Sociedade Brasileira de Educação Matemática

RIPEM

The construction of combinatorics concepts evidenced in mind maps

Cristiane de Arimatéa Rocha
Universidade Federal de Pernambuco
Olinda, PE — Brasil
✉ cristiane.arocha@ufpe.br
iD 0000-0002-4598-2074

Antonio Carlos de Souza
Universidade Estadual Paulista
Guaratinguetá, SP — Brasil
✉ ac.souza@unesp.br
iD 0000-0002-8044-0481

2238-0345

10.37001/ripem.v14i3.3818

Received • 03/03/2024
Approved • 17/04/2024
Published • 20/08/2024

Editor • Gilberto Januario

Abstract: Innovations in teaching practices in higher education are fundamental to promoting opportunities for students to engage in their training process. This study investigated the use of mind maps as an innovative pedagogical practice for constructing concepts about combinatorics by maths undergraduates. Twenty-five groups of prospective teachers from a maths teaching degree course participated in the research, taking an elective subject on combinatorics teaching. As a result, we observed that the different constructions of mind maps provide indications of the breadth, depth, and personal notions of combinatorics presented by the groups, which emphasize the following aspects: the limited variety of combinatorial problems, the evidence restricted to the invariant of order and the representation of formulas. It is clear that these aspects can be remedied by adopting innovative and collaborative teaching practices to produce mind maps on combinatorics, including as an assessment practice.

Keywords: Innovative Teaching Practices. Combinatorial Problems. Mind Map. Mathematics Teaching Degree.

La construcción de conceptos de combinatoria evidenciados en mapas mentales

Resumen: Las innovaciones en las prácticas docentes en la enseñanza superior son fundamentales para promover oportunidades para que los estudiantes se impliquen en su proceso de formación. Este estudio investigó el uso de mapas mentales como práctica pedagógica innovadora para la construcción de conceptos sobre combinatoria por parte de estudiantes universitarios de matemáticas. Participaron en la investigación 25 grupos de estudiantes de una licenciatura en matemáticas, que cursaban una asignatura optativa sobre la enseñanza de la combinatoria. Como resultado, se observó que las diferentes construcciones de mapas mentales proporcionan indicios de la amplitud, profundidad y nociones personales de combinatoria presentadas por los grupos, que enfatizan los siguientes aspectos: la escasa variedad de problemas combinatorios, la evidencia restringida a la invariante de orden y a la representación de fórmulas. Es evidente que estos aspectos pueden remediarse adoptando prácticas pedagógicas innovadoras y colaborativas para elaborar mapas mentales sobre combinatoria, incluso como práctica de evaluación.

Palabras-clave: Prácticas Docentes Innovadoras. Problemas Combinatorios. Mapa Mental. Licenciatura en Matemáticas.

A construção de conceitos de combinatória evidenciados em mapas mentais

Resumo: Inovações em práticas pedagógicas no ensino superior são fundamentais para a promoção de oportunidades de engajamento de estudantes no seu processo de formação. Assim, o presente estudo investigou o uso de mapas mentais como inovação pedagógica para a construção de conceitos sobre combinatória por licenciandos em matemática. Participaram da pesquisa 25 grupos de estudantes de um curso de licenciatura em matemática, que cursavam uma disciplina eletiva sobre o ensino de combinatória. Como resultado, observou-se que as diferentes construções de mapas mentais fornecem indicações para a abrangência, a profundidade e as noções pessoais de combinatória apresentadas pelos grupos. Esses ressaltam os seguintes aspectos: a pouca variedade de problemas combinatórios, a evidência restrita ao invariante de ordem e para representação de fórmulas. Esses aspectos podem ser dirimidos por meio da adoção de práticas pedagógicas inovadoras e colaborativas de produção de mapas mentais sobre combinatória, inclusive como prática avaliativa.

Palavras-chave: Práticas Pedagógicas Inovadoras. Problemas Combinatórios. Mapa Mental. Licenciatura em Matemática.

1 Introduction

Currently, the search for pedagogical innovation in higher education is on the agendas of several institutions in several countries. According to Almeida et al. (2022), pedagogical innovation is adopted as a criterion for institutional qualification in Portugal. The European Commission discussed the need for innovation for professional development in higher education, evaluating successful cases of innovation and indicating the need to develop innovative pedagogical practices (Inamorato dos Santos, Gausas, Mackeviciute, Jotautyte, & Martinaitis, 2019).

Wiebusch and Lima (2019), when investigating scientific productions on pedagogical innovation in higher education in Brazil, chose as analytical categories innovations related to teaching, the curriculum, and those that design pedagogical practices related to teaching methodologies and strategies. Among the innovative practices observed by the authors are conceptual maps, a creative approach through design thinking, mobile devices, Webquest, Radioweb, and active methodologies (Wiebusch & Lima, 2019).

According to Tavares (2020, p.15), "educational innovation practices are not distinguished by being creative, original, or much less technological. They differ from other usual educational practices in a circumscribed social context." In higher education, pedagogical innovation is "an intentional action that aims to improve student learning sustainably" (Walder, 2014, p. 197). Thus, to innovate, higher education institutions must transform traditional teaching practices, focusing on teachers' professional development.

Although the use of digital technologies combined with pedagogical practices enables innovation and flexibility in teaching and learning, Inamorato dos Santos et al. (2019, p. 10) point out that "innovation in teaching at higher education level is happening at a much slower pace than the increased availability of digital technology."

This statement could be observed during the COVID-19 pandemic when emergency remote teaching was adopted as a protective measure for the Brazilian population (Macaya & Jereissati, 2021). Basic education teachers implemented emergency remote teaching with difficulty because of the lack of direct contact with students and the inclusion of technological tools in pedagogical practices (Freitas, Almeida, & Fontenele, 2021) with the insufficient technological infrastructure of those involved in the process (Macaya & Jereissati, 2021).

In higher education, Santos et al. (2022) highlight the difficulties teachers face during remote teaching: problems with the lack of training or provision of resources by colleges, administrative difficulties, aspects related to mental health, and the use of software and digital tools, among others. According to these authors, despite the difficulties faced, there was learning, including the adoption of digital information and communication technologies (DICTs) in higher education, which can help change traditional pedagogical practices and assist in optimizing time and diversification in teaching and assessment practices (Santos et al., 2022).

However, despite the gains in learning, Inamorato dos Santos et al. (2019) highlight the need for higher education teachers to develop teaching skills using DICTs to obtain better student learning. The authors even emphasize the innovative impacts on higher education institutions (HEIs). Almeida et al. (2022, p. 23-24) reiterate that “the exploration of technologies can constitute an important component of pedagogical innovation processes” in higher education.

In the scope of teaching degrees, even before the COVID-19 pandemic, the National Curriculum Guidelines for the Initial Education of Basic Education Teachers (BNC - Formação), established by Resolution CNE/CP n. 2/2019, foresaw the need for knowledge on pedagogical innovation and technologies for teachers. The document ensures that teachers assume as pedagogical foundations the “pedagogical use of innovations and digital languages as a resource for developing prospective teachers’ competencies in tune with those in the BNCC [Base Nacional Comum Curricular] (Brasil, 2018)] and with the contemporary world” (Brasil, 2019, p. 5).

