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Using GeoGebra as a tool for teaching the Padovan board and tiling

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Abstract: This study is part of ongoing doctoral research investigating the teaching of Padovan's combinatorial approach and the scarcity of works on this topic. The objective is to propose an approach to explore combinatorial identities with the help of GeoGebra. This approach aims to provide theoretical support to the teacher to understand and teach the combinatorial approach visually through a didactic situation supported by the construction of activities in the software. The methodology adopted is didactic engineering in its first two phases, given the nature of the ongoing research, and the teaching session was structured following the phases of the theory of didactic situations. This proposal is expected to contribute to the possible integration of GeoGebra into the teaching of the history of mathematics, considering the potential impact of a visual approach to the topic in teaching practice.

Keywords: Didactic Engineering. GeoGebra. History of Mathematics. Padovan Sequence.

El uso de GeoGebra como herramienta para la enseñanza del tablero y del mosaico de Padovan

Resumen: El presente estudio es parte de una investigación doctoral en curso, que investiga la enseñanza del enfoque combinatorio de Padovan y la escasez de trabajos sobre este tema. El objetivo es proponer un enfoque para explorar identidades combinatorias con la ayuda de GeoGebra. Este enfoque tiene como objetivo brindar apoyo teórico al docente para comprender y enseñar el enfoque combinatorio de manera visual, a través de una situación didáctica apoyada en la construcción de actividades en el software. La metodología adoptada es la Ingeniería Didáctica en sus dos primeras fases, dada la naturaleza de la investigación en curso, y la sesión docente se estructuró siguiendo las fases de la Teoría de Situaciones Didácticas. Se espera que esta propuesta contribuya a la posible integración de GeoGebra en la enseñanza de la Historia de las Matemáticas, considerando el impacto potencial de un abordaje visual del tema en la práctica docente.

Palabras clave: Ingeniería Didáctica. GeoGebra. Historia de las Matemáticas. Sucesión de Padovan.

O uso do GeoGebra como ferramenta para o ensino do tabuleiro e dos ladrilhamentos de Padovan

Resumo: O presente estudo é parte de uma pesquisa de doutorado em andamento, que investiga o ensino da abordagem combinatória de Padovan e a escassez de trabalhos sobre esse tema. O objetivo é propor uma abordagem para explorar as identidades combinatórias com o auxílio do GeoGebra. Essa abordagem visa fornecer suporte teórico ao professor para compreender e ensinar a abordagem combinatória de forma visual, por meio de uma situação didática apoiada na construção de atividades no software. A metodologia adotada é a Engenharia Didática em suas duas primeiras fases, dada a natureza da pesquisa em andamento, e a sessão de ensino foi estruturada seguindo as fases da Teoria das Situações Didáticas. Espera-se que essa proposta contribua para a possível integração do GeoGebra ao ensino de História da Matemática, considerando o potencial impacto de uma abordagem visual do tema na prática docente.

Palavras-chave: Engenharia Didática. GeoGebra. História da Matemática. Sequência de Padovan.

1 Introduction

The study of recurrent sequences is widely explored in several areas, providing opportunities for practical applications that enrich the understanding and visualization of mathematical concepts and identities. A well-known example among researchers is the Fibonacci sequence, which has several applications in fields as diverse as biology, physics, and many others (Bergum, Philippou & Horadam, 1996). This wide range of applications highlights the relevance and versatility of recurrent sequences as fundamental tools in different subjects and contexts.

Most authors on the history of mathematics tend to focus on curiosities and illustrations related to the Fibonacci sequence (Burton, 2007). However, many fail to consider crucial aspects, such as the mathematical contributions and pioneering work of the creator of the Fibonacci sequence, as well as the evolutionary process of the sequences.

Thus, this research focuses only on Padovan's combinatorial approach, a third-order sequence that shares similarities with the Fibonacci sequence. To this end, it is worth highlighting the evolution of sequences, in general, integrating with other mathematical content; for example, there is combinatorial interpretation and sequences (Koshy, 2001). Therefore, combinatorial models are created for the recurring sequences, allowing a generalization of the order, according to Benjamin and Quinn's (2003) study.

On the other hand, there is a lack of such content aimed at education, with most studies focusing on pure mathematics. Thus, the proposal is to investigate Padovan's combinatorial approach, specifically aimed at the initial education of mathematics teachers. This study aims to explore the mathematical history behind these numbers and use GeoGebra to enable teachers to visualize mathematical identities more dynamically and interactively.

This work aims to present a teaching proposal that adopts a visual approach to Padovan's combinatorial interpretation, aimed at the teaching degree in mathematics (initial teacher education), using GeoGebra as a resource. To this end, the research is based on the theory of didactic situations (Brousseau, 2002), which guides the planning of a teaching session, and on didactic engineering (Artigue, 2020), chosen as the research methodology due to its close relationship with the theory of didactic situations. These approaches provide a robust theoretical framework for developing the didactic proposal and for investigating the teaching and learning processes involved.

