividing fractions (KoT - register of representation); and knowledge of procedures for carrying out fraction divisions (KoT - procedures).

4 Teachers' knowledge of how to formulate fraction division problems

The teachers faced challenges when formulating problems, as this required knowledge of the mathematical concept and an understanding of what they intended to formulate. The results indicate that the teachers were aware of the sense of sharing for fraction division and therefore formulated problems with this approach in most of the cases requested (Figure 3). We can see that one of the difficulties involves understanding the concept of *dividing*, since the teachers only had knowledge of the sense of sharing.

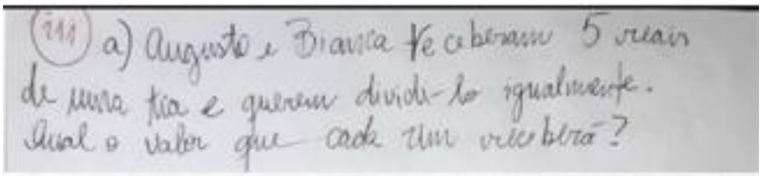
To solve the problems formulated, the teachers used the invert-multiply[2] (IM) procedure, i.e. the concept of the inverse of multiplication, without addressing other ways of solving. Despite this, they obtained correct numerical answers, demonstrating that they knew how to solve the operation of dividing fractions. They also showed that they knew about the equivalence of fractions and decimals.

When formulating their problems, the teachers labeled the units correctly, always indicating the divisor and dividend units clearly. They used real contexts, showing an understanding of fractions in the real world.

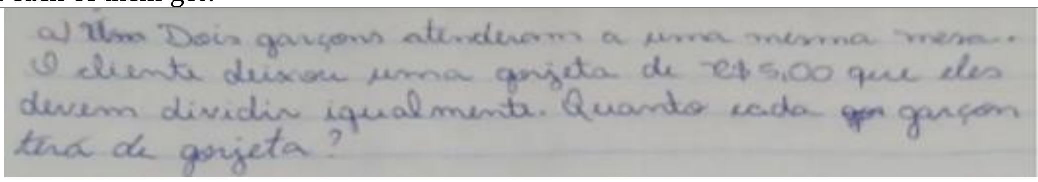
Most of the problems formulated involved dividing equal groups. We observed that, in operations in which the divisor and dividend were fractions, the teachers had greater difficulty formulating problems, as other research has shown (Ma, 1999; Toluk-Uçar, 2009). Although the teachers said that the operation letter d) $\frac{12}{15} \div \frac{3}{5}$ was the most complex to formulate a problem, they also had difficulties with the operations, b) $\frac{2}{5} \div 4$ e c) $7 : \frac{1}{2}$ (Figure 3). Figure 4 below shows some of the productions related to the problems formulated by the teachers:

Figure 4: Teachers' output

a) 5 ÷ 2


(111) a) Augusto e Bianca receberam 5 reais de uma tia e querem dividi-lo igualmente. Qual o valor que cada um receberá?

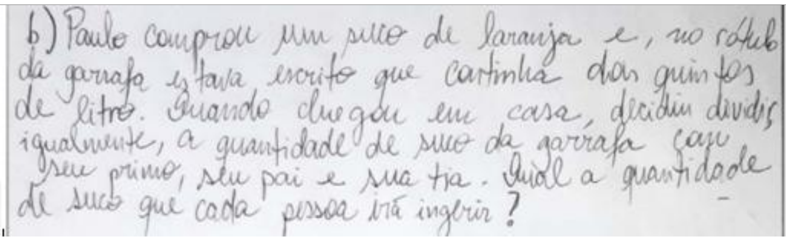
Bruno: Augusto and Bianca received 5 reais from an aunt and want to share it equally. How much will each of them get?


a) ~~Um~~ Dois garçons atenderam a uma mesma mesa. O cliente deixou uma gorjeta de R$5,00 que eles devem dividir igualmente. Quanto cada ~~gar~~ garçom terá de gorjeta?

Ana: Two waiters wait on the same table. The customer has left a tip of R$5.00 which they must share equally. How much will each waiter tip?

b) $\frac{2}{5} \div 4$

[2] IM (invert-multiply) refers to the algorithm by which the fraction corresponding to the divisor is inverted, and the dividend is multiplied by this new fraction. It refers to the algorithm expressed in the relation: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, where "a", "b", "c" e "d" are integers, and "b" and "d" are not null.



b) Paulo comprou um suco de laranja e, no rótulo da garrafa estava escrito que continha dois quintos de litro. Quando chegou em casa, decidiu dividir igualmente, a quantidade de suco da garrafa com seu primo, seu pai e sua tia. Qual a quantidade de suco que cada pessoa irá ingerir?

