Sociedade Brasileira de Educação Matemática

RIPEM

Reflection on practice in teacher training: analysis of Pythagorean Theorem class simulations with Didactic Suitability Criteria

Viviane Beatriz Hummes
Universitat de Barcelona
Barcelona, Espanha
vivihummes@gmail.com
0000-0003-2031-8238

Erich Leighton
Universidad San Sebastián
Concepción, Chile
erichleighton@gmail.com
0000-0001-7319-9469

2238-0345 ISSN

10.37001/ripem.v15i1.4247 doi

Received • 27/06/2024
Approved • 09/11/2024
Published • 01/01/2025

Editor • Gilberto Januario

Abstract: This study explores the integration of class simulations and Didactic Suitability Criteria in the training of future mathematics teachers in Chile, aiming to foster reflective competence from the initial stages of their professional training. Using a qualitative approach, it investigated how future educators use the components of the Epistemic Didactic Suitability Criterion to reflect on and improve class simulations of the Pythagorean Theorem. Despite some confusion in the application of the criterion, all groups demonstrated an active commitment to didactic improvement. This combined approach of class simulations and Didactic Suitability Criteria is revealed as a valuable strategy in initial training, providing meaningful practical experiences and promoting structured reflection on pedagogical practice.

Keywords: Class Simulation. Pythagorean Theorem. Reflection on Practice. Didactic Suitability Criteria. Teacher Training.

Reflexión sobre la práctica en la formación de profesores: análisis de simulaciones de clase del Teorema de Pitágoras con Criterios de Idoneidad Didáctica

Resumen: Este estudio explora la integración de simulaciones de clase y los Criterios de Idoneidad Didáctica en la formación de futuros docentes de matemáticas en Chile, con el objetivo de fomentar la competencia reflexiva desde las etapas iniciales de su formación profesional. Utilizando un enfoque cualitativo, se investigó cómo los futuros educadores utilizan los componentes del Criterio de Idoneidad Didáctica Epistémico para reflexionar y mejorar las simulaciones de clase del Teorema de Pitágoras. A pesar de algunas confusiones en la aplicación del criterio, todos los grupos demostraron un compromiso activo con la mejora didáctica. Este enfoque combinado se revela como una estrategia valiosa en la formación inicial, proporcionando experiencias prácticas significativas y promoviendo la reflexión estructurada sobre la práctica pedagógica.

Palabras clave: Simulación de Clase. Teorema de Pitágoras. Reflexión sobre la Práctica. Criterios de Idoneidad Didáctica. Formación de Profesores.

Reflexão sobre a prática na formação de professores: análises de simulações de aula do Teorema de Pitágoras com Critérios de Adequação Didática

Resumo: Este estudo explora a integração de simulações de aula e os Critérios de Adequação Didática na formação de futuros professores de matemática no Chile, visando fomentar a competência reflexiva desde as etapas iniciais de sua formação profissional. Utilizando uma abordagem qualitativa, investigou-se como os futuros educadores utilizam os componentes do

Critério de Adequação Didática Epistêmico para refletir e melhorar as simulações de aula do Teorema de Pitágoras. Apesar de algumas confusões na aplicação do critério, todos os grupos demonstraram um compromisso ativo com a melhoria didática. Esta abordagem combinada revela-se uma estratégia valiosa na formação inicial, proporcionando experiências práticas significativas e promovendo a reflexão estruturada sobre a prática pedagógica.

Palavras-chave: Simulação de Aula. Teorema de Pitágoras. Reflexão sobre a Prática. Critérios de Adequação Didática. Formação de Professores.

1 Introduction

The ability of teachers to analyze and understand the essential factors that influence the processes of teaching and learning mathematics is crucial for their professional development and for improving educational quality (Aghakhani, Lewitzky & Majeed, 2023; Dyer & Sherin, 2016; Giacomone, Godino & Beltrán-Pellicer, 2018; Godino, Giacomone, Batanero & Font, 2017). However, fostering reflective competence among educators requires the adoption of specific approaches and methodological frameworks from the beginning of their training.