In this sense, this research is biased toward exploring the French tradition of didactics of mathematics, implementing didactic engineering as a research methodology. Thus, as

Calderón and León (2012) stated, didactic engineering presents a correlation between an epistemic analysis and the implementation of a didactic structure. The authors affirm that the essential particularity of this research methodology is the analysis provided during the design of the activities (*a priori*) and the analysis after its implementation (*a posteriori*).

Given the feasibility of visualization and the restrictions of existing research on the topic, we suggested that GeoGebra be used to explore Padovan's combinatorial identities. This approach can enrich the teaching process in the teaching practice of the mathematics teacher, offering a powerful tool to illustrate and understand complex concepts in a more accessible and dynamic way.

Therefore, this is a theoretical work without empirical data, following the itinerary of didactic engineering and the theory of didactic situations. We will use the first two phases, preliminary analysis and design and *a priori* analysis, without prejudice to the research since the other phases can be used in future work.

Based on the above, the following sections outline the first two phases of didactic engineering: preliminary and *a priori* analyses. Given the nature of this didactic proposal, these phases are approached with a theoretical focus.

2 Preliminary analysis

During the preliminary analyses and the other phases of the project, we could perceive the influence of the theory of didactic situations on the research methodology of didactic engineering. The preliminary analysis phase reveals the systematic inclusion of the epistemological component. The design of tasks and situations is carried out in a particular way, giving importance to the search for situations that capture the epistemological essence of mathematics to be learned, to optimize the potential of the environment to study students' autonomous learning and manage the processes of deconcentration and institutionalization (Artigue & Perrin Glorian, 1991). Didactic engineering is based on the theory of didactic situations, fulfilling a dual role: serving as a production approach for teaching based on research results while playing the role of a research methodology (Ferreira, 2016).

Based on the above, the research presents the results of each didactic engineering phase. Initially, we carried out an epistemic analysis of the evolutionary development of the Padovan sequence in light of his combinatorial approach. We also sought to verify whether there are investigations that involve the articulation between Padovan's combinatorial approach and GeoGebra. Next, we establish an *a priori* conception and analysis based on a documentary-type tracking derived from the state of the art, focusing on interpreting the combinatorial approach of these numbers.

Preliminary analyses are the basis for the design phase of the process, involving different dimensions, and especially the following three:

Epistemological analysis of the content in question, addressing the historical part. This analysis helps researchers set the expected objectives and identify possible epistemological obstacles. This analysis also supports searches for mathematical situations that represent the desired knowledge, called the fundamental situation by the theory of didactic situations. A fundamental situation is a problematic situation whose resolution requires this knowledge. Epistemological analysis helps researchers achieve the necessary position concerning their educational environment.

Institutional analysis, identifying the characteristics of the context in which didactic engineering is inserted and the conditions and constraints faced. These conditions and constraints can be situated at different levels, called the hierarchy of co-determination levels (Chevallard, 2002). These may also be associated with curriculum choices, content relating to associated pedagogical practices, more general curriculum characteristics, the teaching of the subject, accessible (technological) resources, assessment practices, school organization, characteristics of the students and teachers involved, and the way the school relates to its environment, among others. Depending on the precise objectives and context of the research, the importance attributed to these different levels may vary.

Didactic analysis, building what the research offers regarding the teaching-learning process of the content in question and whether this will likely guide the design.

3 The Padovan sequence and its combinatorial approach

The analysis of recurring numerical sequences often stands out when examining the Fibonacci sequence, given its historical relevance, while neglecting other essential aspects of its mathematical contribution (Burton, 2007). As a result, other recurring numerical sequences remain unknown, not receiving due attention in initial teacher education courses, as in the Padovan sequence, which is little explored in undergraduate mathematics courses (Vieira, Alves & Catarino, 2022).

The Padovan sequence is a third-order recurrent sequence with a specific recurrence relation. Its name is attributed to the Italian architect Richard Padovan (Stewart, 1996; Vieira, 2020). This relationship is represented by $P(n)=P(n-2)+P(n-3)$, with $n\geq 3$ and $P(0)=P(1)=P(2)=1$, where $P(n)$ is the n-th term of the sequence.

Marohnic, Kovacic, and Radisic (2013) claim that Padovan's study was based on architect Hans van der Laan's (1904-1991) work, who discovered a new irrational number, the plastic number. However, according to research, this number was previously investigated by Gérard Cordonnier (1907-1977), who also made this sequence known as the Hans Van Der Laan or Cordonnier sequence (Alves & Catarino, 2022).