Dina: Paul bought some orange juice and the label on the bottle said that it contained two fifths of a liter. When he got home, he decided to share the amount of juice in the bottle equally with his cousin, his father and his aunt. How much juice will each person drink?

Eva: Of the $\frac{2}{5}$ of a chocolate left over, Manoel ate the fourth part. Which part of the chocolate did Manoel eat?

Célia: John shared $\frac{2}{5}$ of yesterday's pizza with 4 friends. How much did each of them eat?

Carlos: $\frac{2}{5}$ of the wall remains to be painted. 4 painters have been hired and they have to paint the same amount of the wall. How much of the wall will each painter paint?

Source: Prepared by the authors.[3]

The problem prepared by teacher Eva is noteworthy because it corresponds to the operation requested, but some teachers considered that it did not. For example, teacher Eva structures a problem for the operation $\frac{2}{5} \div 4$.

> *Eva: Of the 2/5 of a chocolate left over, Manoel ate the fourth part. Which part of the chocolate did Manoel eat?*

We noticed that the teacher understands the fourth part as dividing by four or multiplying by $\frac{1}{4}$. However, Professor Ana says:

> *Ana: Eva's problem corresponds to* $\frac{2}{5} : \frac{1}{4}$ *and not to* $\frac{2}{5} \div 4$.

Here, teacher Ana makes the mistake of thinking of *eating the fourth part* as dividing by $\frac{1}{4}$ and not by four. In this case, the teacher understands division by four as being by $\frac{1}{4}$ or multiplication by four, which is a common difficulty among teachers, as Ball (1990) and Ma (1999) state.

The teachers had some reflections, such as Ana's on the problems they had prepared that didn't correspond to the requested operation:

> **Ana:** *How are we going to assess the student without putting ourselves in their shoes to know if the mistake they made was their own or ours in terms of verbalization, writing, in other words, how it was passed on.*

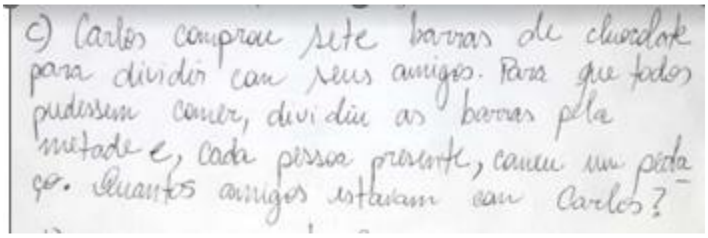
Here, teacher Ana recognizes her difficulty in formulating a problem according to the operation requested. In addition, she wonders how she can assess the student if she herself has difficulty understanding the verbalization and writing of the problem, as well as associating it with the fraction division operation and solving it correctly.

[3] Some problems were sent by the course participants via *Meet chat* and *WhatsApp* during the online training, so they are typed in.

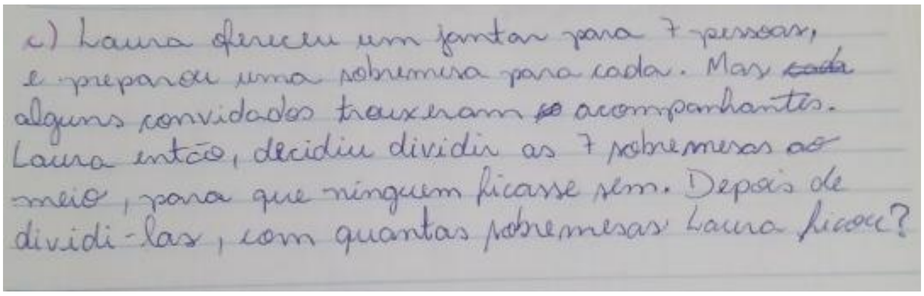
Division is part of the multiplicative structures that cover various situations, such as multiplication, division, fractions and proportionality. Therefore, the question of correspondence between the expression and the problem formulated is essential, as it requires in-depth knowledge of the mathematical topic. Here are some problems formulated by the teachers in relation to a natural number and a fraction (Figure 5).

Figure 5: Teachers' output

c) $7 \div \frac{1}{2}$


c) Carlos comprou sete barras de chocolate para dividir com seus amigos. Para que todos pudessem comer, dividiu as barras pela metade e, cada pessoa presente, comeu um pedaço. Quantos amigos estavam com Carlos?