Researchers such as Pasqualli and Carvalho (2021) and Guerra, Ribeiro, Sousa, and Dias (2003) discuss some processes of pedagogical innovation regarding mathematical teaching and learning and present a series of didactic and/or technological resources that can assist in the development of practices for this purpose. Pasqualli and Carvalho (2021) describe pedagogical innovation as breaking with traditional teaching and learning methods, participatory management, protagonism, and reconfiguration of knowledge.

Based on the scenario described, this research aims to investigate the use of mind maps as a pedagogical innovation for the mathematics undergraduates’ construction of concepts about combinatorics. This article is divided into a brief reflection on mental maps and their use in formative and evaluation processes and in teaching and learning to assist the discussion on the topic. After that, we discuss combinatorics teaching in initial teacher education, specifically from the perspective of concept construction based on Gerard Vergnaud’s theory of conceptual fields. Subsequently, we present the methodological procedures and some results and discuss the pedagogical innovation adopted.

2 Mind maps as a pedagogical innovation in higher education

According to Estrela, Ferreira, Loureiro, and Sarreira (2022), graphic organizers are essential for mathematics teaching and learning. Diagrams, schedules, organization charts, mind maps, concept maps, and flowcharts are examples of graphic organizers that can be used to learn concepts.

Concept maps are graphic organizers capable of representing concepts and connecting phrases between concepts systematized hierarchically (Novak & Cañas, 2010). Wiebusch and Lima (2019) already indicate in their studies the use of concept maps for pedagogical innovation. Like the concept map, other graphic organizers, such as the mind map, can also have characteristics of pedagogical innovation (Pasqualli & Carvalho, 2021).

There are some differences between mind maps and concept maps. The creator of the mind map structure, Toni Buzan, defines it as “a dynamic and stimulating tool that helps thinking and planning become smarter and faster activities” (Buzan, 2009, p. 6). The similarity to a brain structure, the different colors, the systematization of ideas and organization, and its communicative and representative role make mind maps an interesting practice in the classroom that can be widely used in higher education. It presupposes using a “central concept that expands from the inside out, encompassing details” (Buzan, 2009, p. 20).

Debbag, Cukurbasi, and Findan (2021, p. 47) state that the mental map construction technique is “robust and exclusive for transforming all the different functions of the brain, such as words, imagination, numbers, reasons, images, lists, colors, and rhythm in action.” Santos and Santos (2023) highlight the role of mental maps as active methodologies since the student is understood as an active agent in their learning and indicate the use of such maps as an evaluative instrument due to the possibility of a “space for creativity that values self-reflection, promotion, and production of knowledge, [with] access and clarity of evaluation criteria, and [where] self-evaluation is encouraged” (Santos & Santos, 2023, p.5). These authors assert that this methodology can construct critical thinking due to its characteristics. However, it requires students to take responsibility for their learning in creating mental maps since “it demands a reasonable amount of time, as well as breaking with content elaboration styles of traditional written forms” (Santos & Santos, 2023, p. 17).

Several researchers, such as Lima, Santos, and Pereira (2020), Morandini, Anastacio, and Leite (2021), Debbag, Cukurbasi, and Findan (2021), used mind maps in higher education and discussed the benefits of this inclusion in teaching and learning.

Lima, Santos, and Pereira (2020) highlight the use of mental maps (MM) and conceptual maps (CM) during emergency remote teaching in a postgraduate course. The students created the maps using several technological tools (*Google Jamboard®*, *Microsoft PowerPoint®*, *MindMeister®*, *Canva®*, *Cmap Tools®*). According to the authors, through these innovative practices, after the maps had been concluded, they could observe there was learning development, assisting in “assimilating content,” “memorizing content,” and “improving the ability to synthesize the subject” (Lima, Santos & Pereira, 2020, p. 7). Based on experienced practice, the authors reaffirm the potential of this resource in organizing, systematizing, and presenting “relationships of meaning and hierarchy between ideas, concepts, facts or actions, synthesizing and structuring knowledge, and transmitting them quickly and clearly” (Lima, Santos & Pereira, 2020, p. 2).

Morandini, Anastacio, and Leite (2020) reported another use of mind maps in subjects of a mathematics teaching degree course (Fundamentals of Algebra and Arithmetic and Differential and Integral Calculus I). Through the adoption of mind maps in higher education, the authors highlight the opportunity for teachers to understand that “the way of understanding and exemplifying mathematics is different for each person” (Morandini, Anastacio & Leite, 2020, p. 4).

Debbag, Cukurbasi, and Findan (2021) conducted a study with 32 science graduates in Turkey to discuss their opinions on techniques using digital mind maps and mind maps drawn on paper. The study participants presented positive statements about mind maps, highlighting their beneficial use in reinforcing, evaluating, and visualizing learning and underscoring their use as a good practice for teaching different science content.

These graphic organizers have some characteristics of pedagogical innovation (Pasqualli & Carvalho, 2021) as they suggest a better understanding of the central concept and seek to: a) promote a break in the traditional teaching and learning process, as it highlights the student’s prior knowledge and enables the use of concept research to construct the map, whether using pencil and paper, or using technological tools (Santos & Santos, 2023; Debbag, Cukurbasi

& Findan, 2021); b) enable participatory management, since the information is not just with the teacher, providing debate between varieties of mental maps constructed by students (Morandini, Anastacio & Leite, 2020); c) enable protagonism as graphic organizers enable the student to take charge of their learning process by expanding representations and properties of the central concept systematized in a non-linear network of connections, participating in pedagogical decisions (Santos & Santos, 2023) d) discuss the reconfiguration of knowledge, based on the non-linear organization of connections of a concept, enabling openness to different ideas in the classroom, presenting the discussion between scientific, school, and other knowledge (Lima, Santos & Pereira, 2020).

Buzan (2009, p. 79) establishes essential characteristics for the analysis of mental maps in teaching and learning processes as follows: "1. The scope of the subject covered; 2. The depth of the approach to this topic; 3. The inclusion of one's ideas; 4. The adoption of techniques that facilitate learning, such as colors, symbols, and arrows." These characteristics can guide the teacher's work in evaluating those records.

The authors we discuss here ensured in their research that different learning is provided in basic education and higher education through the adoption of mind maps in teaching. They also highlighted their role as an active methodology and pedagogical innovation in initial teacher education. Therefore, they reinforce that this pedagogical practice can be considered innovative in qualification and assessment processes and teaching and learning in several areas of knowledge, especially mathematics.