The definition of morphic numbers emerged from studies related to the properties of the golden number, also known as the divine proportion. Those numbers present only two solutions: the plastic and the golden number. Thus, we observe the relationship between the sequences of Fibonacci and Padovan (Aarts, Follink & Kruijtzer, 2001).

In this context, the investigation of the combinatorial interpretation of the Padovan sequence stands out as a contribution to mathematics teaching, offering a relevant approach based on development via didactic engineering. The academic field has been enriched by several works dedicated to exploring the combinatorial properties of recurrent sequences. Among these studies, some significant contributions stand out, such as those by Benjamin and Quinn (2003), Koshy (2001), Spreafico (2014), Vieira, Alves and Catarino (2022), and Tedford (2019).

These researchers have presented innovative approaches and in-depth analyses, improving the understanding of the Padovan sequence and the pedagogical potential underlying the combinatorial interpretation in recurrent mathematical sequences. The intersection between these investigations enriches the body of theoretical-mathematical knowledge and provides valuable opportunities for curriculum development and more effective pedagogical practices in mathematics teaching.

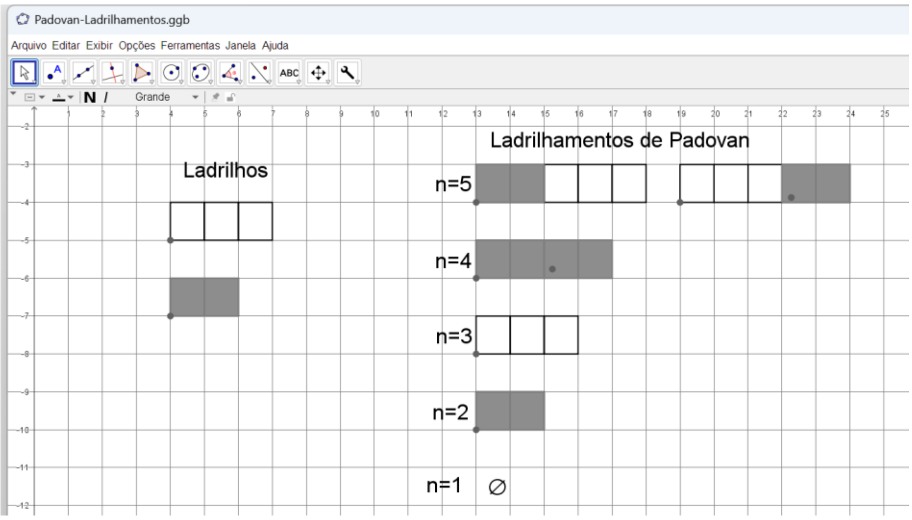
Given the connection between these numbers, we felt motivated to explore other sequences, enhancing their respective epistemological and mathematical developments. Indeed, we noticed the innovative character of Padovan's mathematical evolution, providing research participants with a unique study opportunity. This fact justifies the interest in developing a teaching proposal for the study of this sequence, with emphasis on the combinatorial model of these numbers.

In the context of boards, their formation is given by the composition of squares called *cells*, enumerated to describe specific positions, defined as an n-board. Tiles are the pieces used to fill a board, while tiling is the way to cover it (Spreafico, 2014).

It is worth noting that the combinatorial model deals with the study of the combinatorial interpretation of a sequence, which is carried out in this research through tilings.

Tedford (2019) introduced Padovan tilings using 1 x 2 gray domino tiles and 1 x 3 white trimino tiles to cover an n-board. With this, the research presents the theorem, which he calls p(n), the number of ways to cover an n-Padovan board, and P(n), the n-th term of the sequence. Therefore, we have that p(n)=P(n-2), for n≥ 2. Figure 1 depicts the cases for n=1, 2, 3, 4, 5.

Figure 1: Padovan tilings in GeoGebra



Source: Adapted from Tedford (2019)