Eva: Carlos bought seven chocolate bars to share with his friends. So that everyone could eat, he divided the bars in half and each person present ate a piece. How many friends would be with Carlos?


c) Laura ofereceu um jantar para 7 pessoas, e preparou uma sobremesa para cada. Mas ~~cada~~ alguns convidados trouxeram ~~só~~ acompanhantes. Laura então, decidiu dividir as 7 sobremesas ao meio, para que ninguém ficasse sem. Depois de dividi-las, com quantas sobremesas Laura ficou?

Dina: Laura hosted a dinner party for 7 people and prepared a dessert for each. But some of the guests brought their own side dishes. Laura then decided to divide the 7 desserts in half so that no one would be left without. After dividing them up, how many desserts did Laura have left?

Bruno: I have 7 candies to put in 7 boxes and each box is divided into two parts. I need to fill all these parts of the boxes. What is the possible solution?

Célia: How many people would be served 7 chocolates, knowing that each of them would eat half of each chocolate?

Source: Prepared by the authors.

Bruno's problem (Figure 5) - *I have 7 candies to put in 7 boxes and each box is divided into two parts. I need to fill all these parts of the boxes. What is the possible solution?* - does not correspond to the requested operation, as it asks to divide into two parts, not by a half. In addition, the way the question was asked, *what is the possible solution?*, allows for several answers. One possible solution would be to put a bullet in each box, distributing half in each part (of the box). In other words, the answer doesn't have to include the number 14. This is another point that needs to be considered when formulating problems.

Most of the teachers said that problem formulation was not part of their experience as students or as teachers. We understand that this contributes to their difficulties in formulating problems, especially when the divisor and dividend are fractions.

Question d) $\frac{12}{15} \div \frac{3}{5}$ demanded a greater effort from the teachers to formulate a problem, due to the fact that it involved fractions in both the divisor and the dividend. Some participants expressed the following:

Bruno: I didn't think it would be so difficult to come up with a problem for these questions. We don't think *about designing problems, we just solve them.*

Ana: We don't work on these questions with our students due to lack of time, the curriculum is extensive, so we brush up on these essential questions that could contribute to understanding operations.

Bruno: I don't work on problems with my students, only problem solving. Now that I'm finding it difficult to work out problems, I can see how important this work is.

The teachers presented similar difficulties to those pointed out in other studies. For example, three teachers were unable to formulate a problem for this operation (Utley & Redmond, 2008); one teacher incorrectly formulated a problem about fraction multiplication instead of division, as requested (Ma, 1999; Zembat; 2004).

Two teachers, on the other hand, were able to formulate problems involving ratio and, although they formulated equal group measurement division problems, they did not have knowledge of the meanings of division, sharing and measurement. In this way, the teachers found it difficult to formulate a problem with a sense of measurement, as indicated by their statements:

Bruno: I can't think of any examples of measurement.

Celia: We usually ask "How many times does it fit", but without being aware of whether it's sharing or measuring.

Eva: There is guidance on asking younger pupils this question in the teacher materials at the school where I work, but I didn't know it was because of the meaning of measurement, and I believe many teachers don't know either.

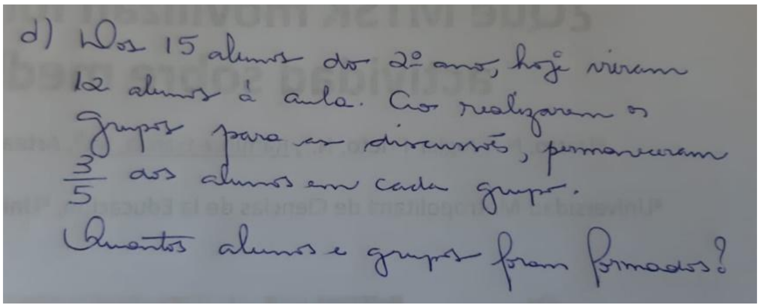
Bruno asks the educator: *Can you think of a measurement problem using fractions?* This is further evidence that the teacher doesn't know how to work out a division problem with a sense of measurement. In this speech, we can see that the difficulty in discussing the problem is not necessarily in the quantities involved, such as the fractions themselves, but in the division operation. So, let's reflect: is the teacher's difficulty related to the fraction or the division?

We have seen the importance of discussing and reflecting on this with teachers not only from the 7th grade, but also from previous years, so that they can work with different types of problems, using natural numbers and, later, fractions.