3 Combinatorics and its insertion in the initial education of teachers who teach mathematics

The development of combinatorial reasoning at different levels of schooling is part of the lines of research of the Study Group on Combinatorial and Probabilistic Reasoning – Geração [Grupo de Estudos em Raciocínio Combinatório e Probabilístico]. Geração, throughout its 15 years of existence, has argued that to promote this reasoning, teaching practices must be included in the initial and/or continuing education of teachers who teach mathematics (Rocha, Montenegro & Borba, 2023). Rocha (2011) points out some difficulties in teaching combinatorics faced by teachers in elementary school (in Brazil, for students aged six to 14) and high school (for students aged 15 to 17). Among the difficulties are the valorization of a single resolution procedure and the choice of repetitive exercises that encourage memorization to the detriment of other resolution procedures, besides the differentiation between combinatorial problems. Martins and Silva (2014), when surveying in-service teachers who had already taught combinatorics in high school, found that to complement their education in this area, the research participants used textbooks and the Internet, as most of them stated that their initial education had gaps to support teaching on this content. To resolve these difficulties, Holanda (2017) defends the inclusion of subjects in teaching degree courses that discuss and enable the construction of knowledge to teach combinatorics.

Combinatorial reasoning develops throughout basic education and is configured as a way of thinking that progresses in the "analysis of situations in which, given specific sets, the elements thereof must be grouped to meet specific criteria (of choosing and/or ordering the elements) and determining –directly or indirectly– the total number of possible groupings" (Borba, 2010, p. 3). Relying on Vergnaud's (2009) theory conceptual fields, Borba (2010, 2013) lists and specifies a series of situations, invariants, and representations related to combinatorics.

Borba (2010, 2013) shows that some of the most frequently addressed problems in combinatorics can be understood from a set of situations (product of measurements, permutation, arrangement, combination), a set of invariants of the concept linked to each one of situations (choice, order, exhaustion of possibilities), and a set of representations (list, tree of possibilities, multiplication, fundamental counting principle, formulas) that offer senses and meanings to combinatorics teaching and learning.

In this way, Borba (2010, 2013) proposes a form of organization for combinatorics that expands understanding and comprehension of these structures, seeking to articulate the development of combinatorial reasoning. This form of organization allows for a broader understanding of combinatorics and can resolve some doubts and difficulties students and teachers face when studying or teaching this subject. Rocha (2019) analyzed combinatorics chapters in high school mathematics textbooks and classified the combinatorial problems and their specific invariants found in those chapters, systematizing them in Chart 1.

Chart 1: Types of combinatorial problems and their invariants/properties

Combinatorial problems	Concept invariants/properties
Product of measures	Choice
Simple arrangement	Choice, ordering
Arrangement with repetition	Choice, ordering, repetition
Simple combination	Choice; properties found in the book: quotient rule and equality between complementary combinations
Simple permutation	Ordering
Permutation with identical elements	Ordering disregarding identical or repeated elements when counting possibilities
Circular permutation	Ordering disregarding identical geometric arrangement in the count
Permutation with repetition	Ordering, repetition

Source: Adapted from Rocha (2019)

According to Rocha (2019), no combination with repetition was observed in the analyzed chapters. The differences found in Borba's (2010, 2013) and Rocha's (2019) research are due to the specificity of high school, which addresses a higher number of combinatorial problems. Regarding the representations or procedures observed in the chapters used in solving combinatorial problems, Rocha (2019) classified them into an enumerative procedure, which is related to natural language or the listing of possibilities; numerical procedure, which uses numerical operations or applies the fundamental counting principle (FCP); graphical procedure, which can be considered as a tree of possibilities or uses a double entry table; algorithmic procedure (formula, factorial) in addition to discussing more than one resolution procedure for the same problem.

Considering the scope of the structure that underpins the construction of combinatorial concepts discussed by the authors, practices are necessary to expand the discussion about this construction in initial teacher education. Rocha (2019, p.35) warns that teaching combinatorics is a complex action that permeates various aspects, such as "the choice of combinatorial problems addressed with greater or lesser emphasis, the complexity of these problems, the most evident procedures, the connections of combinatorics with mathematics and other areas."

The intramathematical relationships of combinatorics were observed in studies by Roa and Navarro-Pelayo (2001), who suggest that this content and the resolution techniques used "have profound implications for the development of other areas of mathematics such as probability, number theory, automaton theory and artificial intelligence, operative investigation, and combinatorial geometry and topology" (Roa & Navarro-Pelayo, 2001, p.1).

In this sense, besides the conceptual structures of combinatorics, we think it relevant that the in-service teacher or the prospective teacher establishes relationships between combinatorics and other content, enabling a more cohesive work in the subject to the detriment of an isolated approach to the content.

Regarding connections with other areas of knowledge, Batanero, Godino, and Navarro-Pelayo (1996) state that combinatorics plays an essential role in physics, such as in listing symmetries in crystallography, in chemistry, in the listing of organic molecules and enumeration of isomers; in biology, in the study of the spread of epidemics, in drug-testing techniques, the transmission of hereditary characters and genetic codes; in computer science, from the discussion of codes and languages, the formulation of algorithms, and the storage of information, among others. From this perspective, we understand that the relationship between combinatorics and other sciences establishes interesting contexts for developing content teaching and learning processes.

4 Methodology

This study was developed to investigate the use of mind maps as a pedagogical innovation for constructing concepts about combinatorics by mathematics teaching degree students. This qualitative research analyzes the combinatorial mental maps produced by participants in their context (Bogdan & Biklen, 1994). The research was developed following the rules of the Human and Social Sciences Research Ethics Committee, established by Resolution N. 510/2016 (Brasil, 2016), specifically Article 1, item VII. Data was collected in 2022 from students regularly enrolled in an elective curriculum component of the mathematics teaching degree course at a federal higher education institution. Elective subjects discuss the development of teaching and learning of combinatorics at different levels of education to promote the consolidation of knowledge of combinatorics and its teaching (Rocha, 2011; 2019) and are often offered to prospective teachers between the third and fourth years of the course.

To study this content, we asked students to build a mental map on combinatorics, organize information related to the concept in the branches, and explain the map briefly in writing. Depending on each student's availability, this activity could be done individually or in pairs. We gave them 15 days for preparation. To collect the activity, we made available a form on Google Classroom for students to insert their mental maps and their short explanations in writing.

As guidance for preparation, different resources were indicated to build the mind map from the beginning: *Google Jamboard, Microsoft PowerPoint, MindMeister, Canva*, and pencil and paper to adapt to the participants' different technological realities. We also recommended that they research combinatorics and its teaching to support the creation of mental maps.

Thirty-four prospective mathematics teachers participated in the activity and produced twenty-five combinatorics mind maps. The mind maps (MM) were numbered from 1 to 25 to guarantee the participants' anonymity.

To analyze the combinatorial mental maps, we chose the following criteria based on Buzan (2009): 1. The *scope of combinatorics*, taking as a basis Borba's (2010; 2013) and Rocha's (2019) discussions on the situations, invariants, and representations of combinatorics evidenced by the undergraduates, indicating aspects of their knowledge of combinatorics; 2. The close *connections between combinatorics and other mathematical concepts* based on the discussion by Roa and Navarro-Pelayo (2001), indicating possibilities for articulated work in mathematics and the presence of applications of combinatorics based on relationships between other sciences, based on Batanero et al.'s (1996) discussions; 3. *Techniques adopted for* *construction* include the type of organization (hierarchical or radial), whether they were built with technological resources or pencil and paper.