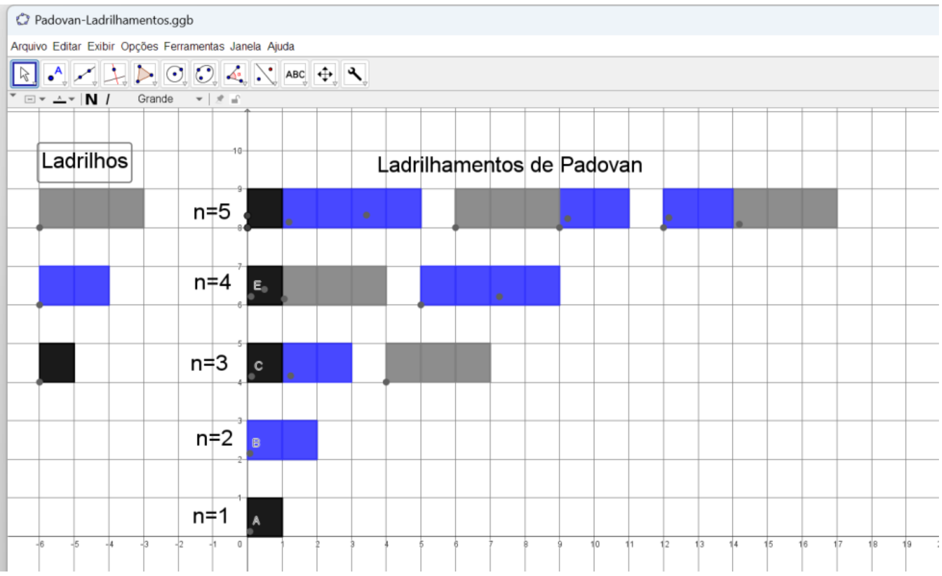
Note that for n=1, there are no possible tilings with the respective tiles available. For n=2, the domino forms only one tiling. For n=3, there is also only one tiling, and only the trimino is present. For the case n=4, one tiling is composed of the concatenation of two dominoes.

Vieira, Alves, and Catarino (2022) studied the generalization of Padovan tilings, presenting the combinatorial model to extend Padovan numbers. Thus, the theorem deals with the number of ways of tiling (p(n)) with the following tiles: black square, blue domino of size 1 x 2, and gray trimino of size 1 x 3. However, for this model, there is a construction rule in which the black square can only be inserted when completing the tilings. Therefore, it can only be inserted once and at the beginning of the board.

Thus, we have that for n≥ 1, the possible tilings of a 1 x n board are given by p(n)=P(n), where P(n) represents the n-th term of the Padovan sequence.

Figure 2 presents several cases addressed by the theorem proposed by Vieira, Alves, and Catarino (2022), providing an enlightening visualization of the tilings and their corresponding configurations regarding the available pieces. This graphical representation allows a more detailed analysis and a visual understanding of the results obtained in the context of the theorem.

Figure 2: Padovan's tilings with GeoGebra (another approach)



Source: Own elaboration (2024)

The demonstration of these theorems occurs through the mapping of sets and subsets. Thus, p(n) is considered the sum of the tilings, in which p(1)=1=P(1); considering the tilings of size n-2, with n≥ 3, we have two cases:

2nd — Consider two subsets of tilings: a subset composed of tilings that start with a black square and another subset formed by the remaining tilings. Now introduce a new set of tilings, starting with a blue domino measuring 1 x 2. This set is obtained by juxtaposing tilings of size n−2 that do not start with a black square with the set of tilings that start with the black square of size 1 x 1. Moreover, it includes the addition of a blue domino measuring 1 x 2 immediately after this square.

2nd — Consider two sets of tilings: one set that starts with a trimino (tiles that do not start with black squares) and another set composed of tilings that start with a black square, followed by a trimino. A new set of tilings has been introduced, starting with a 1 x 3 gray trimino. This new set is formed by juxtaposing the tilings of size n−3 that do not start with a black square with the set of tilings that start with the black square of size 1 x 1 and includes the addition of a gray trimino of size 1 x 3 immediately after this square.

Therefore, based on the principle of the sum of independent cases analyzed (without considering the intersection), we have that p(n)=p(n-2)+p(n-3).

4 Conception and *a priori* analysis

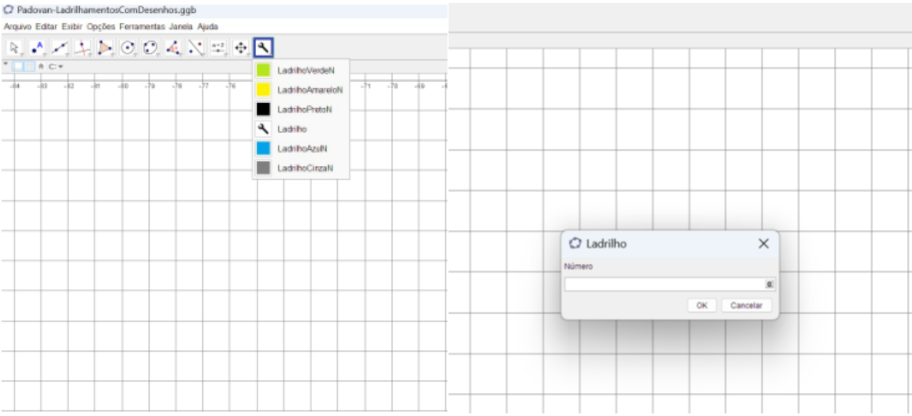
Here we present the proposed didactic situation, followed by an attitudinal prediction of the students based on the phases outlined by the theory of didactic situations.

Considering the Padovan tilings proposed by Vieira, Alves, and Catarino (2022), with your respective pieces available, demonstrate the identity p(n)=p(n-1)+p(n-5), for n≥ 5 through GeoGebra software.