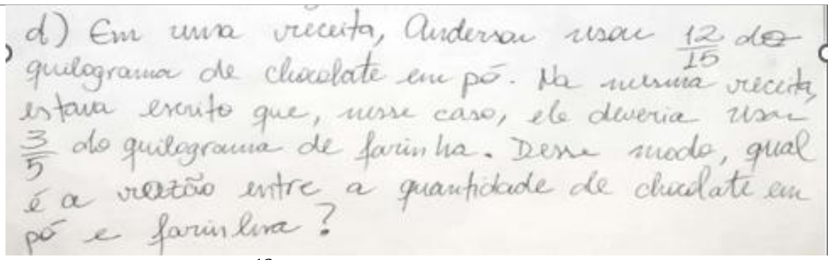
In Figure 6, we present three problems formulated by the teachers for the operation d:

Figure 6: Teachers' output

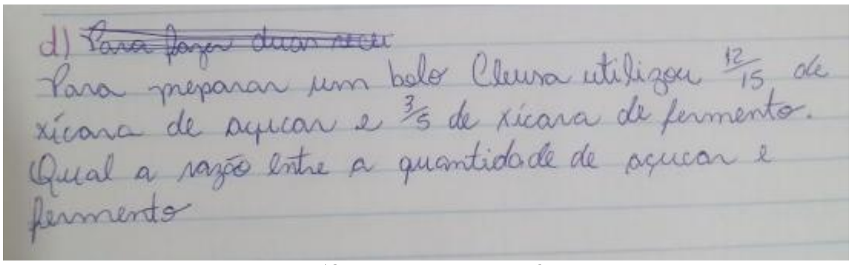
d) $\frac{12}{15} \div \frac{3}{5}$



Eva: Of the 15 2nd year students, 12 came to class today. When they formed groups for the discussion, 3/5 of the students remained in each group. How many students and groups were formed? (Problem sent by the teacher via the *Meet platform*).


d) Em uma receita, Anderson usou $\frac{12}{15}$ do quilograma de chocolate em pó. Na mesma receita, estava escrito que, nesse caso, ele deveria usar $\frac{3}{5}$ do quilograma de farinha. Desse modo, qual é a razão entre a quantidade de chocolate em pó e farinha?

Dina: none recipe, Anderson used $\frac{12}{15}$ of a kilogram of chocolate powder. In the same recipe, it was written that he should use $\frac{3}{5}$ of a kilogram of flour. So, what is the ratio between the amount of chocolate powder and flour?


d) ~~Para fazer duas rece~~
Para preparar um bolo Cleuza utilizou $\frac{12}{15}$ de xícara de açucar e $\frac{3}{5}$ de xícara de fermento. Qual a razão entre a quantidade de açucar e fermento

Bruno: To make a cake, Cleuza used $\frac{12}{15}$ cup of sugar and $\frac{3}{5}$ cup of yeast. What is the ratio between the amount of sugar and yeast?

Source: Prepared by the authors.

Teacher Eva has formulated a problem with an inappropriate context for the requested operation, as she presents $\frac{4}{3}$ of the students as the solution. In addition, we realize that it is only possible to form one group, since 60% of the students are part of it. The other two teachers, Dina and Bruna, created a ratio problem, considering this sense of fraction. The teachers were consciously unaware that these were problems with the sense of division of measurement by comparison (Lo & Luo, 2012). We therefore understand the importance of working on this type of task with teachers so that they can experience it in order to broaden their view of their students' difficulties, as well as their own mathematical knowledge and teaching practice.

With regard to the pictorial representation of the operations required by the task, the teachers presented the one involving the partitioning of equal groups. Most of them used the rectangular area model and had difficulties with the discrete and continuous representations. Furthermore, the representations did not illustrate the complete resolution of the problem, since the teachers relied heavily on the algorithm.

As an example, we'll present the problem of the wall, drawn up by Professor Carlos, for item b) $\frac{2}{5} \div 4$: $\frac{2}{5}$ *of the wall remains to be painted. Four painters have been hired and they have to paint the same amount of the wall. How much of the wall will each painter paint?* The teachers claimed that this problem would be one of sharing and discreetness, as they considered having a fixed part for each person to paint.

Ana: For me, this is a problem of sharing and discreetness.

Educator: when would continuous be for you?

Ana: when the count is not exact?

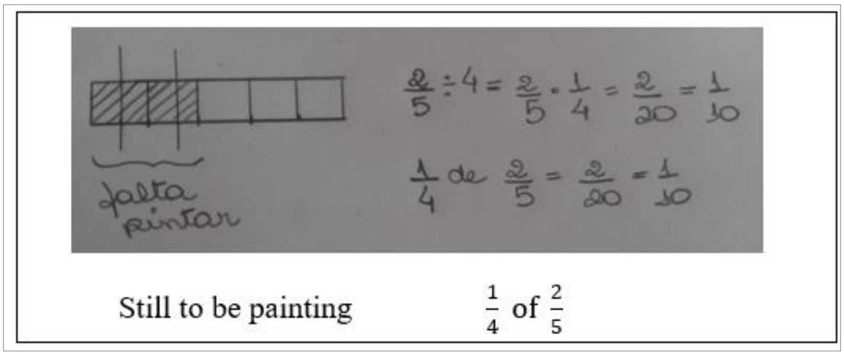
Educator: and this account is not exact?