5 Results and discussions

This section presents and discusses the results found in each criterion adopted by the research. To this end, we considered the following topics: comprehensiveness of combinatorics verified in the constructed mental maps, the depth of connections between combinatorics and other mathematical concepts and relationships between other sciences, and the techniques adopted to facilitate learning.

5.1 Comprehensiveness of combinatorics verified in the constructed mental maps

The scope of combinatorial mental maps was analyzed through keywords correlated with situations, invariants, and representations related to this concept, as shown in the discussion of the theory of conceptual fields. Borba's (2010; 2013) and Rocha's (2019) discussions were used to classify these keywords. The MM mentioned the situations of simple arrangement (SA), simple combination (SC), simple permutation (SP), product of measures (PM), circular permutation (CP), permutation with identical elements (PI), permutation with repetition (PR), arrangement with repetition (AR) and combination with repetition (CR). Table 1 presents the frequency of situations mentioned to verify the coverage in each map.

Table 1: Frequency of combinatorial situations mentioned in MM

	SA	SC	SP	PM	PC	PI	PR	AR	CR	Total
MM1	1	1	1	0	1	0	1	0	0	5
MM2	1	1	1	0	0	0	0	0	0	3
MM3	1	1	1	0	0	0	0	0	0	3
MM4	1	1	0	1	0	0	0	0	0	3
MM5	1	1	1	1	0	0	0	0	0	4
MM6	1	1	1	1	1	1	0	0	0	6
MM7	1	1	1	0	0	0	0	0	0	3
MM8	1	1	1	0	0	0	0	0	0	3
MM9	1	1	1	0	0	0	0	0	0	3
MM10	1	1	1	1	1	0	1	0	0	6
MM11	1	1	1	1	0	0	0	0	0	4
MM12	1	1	1	0	0	0	1	1	1	6
MM13	1	1	1	0	0	0	0	0	0	3
MM14	1	1	1	1	0	0	0	0	0	4
MM15	1	1	1	0	0	0	1	1	0	5
MM16	1	1	1	1	1	0	1	1	1	8
MM17	1	1	1	0	0	0	0	0	0	3
MM18	1	1	1	1	0	0	0	0	0	4

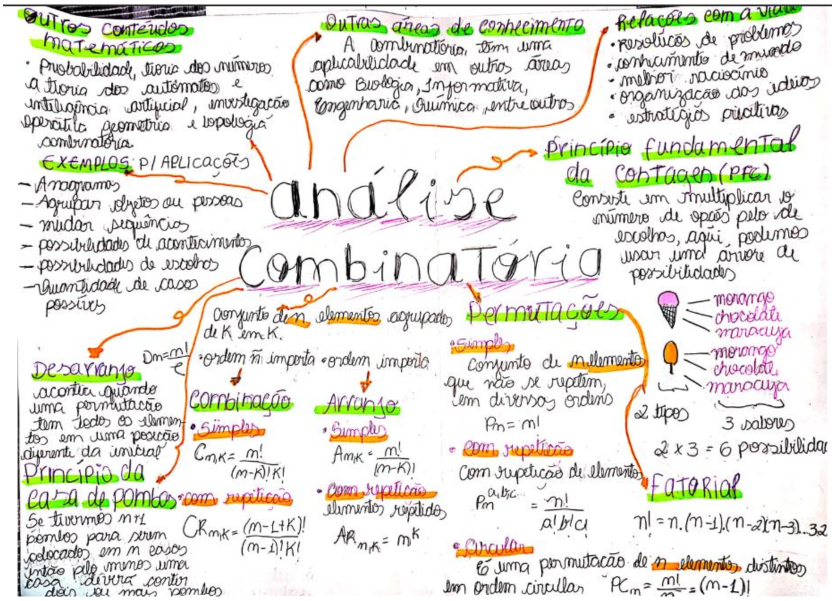
MM19	1	1	1	0	1	0	0	0	0	4
MM20	1	1	1	1	0	1	0	0	0	5
MM21	1	1	1	0	0	0	0	0	0	3
MM22	1	1	1	0	0	1	0	1	1	6
MM23	1	1	1	0	0	0	0	0	0	3
MM24	1	1	1	0	0	0	0	0	0	3
MM25	1	1	1	0	0	0	0	0	0	3
Total	25	25	24	9	5	3	5	4	3	

Source: Prepared by the authors.

We noticed that the situations of a simple arrangement and simple combination were mentioned in all maps. Simple permutation was frequent in 24 maps. 48% (12) of the mind maps mentioned only those three situations, indicating little coverage of that characteristic. Only MM12 and MM16 exhibited situations with the repetition invariant (arrangement with repetition, combination with repetition, and permutation with repetition). Few maps mentioned circular permutation (5) and permutation with identical or repeated elements (3). We verified in Table 1 that MM6, MM10, MM12, MM16, and MM22 cited at least six situations, which denotes better coverage concerning situations.

The product-of-measures situation was not mentioned, but examples of problems in this situation were included, which involved the choice of looks (clothing), meals, and ice-cream flavors, among others. Figure 1 presents the mental map with the widest scope of combinatorial situations.

Figure 1: MM16 combinatorics mind map



Source: Research collection (2024)

Furthermore, Figure 1 shows an example of how products of measures were represented through drawings that indicate the choice of ice cream sticks or cones in strawberry, chocolate, or passion fruit flavors. In the MM16 example, students draw the possibilities of ice cream flavors. They use a tree of possibilities and solve it using multiplication. Non-prototypical combinatorial situations such as derangement and the pigeonhole principle are also observed.

From Figure 1 and the choice of colors to underline the words, the fundamental principle of counting and the factorial seem to be regarded as situations by the authors of MM16. This mistake can also be observed in other maps, in which the fundamental counting principle (FCP), also called the multiplicative principle or counting principle, is considered a situation. For Borba (2010, 2013) and Rocha et al. (2023), this type of situation is called a discrete product of measures or a Cartesian product. Mekhmandarov (2000) highlights the importance of some fundamental points of the product of measures for understanding this structure: the choice of a single element from each set, the identification of a possibility as a new element of the product set, the recognition that an element of the original sets can appear in several possibilities and the understanding that the same possibility can only occur once in the product set.

Borba (2010, 2013) and Rocha et al. (2023) identify the FCP as a way of solving combinatorial problems –considered a type of representation, as it can be used to solve different types of problems. Similarly, the factorial is also used to answer combinatorial problems, being present in most of the formulas used to solve them; in this case, it is understood only as a representation. These errors were quite present in the mental maps created.

Based on what was exposed in MM16, we verified representations using the fundamental counting principle, the factorial, drawings, the tree of possibilities, multiplication, and formulas. In MM16, we found that situations are presented based on the absence or presence of order or repetition invariants. The choice invariants and geometric arrangement (position) are presented using examples in this map.