Before starting the phases of the theory of didactic situations, the teacher should explain a little about the resource developed in GeoGebra, allowing visualization of Padovan's tilings, thus facilitating the understanding and development of the activity. Therefore, Figure 3 shows the home screen of the tool, with the buttons available in my development. Initially, the user must click on the *Tile* [Ladrilho] option and then click on the vertex of the square in the software mesh. Soon after, a window will open, asking us to enter the size of the tile, after which we must click again on the vertex for the tile to be drawn. For tiles with the available colors, as with LadrilhoVerdeN [green], the user can insert a green tile according to the size entered. It is inserted by clicking on the vertex or within the mesh where we want the tile to be drawn; then, a window opens for us to enter the size of the tile. The piece is then drawn in the location suggested by the user. The script happens the same way for the other colors, LadrilhoAmareloN, LadrilhoPretoN, LadrilhoAzulN, and LadrilhoCinzaN [yellow, black, blue, and gray].

On the left side are the tiles (pieces) available for the proposed activity, allowing users to understand that they can only use them. Below is the semi-filled mesh with some Padovan tiling to familiarize the participant with the resource. It should be noted that for this activity, only the options Ladrilho, LadrilhoPretoN, LadrilhoAzulN, and LadrilhoCinzaN [tile, black, blue, and gray] will be used since only the Padovan sequence and its identity will be addressed.

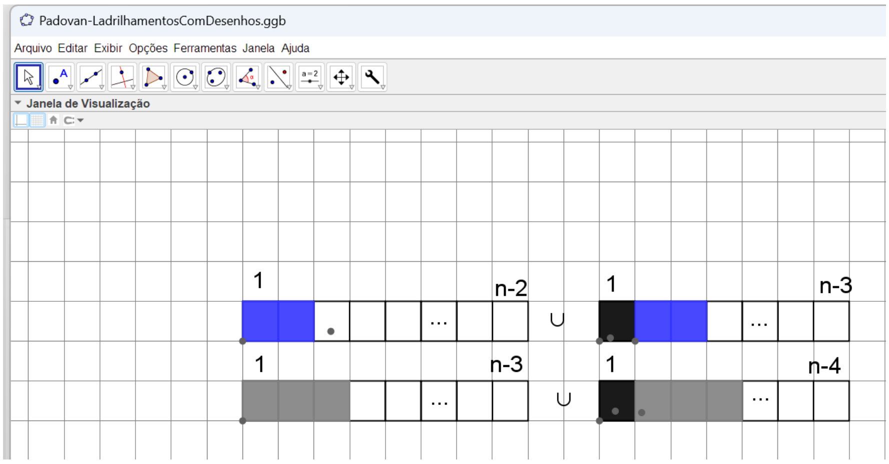
Figure 3: Presentation of the resource developed in GeoGebra



Source: Own elaboration (2024)

Action Phase: The aim is for students to review their prior knowledge of boards, tiles, and tiling to deepen their understanding of Padovan's combinatorial model. In this process, it is possible to revisit Vieira, Alves, and Catarino's (2022) demonstration, highlighting the importance of validation by carefully mapping the sets involved. This approach not only strengthens the theoretical foundation but also encourages a deeper understanding of the interaction between the elements of the problem. Figure 4 shows this demonstration performed in the GeoGebra software.

Figure 4: Demonstration of Padovan tilings in GeoGebra

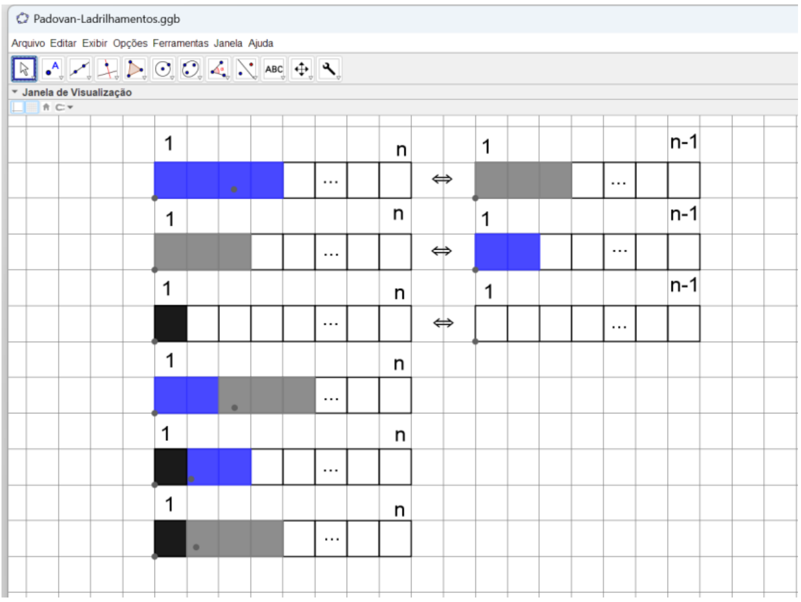


Source: Own elaboration (2024)