Bruno: We usually only work with discrete and continuous in Statistics.

Dina: This was something new for me, the meanings of measurement and sharing and representation, this *discrete and continuous thing.*

The teachers acknowledged that representation is a challenge, but said that it contributes to the student's understanding of the problem, as it helps them understand what is being done. However, we noticed that this facility pointed out by the teachers also referred to their own understanding of the problem solution. Teacher Dina's representation and the teachers' dialog (Figure 7) show the difficulty in carrying out the representation with understanding:

Figure 7: Teacher Dina's production



Source: Prepared by the authors.

Educator: Where is the $\frac{2}{20}$ *representation?*

Dina: $\frac{2}{20}$ *has been simplified, we've found* $\frac{1}{10}$.

Educator: Yes, but does the answer you found correspond to your pictorial representation?

Teachers: [They couldn't explain why or how to do it].

Educator: Where is the pictorial representation of $\frac{2}{20}$*?*

Dina: I'm trying to relate it to the area of the chocolate

It is essential that mathematical language is associated with what is represented, because only then will it be possible to effectively understand what is being done. Representations help students understand mathematical concepts, as well as teachers. For this reason, the focus should be on developing mathematical knowledge, exploring ways of interpreting and representing different solutions.

Explicitly listing the strategies (mental calculation, pictorial representation, procedures) and the steps involved in solving the problem allows students and teachers to broaden their understanding of the meaning of division and operations with fractions. Based on these strategies, it becomes more feasible to think about which unit of reference allows us to compare $\frac{2}{5}$ with half of $\frac{1}{5}$ (which corresponds to each of the parts of each painter - indicated by the painted part). This is one of the students' difficulties, so it is necessary to propose situations that develop the idea of the unit of reference (Lo & Luo, 2012; Lee *et al.*, 2011).

The teachers had difficulties with the unit because of the gap in the sense of measure in fraction division, since the concept of unit is part of the knowledge of the senses of division. In the sense of measurement, you need to be able to conceptualize the unit and know how to use it. So, in order to understand fraction division, you need to know the meanings of division.

Therefore, it is important to know the meanings of division and different representations, i.e. different ways of externalizing a mental image, such as pictorial (drawings) and symbolic. These representations help in the choice of teaching strategies, making it possible to identify which are more powerful for certain contents when working with students, as well as the appropriate context for presenting the problem. In this study, the majority of teachers demonstrated knowledge related to generating situations with decimal numbers, especially in the context of money and cooking. We present a summary of the knowledge revealed by the teachers (Table 2) when carrying out the problem formulation task in the context of fraction division:

Table 2[4]: Knowledge revealed by teachers when carrying out the problem formulation task in the context of fraction division

MTSK indicators		Bruno	Dina	Carlos	Ana	Célia	Eva	Requested operation
KoT - phenomenology	Equal group measurement division (M); Equal group split division (P); Comparison measurement division (MC); Comparison partition division (PC); Rectangular area division (A).	P	P	P	P	P	M	a) $5 \div 2$
		P	P	P	P	P	P	b) $\frac{2}{5} \div 4$
		P	M	P	P	M	M	c) $7 \div \frac{1}{2}$
		MC	MC	MC				d) $\frac{12}{15} \div \frac{3}{5}$
	Knowledge of situations that generate decimal numbers: involving money and cooking.	x	x	x	x	x		a) $5 \div 2$
		x	x	x	x	x		b) $\frac{2}{5} \div 4$
		x	x	x	x	x		c) $7 \div \frac{1}{2}$
		x	x	x	x	x		d) $\frac{12}{15} \div \frac{3}{5}$
KoT - properties, definitions and fundamentals	Division and multiplication as inverse operations (Dividing by 4 is the same as multiplying by 1/4).		x	x		x		a) $5 \div 2$
			x	x		x		b) $\frac{2}{5} \div 4$
			x	x		x		c) $7 \div \frac{1}{2}$
			x	x		x		d) $\frac{12}{15} \div \frac{3}{5}$
	Knowledge of the equivalence of fractions and decimal numbers.	x	x	x	x	x	x	a) $5 \div 2$
		x	x	x	x	x	x	b) $\frac{2}{5} \div 4$
		x	x	x	x	x	x	c) $7 \div \frac{1}{2}$