In general, focusing on the concept invariants presented by mental maps, ordering is explicitly indicated in most maps as a way of differentiating between arrangement and combination problems. The notion of repetition appears in few maps, basically, in the nomenclature of problems. It can be seen that the choice invariant appears in some situations in which students differentiate between the permutation problem and the arrangement problem.

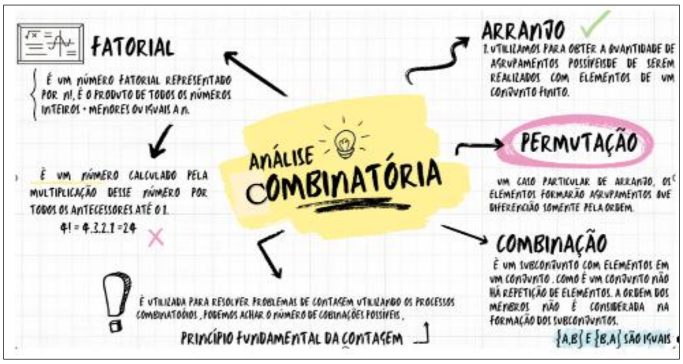
The representations emphasize the choice of formula to differentiate and exemplify combinatorial problems. Rocha and Souza (2021) state that working with combinatorial problems can provide an "environment for discussion about mathematics teaching and learning processes, even about the harm of presenting –without due initial discussions– mathematical formulas." In this sense, other explorations that enhance learning can be carried out before presenting formulas, such as valuing personal resolution strategies, proposing an explanation for the resolution with peers, or creating combinatorial problems.

5.2 The close connections between combinatorics, other mathematical concepts, and other areas of knowledge

In this subsection, we discuss the connections between mathematical content and combinatorics and between this content and other areas of knowledge presented in the maps created. At first, we observed that eight maps did not present intramathematical references (MM1, MM6, MM10, MM14, MM18, MM21, MM24, MM25).

Figure 2 presents an example of these maps that focused only on simple combinatorial problems without relating them to other mathematical content. MM24 is an example of a digital mind map built on *Canva*, which adequately addresses only situations of simple arrangement, simple combination, and simple permutation, following the definitions found in mathematics textbooks. It does not establish connections between the problems or integrate them with other mathematical content, presenting them in isolation.

Figure 2: MM24 combinatorics mind map

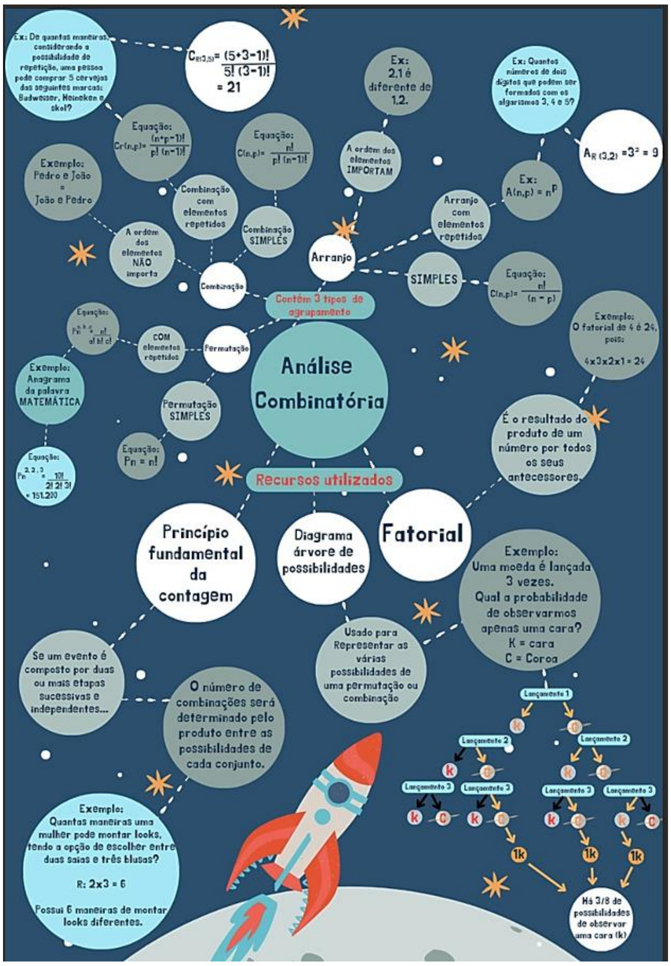


Source: Research collection (2024)

Godino and Batanero (2016) warn about the teaching and learning processes of combinatorics in isolation, highlighting that the separation of this content from the others ends up emphasizing teaching centered on formulas, solving exercises that focus only on combinatorial expressions, or differentiating combinatorial operations in a natural language statement.

Figure 3 presents another MM22 digital mind map built in *Canva*, and in it, situations, representations, and invariants are displayed in the context of planets in space.

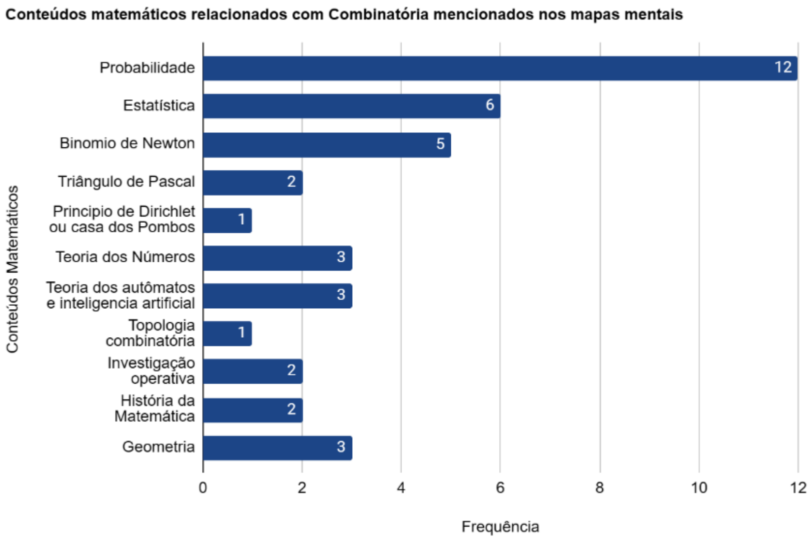
Figure 3: MM22 combinatorics mind map



Source: Research collection (2024)

In Figure 3, the construction of a tree of possibilities for coin tossing that articulates combinatorics and probability. According to Rocha (2019, p. 80), the representation of a "tree of possibilities assumes total relevance when carrying out enumerative actions to solve combinatorial problems." Notably, the probability content was indicated in twelve mental maps, with the students highlighting the most critical points. Graph 1 shows the frequency of mathematical content related to the mental maps produced by the prospective teachers.

Graph 1: Mathematical content related to combinatorics mentioned in mind maps.