Formulation Phase: students are expected to begin mapping for each case presented in the identity. Therefore, it is necessary to check the possibility of starting each piece on the board. Indeed, we present some examples of a board of size n mapping to a board of size n-1: starting with a pair of dominoes, map them to a board of size n-1 by replacing the pieces with a trimino. Starting with a trimino, replace the piece with a domino. Start with a black square, and remove the square. Start with a domino followed by a trimino, then remove the domino. Start with a black square followed by a trimino, then replace the two pieces with a domino. Start with a black square followed by a domino; remove the domino.

Validation Phase: To demonstrate identity using the notion of a board, it is necessary to construct a bijection between the sets that contain the tilings of p(n) U p(n-4) and p(n-1) U p(n-2). Figure 5 depicts the mapping from p(n) to p(n-1):

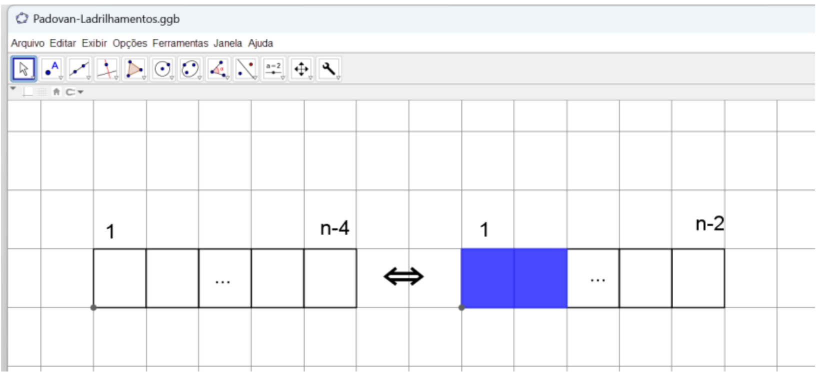
Figure 5: mapping of p(n) and p(n-1)



Source: Own elaboration (2024)

Figure 6 depicts the mapping from p(n-4) to p(n-2):

Figure 6: Mapping from p(n-4) to p(n-2)



Source: Own elaboration (2024)

The models presented in Figures 3 and 4 are validation possibilities that students can present.

Dialectics of Institutionalization: When revealing the intention of the activity, the mediating teacher is expected to create an environment conducive to the in-depth exploration of the identity of Padovan's combinatorial model through the approach with boards. Such openness allows theoretical understanding and a practical analysis through problem solving. Additionally, when discussing participants' resolutions, the aim is to provide an opportunity to explore the different paths followed, clarify doubts, correct mistakes, and reinforce successes, promoting more robust and participatory learning.

5 Experimentation, *a posteriori* analysis and validation

During the experimentation phase, data is collected for *a posteriori* analysis. The nature of these extracted data depends on the established objectives of didactic engineering, the hypotheses put to the test, and the conjectures made in the conception and *a priori* analysis. However, special attention is given to data collection, allowing the researcher to understand the students' interaction with the environment and to what extent this interaction supports their autonomous transition from the initial to the targeted strategies. Furthermore, an analysis of the processes of deconcentration and institutionalization is carried out (Artigue, 2014a).

These collected data encompass student productions, including computer files, when technology is used, researcher observations, and audio and video recordings. The data collected during the experimentation can be complemented by additional data (questionnaires, interviews with students and teachers, and tests), allowing for a better evaluation and analysis of the results of didactic engineering (Artigue, 2020).

In the a *posteriori* analysis phase, the collected data is organized and compared with the data from the design and the *a priori* analysis phase. Thus, some questions are raised: To what extent do the data collected during the experimentation phase support the conception and *a priori* analysis? What are the significant convergences and divergences, and how can they be interpreted? What happened that was not foreseen, and how can this be interpreted?

The hypotheses underlying the research structure are validated through this connection between conception and *a priori* analysis and *a posteriori* analysis (Artigue, 2014b). One must be aware of the always-existing differences between the reference provided by the conception and *a priori* analysis and the contingency analyzed in the *a posteriori* analysis. As noted, the conception and *a priori* analysis approaches students in a generic and epistemic way.

It is worth noting that these last two phases of didactic engineering were not carried out for the present research, and this did not harm the investigation.