[4] Empty space: teachers were unable to present problems, solutions or representations.

		x	x	x	x	x	x	d) $\frac{12}{15} \div \frac{3}{5}$
KoT - register representation of division of fractions	Discrete (D) and continuous (CO) fractions; Area (A), length (CP) and set (C).	A	A	A	A	A	A	a) $5 \div 2$
		A	A	A	A	A	A	b) $\frac{2}{5} \div 4$
		A	CP	A	A	A	A	c) $7 \div \frac{1}{2}$
			CP	A	A			d) $\frac{12}{15} \div \frac{3}{5}$
KoT - procedures	Knowledge of the algorithm for dividing fractions: inverting and multiplying, correct numerical answers.	x	x	x	x	x	x	a) $5 \div 2$
		x	x	x	x	x	x	b) $\frac{2}{5} \div 4$
		x	x	x	x	x	x	c) $7 \div \frac{1}{2}$
		x	x	x	x	x	x	d) $\frac{12}{15} \div \frac{3}{5}$

Source: Prepared by the authors.

The teachers reveal knowledge of the sense of sharing for dividing fractions; reason for fractions; the structure of problems of partitioning equal groups, measurement by equal groups and by comparison (even if they are unaware of the sense of measurement). In addition, they know situations that generate decimal numbers; equivalence of fractions; decimal numbers; multiplicative inverse; figural representation registers (line and region) and arithmetic language and algorithms such as MI. It can be seen that the specialized mathematical knowledge that the teachers have impacts their teaching practice on the topic, as well as their difficulties and challenges.

This presents some of the difficulties encountered by teachers in relation to fraction division, including:

- the meaning of measure;
- the concept of unit;
- discrete and continuous representation;
- other models, such as length and sets and mathematically correct representation that corresponds to the solution of the problem;
- the formulation of problems with divisor and dividend containing fractions;
- the formulation of problems that correspond to division rather than multiplication of fractions;
- formulating problems involving problem structures other than partitioning by equal groups.

The difficulty in the teachers' mathematical knowledge (related to fraction division using pictorial representation) seems to interfere with their knowledge about teaching fraction division, which may indicate knowledge of strategies, such as avoiding the use of pictorial representation to understand this operation and using the IM procedure) - relationship between KoT and KMT. Difficulty with regard to the meanings of division also seems to have an impact on the teaching of this topic, since it is a common (cross-cutting) characteristic between division of natural numbers and division of fractions, namely: in both cases, this operation can be interpreted with the ideas of sharing and measurement (KSM).

The challenges, difficulties and gaps revealed by the teachers point us in the direction of tasks to be considered in teacher training, especially in relation to the specialized knowledge of the teacher in the context of fraction division, in order to know and expand the specialized knowledge needed by the math teacher to teach this topic.

5 Conclusion

The results of this investigation corroborate research already carried out (Ball, 1990; Ma, 1999; Iskenderoglu, 2018), as they consider that teachers present challenges and difficulties when formulating problems related to the division of fractions.

Our study brings different results from those pointed out by Lo & Luo (2012), who claim that teachers are more likely to formulate problems with a sense of measurement than with a sense of partition when it comes to dividing fractions. The authors reveal that prospective elementary school teachers in Taiwan did not use the concept of partitive division when solving or proposing word problems with the divisor being a proper fraction, which was a reflection of the emphasis given in Taiwanese mathematics textbooks.

However, our research showed that teachers were not aware of the sense of measurement for division and therefore formulated problems with a sense of sharing. We believe that this is due to the different contexts in different countries or locations, since in one case the focus is on teaching the sense of measure, and in the other the emphasis is on the sense of sharing, which may explain the discrepancy between the investigations.

Thus, while some studies indicate that teachers have a conception of division in relation to the sense of measure (Tirosh, 2000; Zembat, 2004; Lo & Luo, 2012), our investigation presents a different result, in which teachers only demonstrated knowledge of the partitive sense.

Returning to our research question: What specialized mathematical knowledge do teachers reveal when carrying out a problem formulation task on fraction division? In Table 3, we briefly present the knowledge revealed by the teachers:

Table 3: Teachers' knowledge of fraction division

Specialized knowledge revealed by teachers in relation to fraction division		
KoT Knowledge of topics	Phenomenology and applications	- Knowledge of one of the meanings of division: sharing; - Sense of ratio for fractions; - Structure of problems: partitioning by equal groups, measurement by equal groups and measurement by comparison (even if they don't know the meaning of measurement).
	Properties	d measurement by comparison (even if they don't know the meaning of measurement). - Knowledge of situations that generate decimal numbers: involving money and cooking; - Knowledge of the equivalence of fractions, decimal numbers, multiplicative inverse; - Dividing by 4 is the same as multiplying by ¼.
	Register of representation	- Knowledge of arithmetic, common and figural language (line and region).
	Procedures	- Knowledge of the algorithm for dividing fractions: inverting and multiplying, knowledge of solving fraction

		division.