In this context, class simulation emerges as a valuable strategy in the initial training of educators. This methodology offers future teachers the opportunity to actively participate in situations that replicate the educational reality, allowing them to expand their knowledge about the nature of simulated processes and providing them with meaningful practical experiences and learning opportunities (Bradley & Kendall, 2014; Gibson, Knezek, Redmond & Bradley, 2014; Speed, Bradley & Garland, 2015).

However, to effectively cultivate reflective competence, it is essential to integrate additional concepts and methodological frameworks, such as Lesson Study (Huang, Takahashi & da Ponte, 2019), Concept Study (Davis, 2008), Professional Noticing (Mason, 2002), and the Didactic Suitability Criteria (DSC) proposed by the Onto-semiotic Approach (OSA) (Godino, Batanero & Font, 2007, 2019).

From this perspective, the specific aim of this study is to assess how future teachers use the components of the Epistemic DSC when reflecting on simulated lessons and to examine the relationship between these reflections and their proposals for improving these lessons. To this end, a training module was developed that combines lesson simulation with the DSC, targeting primary education students specializing in mathematics at a university in Chile.

This research aims to answer two fundamental questions: How do future teachers apply the components of the Epistemic DSC when reflecting on simulated lessons? and What connections can be identified between future teachers’ reflections and their proposals to improve these lessons?

In the next chapter, the DSC is presented as the theoretical framework of this research. The third section details the methodology used, while the fourth explores the results obtained. Finally, the fifth section provides a brief discussion, followed by conclusions..

2 Didactic Suitability Criteria (DSC)

Within the theoretical framework of the Model of Competencies and Didactic-Mathematical Knowledge of the Mathematics Teacher, based on the principles of the Onto-Semiotic Approach (OSA) to mathematical cognition and instruction (Godino *et al.*, 2019), it is argued that the crucial skills of the mathematics teacher are mathematical competence and competence in analysis and didactic intervention. The essential core of the latter competence lies in the ability to design, implement, and evaluate learning sequences, whether one's own or others', using didactic analysis techniques and suitability criteria (Breda, Font & Pino-Fan, 2018; Breda, Pino-Fan & Font, 2017). The objective is to establish cycles of planning, implementation, evaluation, and suggesting improvements.

The DSC (Breda *et al*., 2018) propose that the norms or principles guiding teaching practice should be a matter of consensus, offering guidelines to direct such practice. These principles act as a priori criteria, agreed upon with the aim of achieving higher quality classes. Despite their importance, there is a recognition of the need to consider the reality and individual context of each teacher, relativizing the importance of each principle (Godino, Font, Wilhelmi & De Castro, 2009).

This approach suggests the consideration of various specific partial aspects or criteria, among which the following stand out: epistemic, ecological, cognitive, affective, mediational, and interactional suitability (Font, Planas & Godino, 2010; Godino, Batanero & Burgos, 2023).

According to Breda *et al*. (2017, 2018), when addressing the notion of a *good class* or *good sequence of mathematics classes*, the importance of providing quality teaching that enables students to master meaningful mathematical activities is highlighted, which is classified as the epistemic dimension. In this sense, the aim is to ensure not only the transmission of solid mathematical knowledge but also a deep and meaningful understanding by the students.

However, ensuring mathematical quality alone is not sufficient; a good teaching and learning process must be considered to ensure that students truly internalize concepts and develop effective mathematical skills, which relates to the cognitive dimension. In this context, epistemic and cognitive principles emerge, seeking to impart and learn mathematics effectively.

Furthermore, the importance of mathematics being relevant and useful in students' social and work environments is recognized, which is classified as the ecological dimension. This aspect emphasizes the need to link mathematical content with practical situations and applications, contributing to the utility and relevance of the discipline.

While teaching and learning *good mathematics* is crucial, the incorporation of an emotional principle (affective dimension) is also emphasized. This ensures that students not only acquire useful mathematical knowledge but also develop a positive attitude towards the subject and enjoy the learning process, avoiding possible aversions.