6 Planning for future research using empirical data

This section is dedicated to planning for collecting empirical data, which can be done in mathematics teaching degree courses in higher education institutions. The application aims to comprise four meetings lasting 2 hours/class each. Chart 1 presents a summary of the activities to be developed in each meeting:

Chart 1: Summary of experimentation activities

1st meeting	Review of the Fibonacci sequence, previously discussed with the subject teacher, recovering historical and evolutionary aspects of these numbers. In this context, mathematical advances related to the Fibonacci sequence and the notion of board, tiles and tiling are explored.
2nd meeting	Introducing the concept of a board, presenting Fibonacci's combinatorial interpretation, its corresponding pieces, and the demonstration of the theorem associated with the combinatorial approach.
3rd meeting	Study of Padovan's combinatorial model, including the demonstration of the

	theorem.
4th meeting	Implementing the proposed problem situation, allowing students to explore Padovan's combinatorial identity in relation to their combinatorial model.

Source: Own elaboration (2024)

The meetings should be conducted based on the theory of didactic situations, using a didactic approach developed through a teaching situation and accompanied by a list of exercises. During these meetings, participants should engage in discussions to develop resolution strategies for the proposals presented in Chart 1. In this context, the didactic contract will be formalized, establishing the teacher's expectations concerning the students and vice versa, including the relationships with knowledge and how knowledge will be approached and treated by both parties.

This stage is characterized by the application of the previously organized structure, observing learning situations, and incorporating the concepts outlined in the didactic research. This process does not follow the dynamics of traditional classes. The class organization at this time is based on the theory of didactic situations and directed toward collecting data from students, allowing a more in-depth analysis of the teaching-learning process.

The theoretical approach can be implemented in practice using the resource provided, allowing challenges surrounding the use of technological resources in the classroom to be overcome, as is the case with GeoGebra. To this end, one solution is to provide a brief explanation of the tools available in the software to familiarize students with the tool. Furthermore, another challenge is the demonstration of identity, which can solved with a good foundation regarding the Padovan sequence through the presentation of other similar identities and their respective demonstrations.

During the application, we suggest that readers study the work of Vieira et al. (2023), Vieira (2020), and Vieira, Alves, and Catarino (2024; 2023b), dealing with the application of specific problem situations in the area of sequences, involving didactic engineering and the theory of didactic situations. In their research, the authors explore methodologies based on the problem situation developed and applied in initial education courses for mathematics teachers, strengthening the experience with recurring numerical sequences in the classroom.

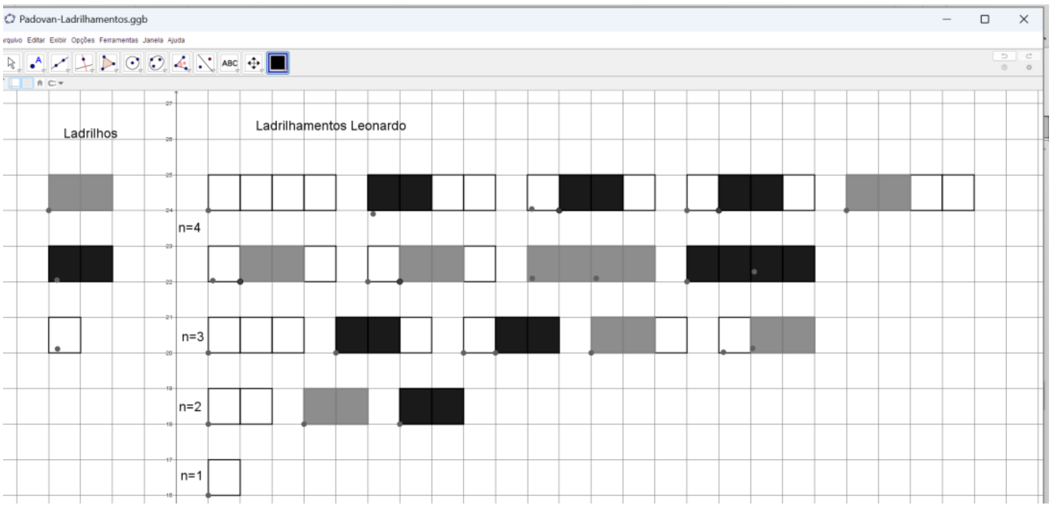
Given the above, the authors address the difficulties and challenges experienced, making the application of these methodologies with the content covered a valid experience. Thus, the last two phases of didactic engineering and the dialectics of the didactic situation discussed are highlighted, making it an excellent tool for the reader who wants practical validations, thus strengthening the justification for the need for future experimental research.

7 Applications for other sequences and the board with concrete material

The approach developed in GeoGebra allows for addressing other numerical sequences involving the notion of board and tiles. Thus, with the tools made available in the resource discussed in this research, tiles can be used for some sequences, such as Fibonacci, Pell, Jacobsthal, Mersenne, and Leonardo sequences.