Source: Prepared by the authors.

It can be seen that most of the teachers solved the problems using the invert-multiply algorithm and did not show flexibility with pictorial representation or the use of the unit concept. They were aware of the meaning of sharing and used this understanding to formulate most of the problems, opting for continuous representation and the area model over other models. Regarding the structure of the problems, the teachers preferred the presentation of division of partitioning of equal parts.

With this investigation, it was possible to identify that there are difficulties prior to fractions, related to the concept and meaning of division, as well as formulating problems with a sense of measurement, even with natural numbers. There was also a difficulty in understanding division by $\frac{1}{2}$ as half, in other words, understanding it as multiplication by two. In addition, difficulties were identified in discrete and continuous representation; in the concept of unit of reference; and in the correspondence and connection between the problem formulated and its solution as a pictorial representation.

The task addressed in this investigation proved powerful in revealing teachers' knowledge of fraction division, as well as highlighting other specialized knowledge. However, we cannot consider that the knowledge relating to these issues has been developed by the teachers or that it will be integrated into their practices. This is because the methodology, the analysis tool used (MTSK) and the time frame of the investigation do not allow the inference of analyzing the knowledge developed by the teacher, but only the knowledge revealed.

We corroborate the studies by Toluk-Ucar (2009), Isik & Kar (2012), who state that by providing teachers with opportunities to make sense of problems and develop reasoning for their justifications, they will improve their mathematical knowledge, thus helping them to teach with understanding and reasoning. We therefore stress the importance of working on solving and formulating problems from natural numbers onwards, as it is through these experiences that skills will be developed and mathematical knowledge expanded. We stress the importance of working on problem formulation in initial and continuing teacher training, so that they can expand their specialized knowledge of fraction division.

As possibilities for future research, we suggest investigations related to the teacher's interpretative knowledge and problems on fraction division. As well as studies that deepen knowledge regarding representation (discrete and continuous) for solving fraction division problems.

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References

Abu-Elwan, R. (2002). Effectiveness of problem posing strategies on prospective mathematics teachers' problem-solving performance. *Journal of Science and Mathematics Education in Southeast Asia*, *25*(1), 56-69.

Ball, D. L. & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In: J. Boaler (Ed.). *Multiple Perspectives on Teaching and Learning.* (pp. 83-104). Springer.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, *21*(2), 132-144.

Ball, D. L., Lubienski, S. & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In: V. Richardson (Ed.). *Handbook of Research on Teaching.* (pp. 433-456). American Educational Research Association.

Behr, M. J., Harel, G., Post, T. R. & Lesh, R. (1992). Rational number, ratio, and proportion. In: D. A. Grouws (Ed.). *Handbook of Research on Mathematics Teaching and Learning.* (pp. 296-333). Macmillan.

Cai, J., Hwang, S., Jiang, C. & Silber, S. (2015). Problem-posing research in mathematics education: Some answered and unanswered questions. In: F. M. Singer, N. F. Ellerton & J. Cai (Ed.). *Mathematical problem posing. From research to effective practice.* (pp. 3-34). New York: Springer.

Carrillo, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Àvila, D., ... & Muñoz-Catalán, M. C. (2018). The mathematics teacher's Specialized knowledge (MTSK) model. *Research in Mathematics Education*, *20*(3), 236-253.

Climent, N. (2002). *El desarrollo profesional del maestro de primaria respecto de la enseñanza de la matemática*. 2002. 185f. Tese (Doutorado). Universidad de Huelva.

Dehaene, S. (1997). *The number sense*. Oxford University Press.

DeWolf, M., Grounds, M. A., Bassok, M. & Holyoak, K. J. (2014). Magnitude comparison with different types of rational numbers. *Journal of Experimental Psychology: Human Perception and Performance*, *40*, 71-82.

Ellerton, N. F. (2013). Engaging pre-service middle-school teacher-education students in mathematical problem posing: Development of an active learning framework. *Educational Studies in Mathematics, 83*(1), 87-101.

Ervin, H. K. (2017). Fraction multiplication and division models: A practitioner reference paper. *International Journal of Research in Education and Science*, *3*(1), 258-279.