Other criteria, such as the appropriate use of institutional resources, manipulative materials, or teaching technologies (mediational dimension), and efficient management of the teaching-learning process (interactional dimension), must be considered to achieve positive and holistic results in students' mathematical education. These additional elements emphasize the importance of adaptability and effectiveness in teaching practice, beyond mere content transmission.

2.1 Epistemic Didactic Suitability Criterion

The components and indicators of the DSC have been developed taking into account trends, principles, and research findings in the field of mathematics didactics (Godino, 2013; Breda *et al*., 2018). Constructing an effective class involves considering various aspects to guide the teacher in its delivery. In particular, epistemic suitability focuses on the degree of representativeness and interconnection of the institutional meanings implemented (or intended) in relation to a reference meaning.

From the consensus perspective, it is established that the mathematics taught must meet specific criteria to be considered suitable or adequate (Breda *et al*., 2018). Firstly, it is expected to be free of errors, thus ensuring the accuracy and reliability of the transmitted information. Additionally, the presentation of mathematical content is expected to avoid ambiguous explanations that may generate confusion in students' understanding. It is also valued that the teaching of mathematics enriches mathematical processes, promoting the development of skills and a deep understanding of concepts (Breda *et al.*, 2017).

Finally, it is advocated that the mathematics presented in the classroom provides a representative sample of the concept or notion intended to be taught, offering a comprehensive and contextualized view to facilitate a holistic understanding by students (Breda *et al.*, 2017). These fundamental principles aim to ensure the quality and effectiveness of the mathematics teaching process.

Breda *et al*. (2017) propose a system of components and indicators as a guide for the analysis and assessment of didactic suitability, adaptable to instructional processes at any educational stage. The components and indicators of the Epistemic DSC are summarized in Table 1 below.

Table 1: Components and Indicators of the Epistemic DSC

Components of the Epistemic DSC	Indicators
Errors	There are no practices observed that are considered incorrect from a mathematical standpoint.
Ambiguities	There are no ambiguities observed that could lead to confusion among students: definitions and procedures are clearly and correctly stated, adapted to the educational level they are addressing; explanations, checks, demonstrations are tailored to the educational level they are addressing, controlled use of metaphors, etc.
Richness of processes	The sequence of tasks includes the performance of relevant processes in mathematical activity (modeling, argumentation, problem-solving, connections, etc.).
Representativeness of the complexity of the mathematical object	The partial meanings (definitions, properties, procedures, etc.) are a representative sample of the complexity of the mathematical notion intended to be taught (as outlined in the curriculum). For one or several partial meanings, there is a representative sample of problems. For one or several partial meanings, there is a use of different modes of expression (verbal, graphical, symbolic, etc.), and treatments and conversions between them.

Source: Breda *et al.* (2017, p. 190)

3 Methodology

This research employs a qualitative methodological approach, aiming to understand and observe a situation, emphasizing systematic data analysis to interpret social phenomena, especially those related to educational experiences. (Corbin & Strauss, 2014; Ortíz, 2023). Specifically, this study investigates the integration of class simulations and the Didactic Suitability Criteria in initial teacher education to promote reflection on practice.

The research was conducted with a group of 20 students from the Bachelor of Education in Elementary Education with a specialization in Mathematics program at a Chilean university.

The training module was implemented within the framework of a course called *Reasoning in Mathematics* taken during the seventh academic semester of a ten-semester program. The course aims to help students understand what it means to learn mathematics from various theoretical frameworks of Mathematics Didactics and how to manage a class, providing a specific educational context for the research development. The second author of this article served as the responsible teacher for the course. All participants provided informed consent by signing the relevant document.

3.1 Design of the Training Module

The training module consisted of a total of 8 sessions, which were conducted between May and July 2023. In the first two sessions, students, divided into 6 groups (G1, G2, G3, G4, G5, and G6), were responsible for designing a learning experience on the Pythagorean Theorem (sessions 1 and 2).