As an example, Figure 7 shows the approach to Leonardo's sequence studied by Vieira, Alves, and Catarino (2023a), which uses tiles with white squares and gray and black dominoes, knowing that gray dominoes and black dominoes cannot occupy the same board, and can only appear alone or with squares. Therefore, the number of Leonardo's ways of tiling equals the number of Leonardo's sequence, represented by $l_n=L_n$, where l_n is the number of ways of tiling, and L_n is the n-th term of the sequence.

Figure 7: Leonardo's tilings



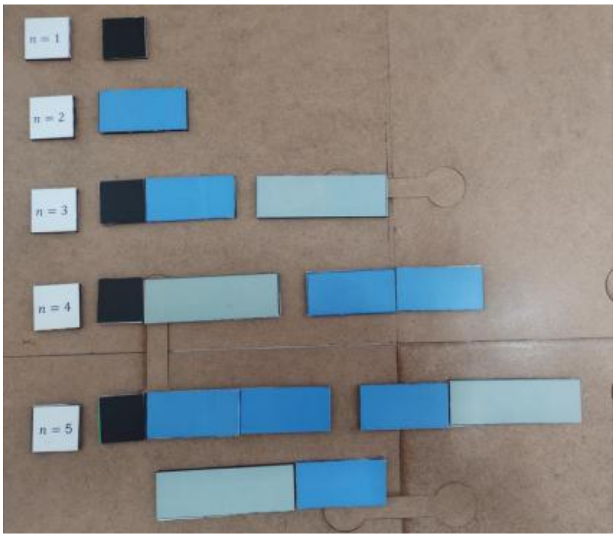
Source: Own elaboration (2024)

The ability to represent numerical sequences through tiles in GeoGebra enables new possibilities for research and development in mathematics and its applications. As a result, this tool can be used to test hypotheses, visualize results, and communicate findings more efficiently. The visual approach can also reveal new patterns and connections that are not immediately apparent through traditional analytical methods.

The application of technological tools such as GeoGebra to explore numerical sequences and tiling patterns not only enriches mathematics education but also advances research in applied mathematics, allowing us to address possible strategies for teaching certain content in the classroom. This approach combines visualization, interactivity, and mathematical rigor, providing a platform with great potential for understanding and investigating recurring numerical sequences integrated with mathematical and technological content.

Nevertheless, concrete material has been developed to study tilings that address the Fibonacci and Padovan sequences, as discussed by Vieira, Alves, and Catarino (2023b). Figure 8 shows tilings using MDF pieces, which helps us visualize the colors, as in Vieira, Alves, and Catarino's (2022) rule. However, the material can also be applied using the technique developed by Tedford (2019).

Figure 8: Padovan tilings with concrete materials



Source: Vieira, Alves, and Catarino (2023b)

Finally, Figure 9 presents an experiment with concrete materials in the classroom in the History of Mathematics subject with the Padovan sequence approach for students of the mathematics teaching degree (initial teacher education). In Vieira, Alves, and Catarino (2023b), participants carry out the proposed activity using the theory of didactic situations, building knowledge and defining the recurrence of the Padovan sequence through the combinatorial approach.

Figure 9: Application of Padovan tilings with concrete materials

Source: Vieira, Alves, and Catarino (2023b)

8 Final Considerations

Given the teaching proposal presented to work with the Padovan sequence, the relevant contribution provided by the adopted combinatorial approach is notable. This approach focuses on exploring identities aimed at the combinatorial interpretation of the Padovan sequence. The initial theoretical contextualization, which starts from constructing a historical and epistemological evolution of these numbers, provided a solid basis for elaborating a teaching session to construct knowledge.

Our approach to the theme was developed from a didactic situation organized based on the theory of didactic situations, emerging from a perspective that values visualization in the history of mathematics. To this end, we used didactic engineering, structuring its first two phases to implement it in the classroom and collect data.

When considering the implications for teaching, we reflected on how this didactic proposal can positively impact the approach to the topic, especially in the context of initial teacher education. By offering an alternative for understanding the combinatorial approach of the Padovan sequence from a visual perspective, we seek to contribute to the understanding of the combinatorial identities of other sequences. This construction is suggested as teaching material, and we present concrete suggestions for its integration into the curriculum, aiming to improve students' understanding and practical application.

As highlighted in the preliminary analysis of this work, undergraduate students often encounter difficulties and do not study specific themes in the history of mathematics. We expect that approaching the topic with GeoGebra can help them overcome these difficulties through interactive visualization, offering a more concrete representation and facilitating the understanding of concepts. We believe that using GeoGebra provides an experience that promotes active learning and helps students internalize mathematical concepts.

Finally, we hope that this didactic proposal contributes to the teaching of group theory and the understanding of the combinatorial identities of the Padovan sequence, providing support for more effective pedagogical practices. Aligned with didactic engineering and the theory of didactical situations, such practices can enrich the initial education path of mathematics teachers.

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