Fazio, L., & Siegler, R. (2011). *Educational practices series 22: teaching fractions*. Unesco.

Fischbein, E.; Deri, M., Nello, M. S. & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, *16*(1), 3-17.

Greer, B. (1992). Multiplication and division as models of situations. In: D. A. Grouws (Ed.), Handbook of research on *mathematics teaching and learning.* (pp. 276-295). Macmillan.

Isik, C. & Kar, T. (2012). An error analysis in division problems in fractions posed by preservice elementary mathematics teachers. *Educational Sciences: Theory & Practice*, *12*(3), 2289-2309.

Iskenderoglu, A. T. (2018). Fraction multiplication and division word problems posed by different years of pre-service elementary Mathematics teachers. *European Journal of Educational Research*, *7*(2), 373-385.

Izsák, A. (2003). "We want a statement that is always true". Criteria for good algebraic representations in the development of modeling knowledge. *Journal for Reaserch in Mathematics Education*, *34*(3), 191-227.

Janvier, C. (1987). Representation and understanding: The notion of function as an example. In: C. Janvier (Ed.). *Problems of representation in the teaching and learning of mathematics.* (pp. 67-71). Lawrence Erlbaum Associates Inc.

Kilic, C. (2015). Analyzing pre-service primary teachers' fraction knowledge structures through problem posing. Eurasia *Journal of Mathematics, Science & Technology Education*, *11*(6), 1603-1619.

Lamon, S. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In: F. K. Lester Jr (Ed.), *Second handbook of research on mathematics teaching and learning.* (pp. 629-667). National Council of Teachers of Mathematics.

Lee, K. H., Lee, E. J. & Park, M. S. (2016). Task modification and knowledge utilization by Korean prospective mathematics teachers. *Pedagogical Research*, *1*(2), 1–13.

Lee, S. J., Brown, R. E. & Orrill, C. H. (2011). Mathematics teachers' reasoning about fractions and decimals using drawn representations. *Mathematical Thinking and Learning*, *13*(3), 198-220.

Lesser, L. M. & Tchoshanov, M. A. (2005, 2, October). *The effect of representation and representational sequence on students' understanding.* 27º Annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Roanoke.

Lo, J. J. & Luo, F. (2012). Prospective elementary teachers' knowledge of fraction division. *Journal of Mathematics Teacher Education*, *15*(6), 481-500.

Ma, L. (1999). K*nowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Earlbaum. National Council of Teachers of Mathematics. (2000). Principles and standards of school mathematics.

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standard for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Ribeiro, M., Almeida, A. & Mellone, M. (2021). Conceitualizando Tarefas Formativas para Desenvolver as Especificidades do Conhecimento Interpretativo e Especializado do Professor. *Perspectivas da Educação Matemática*,*14*(35), 1-32.

Rojas, N. & Moriel-Junior, J. G. (2016). Fracciones y decimales. In: J. Carrillo, L. C. Contreras, N. Climent, M. Montes, D. I. Escudero, E. Flores & M. C. Muñoz-Catalán (Ed.). *Didáctica de las Matemáticas para maestros de Educación Primária.* (pp. 75-98). Ediciones Paraninfo, S.A.

Silva, F. A. F., Vidal, F. A. & Carvalho Filho, E. A. de . (2023). Análise da compreensão de professores de Matemática sobre as características visuais de figuras geométricas para o estabelecimento da relação parte-todo dos números racionais. *Revista Internacional de Pesquisa Em Educação Matemática, 13*(2), 1-16.

Simon, M. A. (1993). Prospective Elementary Teachers' Knowledge of Division. *Journal for*

Research in Mathematics Education, *24*(3), 233-254.

Stake, R. E. (1995). *The art of case study research.* Sage Publications, Inc.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research of Mathematics Education*, *31*(1), p. 5-25.

Toluk-Uçar, Z. (2009). Developing pre-service teachers understanding of fractions through problem posing. *Teaching and Teacher Education, 25*(1), 166-175.

Utley, J. & Redmond, A. (2008, March). P*rospective Elementary Teachers' Attitudes Towards and Knowledge of the Division of Fractions*. [Paper presentation]. Annual meeting of the Research Council on Mathematics Learning, Oklahoma City.

Xie, J. & Masingila, J. O. (2017). Examining interactions between problem posing and problem solving with prospective primary teachers: A case of using fractions. *Educational Studies in Mathematics*, *96,* 101-118.

Zembat, I. O. (2004). *Conceptual development of prospective elementary teachers: The case of division of fractions*. 2004. [Ph.D. dissertation]. The Pennsylvania State University.