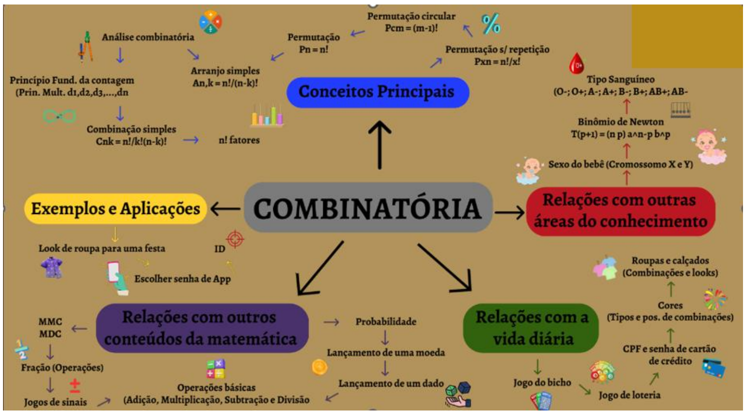
Source: Prepared by the authors (2024)

We observed the variety of proposed articulations, which can enable more articulate work in combinatorics. Rocha (2019) discusses the possibility of using combinatorics in various domains, including the formation of words in the language, the harmonization of notes in music, the representation of electrical circuits according to Kirchhoff's laws in physics, the investigation of genetic patterns in the DNA in biology, the structural organization of organic compounds in chemistry, and the resolution of optimization challenges linked to production, logistics, and distribution in economy

Regarding the extra-mathematical applications of combinatorics mentioned in the mental maps created by the undergraduates, more than half of the maps did not make extra-mathematical mentions: MM1, MM6, MM10, MM11, MM13, MM14, MM17, MM18, MM20, MM21, MM22, MM23, MM24, MM25.

Figure 4 shows areas related to combinatorics and examples of everyday contexts generally related to combinatorial problems. We found that unlike the relationships between combinatorics and mathematical content, in which examples of probability problems and their resolution were made available (MM22), on the MM19, other areas of knowledge are mentioned only in words and a list of possibilities, without situations that exemplify how the areas relate to combinatorics. In this way, the layout of mental maps can somehow restrict the presence of contextualization of combinatorics and its problems, as it is a proposal with few characters. Despite this, as shown in Figure 4, MM19 links combinatorics with the area of biology.

Figure 4: MM19 combinatorics mind map



Source: Research collection (2024)

Therefore, information about the areas of knowledge mentioned by the undergraduates' mental maps was systematized in Graph 2. When examining this graph, we see that health-related areas, such as biology, sciences, and genetics, are mentioned more frequently than the others. IT and chemistry areas were mentioned in five maps.

Graph 2: Areas of knowledge related to combinatorics mentioned in mind maps

Source: Prepared by the authors (2024)

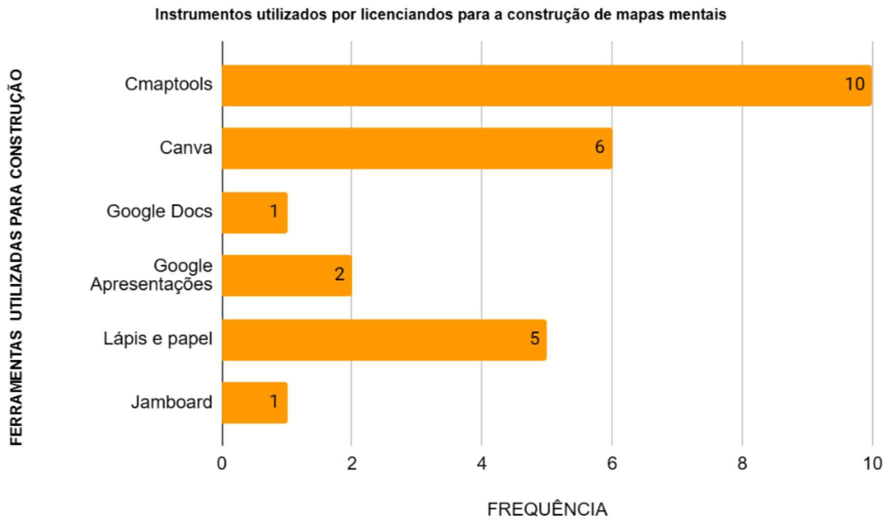
We also note the diversity of the areas expressed on the maps; however, more coordinated work between the different areas of knowledge and combinatorics is considered relevant, especially in the initial education of teachers who teach mathematics.

Regarding the inclusion of relationships between combinatorics and other areas of knowledge, some mental maps revealed the insertion of everyday content that concerns combinatorics, as presented in MM19. The mind maps displayed ideas that reference contexts used in combinatorial problems or ideas related to everyday activities: bank passwords, security systems, CPF numbers, car license plates, color combinations, stock market investments, games, lottery, clothing, snack menus, team organization, and sports championships. In explaining the MM22 map (Figure 3), the undergraduates consider: "No doubt, the application of this theme in our daily lives is comprehensive. We can experience it in the most varied situations involving decision-making or counting possibilities" (Research collection, 2024).

5.3 Techniques adopted to facilitate learning

This study found that the combinatorics mental maps presented by the prospective teachers were made through several instruments, such as pencil and paper, and technological tools, such as *Google Jamboard, Google Presentations, Canva, Cmap Tools, and Google Docs.* Graph 3 presents the frequency of instruments used to create mental maps. BNC-Formação defines as a general teaching competency the action of "understanding, using, and creating digital information and communication technologies in a critical, meaningful, reflective, and ethical way in the different teaching practices, as a pedagogical resource and as a training tool" (Brasil, 2019, p. 13).

Graph 3: Instruments used by undergraduate students to construct mind maps



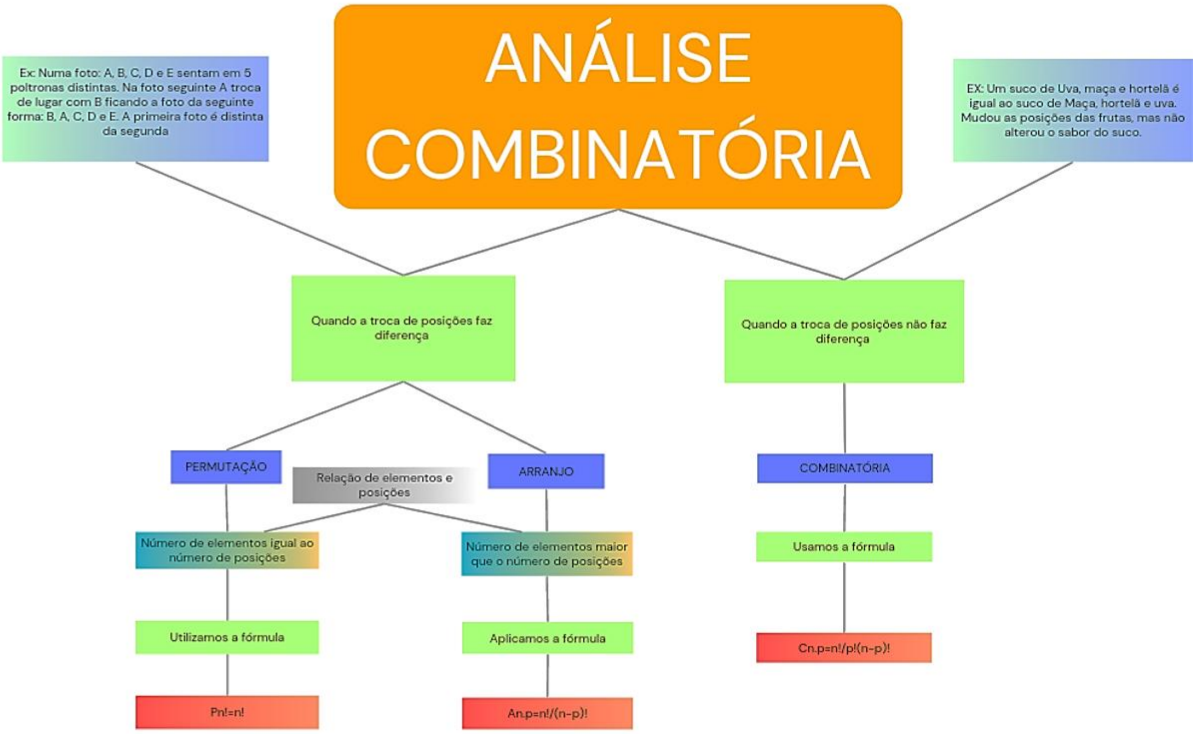
Source: Prepared by the authors (2024)

The prospective teachers' techniques for organizing mental maps were hierarchical and radial. In the hierarchical form, the concepts (situations, invariants, and representations) followed a top-down order, as shown in Figure 5. MM1, MM3, MM4, MM7, MM8, MM9, MM10, MM13, MM14, MM15, MM20, and MM25 used the hierarchical form in their mental maps.

If we look at Figure 5, MM25 not only presented the nomenclature for different combinatorial situations (permutation and arrangement), it also presented a definition and/or characteristic of the nature of combinatorial problems, displayed the invariant of order (position), highlighted as a symbolic representation the formulas, and explained two combinatorial problems. We also noticed an error in the use of combinatorial nomenclature as the type of combination problem.

From the analysis, it is clear that MM25 indicates students' knowledge of the construction of concepts in combinatorics, mainly when explaining the formulas as a representation in each of the proposed situations. We verified gaps in various combinatorial situations, presenting only the simplest ones.

Figure 5: MM25 combinatorics mind map



Source: Research collection (2024)

The remaining maps were organized radially. Buzan (2009) states that the main ideas on the map generally occupy the central position and are related to other ideas that complement each other (Figures 1, 2, 3, and 4). The examples presented suggest that different ways of understanding and relating combinatorics occur, indicating that the choices produced in each map provide clues to relationships that can be expanded, constructed, and consolidated to promote a broad development of combinatorial reasoning.

6 How can adopting mind maps as an innovative pedagogical practice in higher education help construct concepts about combinatorics by prospective mathematics teachers?

The combinatorics mental maps constructed by undergraduate students allowed discussion about the scope of combinatorics, inserting other problems that need to be understood by mathematics teaching degree students into simple combinatorial situations (arrangement, combination, and permutation). The practice made it possible to discuss the properties/invariants of the concept in each situation, bringing more elements to differentiate between the types of combinatorial problems, besides enabling the understanding of different representations beyond the formula.

Mind maps reference the relationships between combinatorics, other mathematical concepts, and other areas of knowledge or their absence. This knowledge can help choose the applications of combinatorics and the activities that can be part of a teacher's actions. Articulated combinatorial work with different concepts and areas of knowledge can enable a deeper understanding of this knowledge. We suggest further studies to discuss how these articulations can occur in the classroom. Due to the low frequencies of other content in mental maps, it is still necessary to reinforce the articulated discussions of combinatorics with other mathematical content and other areas of knowledge.

This type of record allowed us to notice the similarities and differences in each mental map, which can be associated with understanding and learning the concept of combinatorics. From this perspective, the teacher can, based on the information, reflect on the path to be taken to resolve the difficulties encountered. An example of this path could be the socialization of mental maps created, and a posteriori comparison or self-evaluation of each record can be carried out, debates about different points of view, or even other research that substantiates the arguments, providing opportunities for new learning.

In this way, we advocate that such innovative pedagogical practices should be applied in basic education and higher education, which could help build connections between different areas and combinatorics and strengthen the identification of applications in contexts of this content that are part of everyday life.

It is essential to develop a work in initial teacher education that articulates combinatorics with other areas of knowledge and contexts that are part of everyday life that arouse motivation and interest in students. This action can support the broader construction of combinatorial reasoning, which may result in the development of knowledge to teach combinatorics.

References

Almeida, L., Gonçalves, S., do Ó, J. R., Rebola, F., Soares, S. & Vieira, F. (2022). *Inovação Pedagógica no Ensino Superior: cenários e caminhos de transformação*. Lisboa: Agência de Avaliação e Acreditação do Ensino Superior. Coleção A3es readings.

Batanero, C. M., Godino, J. D. & Navarro-Pelayo, V. (1996). *Razonamiento Combinatório*. Madri, Ed.Síntesis.

Borba, R. (2010, julho). O raciocínio combinatório na Educação Básica. In: Anais do 10 Encontro Nacional de Educação Matemática – X ENEM. (pp. 1-12), Salvador, BA.

Borba, R. (2013, julho). Vamos combinar, arranjar e permutar: aprendendo Combinatória desde os anos iniciais de escolarização. In: Anais do 11 Encontro Nacional de Educação Matemática – XI ENEM. (pp. 1-12), Curitiba, PR, Brasil.

Bogdan, R. & Biklen, S. (1994). *Investigação qualitativa em educação: uma introdução à teoria e aos métodos*. Lisboa: Porto Editora.

Brasil. Ministério da Saúde, Conselho Nacional da Saúde. (2016). *Resolução nº 510/2016.* Normas aplicáveis a pesquisas em Ciências Humanas e Sociais. Brasília, DF.

Brasil. Ministério da Educação. Secretaria de Educação Básica. (2018). *Base Nacional Comum Curricular*. Brasília, DF.

Brasil. Ministério da Educação, Conselho Nacional de Educação, Conselho pleno. (2019). *Resolução CNE/CP nº 2*. Diretrizes Curriculares Nacionais para a Formação Inicial de Professores para a Educação Básica. Brasília, DF.

Buzan, T. (2009). *Mapas Mentais*. Tradução por P. Polzonoff Jr. Rio de Janeiro, RJ: Editora Sextante.

Debbag, M., Cukurbasi, B. & Findan, M. (2021). Use of digital mind maps in Technology Education: a pilot study with pre-service Science Teachers. *Informatics in Education*, *20*(1), 47-68.

Estrela, A., Ferreira, P., Loureiro, C. & Sarreira, P. (2022). Organizadores gráficos: ferramentas essenciais no ensino e aprendizagem de matemática e de ciências físicas e naturais. In: A. Dias & C. Loureiro. *Práticas de Integração Curricular nos 1.º e 2.º Ciclos do Ensino Básico*. (pp. 145-178). Escola Superior de Educação de Lisboa: Instituto Politécnico de

Lisboa.

Freitas, A. C. S., Almeida, N. R. O. de & Fontenele, I. S. (2021). Fazer docente em tempos de ensino remoto. *Ensino em Perspectivas*, *2*(3), 1–11.

Godino, J. D & Batanero, C. (2016). Implicaciones de las relaciones entre epistemologia e instrucción matemática para el desarrollo curricular: el caso de la Combinatoria. *La matematica e la sua didattica*, Università di Bologna, *24*(1-2), 17-39.

Guerra, A. de L. e R., Ribeiro, G. A., Sousa, M. A. de M. A., & Dias, E. C. V. (2023). Inovação e Matemática: como as tecnologias estão revolucionando o ensino. *Revista Ibero-Americana de Humanidades, Ciências e Educação*, *9*(8), 2298–2306.

Holanda, D.S. (2017). *Investigando uma proposta de formação inicial de professores de matemática no interior de Pernambuco: conhecimentos docentes de combinatória*. 2017. 120f. Dissertação. (Mestrado em Educação Matemática e Tecnológica). Universidade Federal de Pernambuco. Recife, PE.

Inamorato dos Santos, A., Gausas, S., Mackeviciute, R., Jotautyte, A.& Martinaitis, Z. (2019). Innovating Professional Development in Higher Education: an analysis of practices, EUR 29676 EN, *Publications Office of the European Union*, Luxembourg, 1-75.

Lima, A.C.B., Santos, D.C.M. & Pereira, A.P.S. (2020). Mapas Mentais e Conceituais como Ferramentas para a aprendizagem significativa no Ensino Remoto. In: *Anais do 3º IntegraEAD*. Educação e tecnologias digitais em cenários de transição: múltiplos olhares para aprendizagem. (pp.1-10). UFMS, MS.

Macaya, J.F.M. & Jereissati, T. (2021). Continuity of learning during the COVID-19 pandemic: The use of ICT in Brazilian public schools In: *Education and digital technologies: Challenges and strategies for the continuity of learning in times of COVID*-19. Núcleo de Informação e Coordenação do Ponto BR. São Paulo, SP: Comitê Gestor da Internet no Brasil, 2021.

Martins, G.G. & Silva, J.D. da. (2014). Reflexão sobre o ensino de análise combinatória no Ensino Médio: percepções de professores formados no CEUNES–UFES. *Amazônia. Revista de Educação em Ciências e Matemática*, *11*(21), 44-52.

Mekhmandarov, I. (2000). Analysis and synthesis of the Cartesian Product By Kindergarten Children. In. *24th Proceedings of the Conference of the International Group the Psychology of Mathematics Education*(PME), Hiroshima, Japan, July 23-27, 3, 295-301.

Morandini, D. C. W., Anastacio, L. R. & Leite, M. G. de L. M. (2021). Mapas Mentais: Experiências no Ensino Remoto Emergencial Universitário de Licenciatura Plena em Matemática. In *Anais do Encontro Virtual de Documentação em Software Livre e Congresso Internacional de Linguagem e Tecnologia Online*, [S. l.], *9*(1).

Novak, J. D. & Cañas, A. J. (2010). A teoria subjacente aos mapas conceituais e como elaborá-los e usá-los. *Práxis Educativa*, Ponta Grossa, *5*(1), 9-29.

Pasqualli, R. & Carvalho, M.J.S. (2021). As inovações pedagógicas nos cursos de licenciatura em ciências naturais e matemática a distância dos Institutos Federais de Educação, Ciência e Tecnologia no Brasil. *Revista Humanidades e Inovação*. *8*(50), 132-146.

Roa, R & Navarro – Pelayo, V. (2001). Razonamiento combinatorio e implicaciones para la enseñanza de la probabilidad. In: *Actas de Jornadas Europeas de Estadística: La enseñanza y la difusión de la estadística*. Instituto Balear de Estadística. Consejaría de

Economía, Comercio e Industria. Gobierno de las Islas Baleares. Palma (Islas Baleares), 253-264.

Rocha, C. A. (2011). *Formação docente e o ensino de problemas combinatórios: diversos olhares, diferentes conhecimentos*. 2011. 191f. Dissertação (Mestrado em Educação Matemática e Tecnológica). Universidade Federal de Pernambuco. Recife, PE.

Rocha, C. A. (2019). *Estudo de combinatória no ensino médio à luz do enfoque ontossemiótico: o que e por que priorizar no livro didático e nas aulas?*. 2019. 380f. Tese (Doutorado em Educação Matemática e Tecnológica). Universidade Federal de Pernambuco. Recife, PE.

Rocha, C.A. & Souza, A.C. (2021). Conhecimento de crianças pequenas da Educação Infantil e alunos dos anos iniciais do Ensino Fundamental sobre Combinatória: O que apontam as pesquisas brasileiras no período de 2010 a 2019? *Educ. Matem. Pesq.*, São Paulo, *23*(4), 452-484.

Rocha, C. A.; Montenegro, J.A. & Borba, R.E.S.R. (2023). Desafios de Ensinar Combinatória: contribuições do Geração- UFPE sobre Conhecimentos Docentes. In: S.S. Santos, G. C. Barbosa & P. B. Martins (Orgs.) *Ações mobilizadas por professores que ensinam combinatória, estatística e probabilidade* [recurso eletrônico]: reflexões, proposições e desafios. (81-103). Santo Ângelo, SP: Metrics.

Santos, J.T.T.T. dos, Assunção-Luiz, A. V., Eburneo Pereira, A. L., Barbosa Luciano, A. R. M., Carvalho Filho, I. F. P. de, Lima Santos, M. J. S. F., & Garbin, M. C. (2022). Dificuldades enfrentadas por docentes do ensino superior frente ao contexto da pandemia de COVID-19. *Revista Ibero-Americana de Educação, 88*(1), 111-126.

Santos, P.R. dos, Santos, E. F. dos. (2023). *Mapas Mentais como Instrumento avaliativo: uma proposta de aplicação para a avaliação da aprendizagem.* Lorena, São Paulo: Ed. dos autores.

Tardif, M. (2002). *Saberes docentes e formação profissional.* Petrópolis, RJ: Editora Vozes.

Tavares, F. G. de O. (2020). *Práticas educacionais inovadoras e costumeiras: fatores de diferenciação.* Dissertação (Mestrado em Educação). Universidade de São Paulo. São Paulo, SP.

Walder, A. M. (2014). The concept of pedagogical innovation in higher education. *Education Journal*, *3*(3), 195-202.

Wiebusch, A., & Lima, V. M. do R. (2019). Inovação nas práticas pedagógicas no Ensino Superior: possibilidades para promover o engajamento acadêmico. *Educação Por Escrito, 9*(2), 154–169.