During sessions 3 and 4, the future teachers implemented the designed experiences, with those leading the experience taking on the role of teachers, while the other participants acted as students. The professor in charge of the subject played the role of evaluator during this stage.

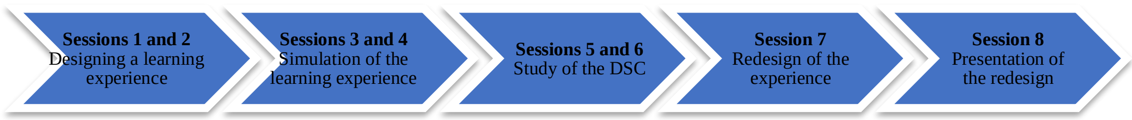
In the following two sessions, two classes focused on the DSC theme were taught. These sessions were led by the first author of this work, who is an expert in the field. During these classes, theoretical aspects, components, and indicators were addressed and exemplified, particularly focusing on the context of the Pythagorean Theorem.

After these theoretical sessions, participants were tasked with reformulating their simulated experiences based on the DSC. This was done with the purpose of identifying errors or obstacles evidenced during the work done in contrast with these criteria and establishing substantial improvements. To facilitate this process, a template was provided that included components and indicators, along with guiding questions, to direct their reflections towards specific aspects requiring attention.

To conclude, a session was organized to present the redesigned experiences, highlighting the differences between the initial proposals and the modified ones after the redesign process with the DSC. This exercise was complemented with a roundtable discussion where participants shared their final ideas and reflections on the experience, supported by the professor in charge of the subject and the expert in DSC theory.

Figure 1 below summarizes the contents addressed during the course sessions.

Figure 1: Sessions of the training module



Source: Own elaboration

3.2 Designs of the Simulated Class Experiences

The designs developed by the future mathematics teachers for primary education considered the assessment indicators described in the learning objective corresponding to the Pythagorean Theorem in the curriculum for 8th grade level in the Chilean school system: "To explain, in a concrete, pictorial, and symbolic way, the validity of the Pythagorean Theorem and to apply it to solving geometric and everyday life problems, manually and/or using educational software".

In addition to the above, the Chilean school curriculum establishes Evaluation Indicators (EI) as a way to evidence students' performance in the learning process. Each working group was assigned one of these indicators; given the number of participating groups, proposals were only presented for the following:

- EI1: They discover the Pythagorean Theorem concretely or pictorially, by decomposing or composing squares and right triangles.
- EI2: They recognize that with two sides of the right triangle given, the third side can be calculated.
- EI3: They verify with the given measurements of a triangle whether it is right-angled or not.
- EI4: They calculate the length of the missing side for a triangle to be right-angled and verify it by construction, applying the Thales' Theorem (triangles inscribed in a semicircle).
- EI5: They solve everyday problems to calculate the length of unknown and inaccessible sides in the plane and in space, first determining the respective right triangles.

The organization of the IE was as follows, as shown in Table 2.

Table 2: Organization of groups according to EI and number of members

Evaluation Indicator (EI)	Group Number	Number of members in the group
1	1	4
2	2	2
3	3 and 5	3
4	6	4
5	4	4

Source: Own elaboration

Each group carried out the task of developing a class proposal with three stages (beginning, development, and closure), according to a defined format and focusing primarily on meeting the designated evaluation indicator; for this purpose, two work sessions were considered in which the professor in charge of the subject conducted a monitoring and review process of the proposed task. In the subsequent stage, they carried out the process of simulating the planned activity to the group of classmates; given the number of participating groups, two class sessions were considered.

Below, Figure 2 presents the format of the design of the simulated classes.

Figure 2: Format of the simulated class design

Curriculum learning objective:	
Learning outcomes:	Concepts and/or skills to develop:
Describe the connections between the learning outcomes and the concepts and/or skills:	
Beginning: Time: Short description:	
Development: Time: Short description:	
Closure: Time: Short description:	
Student´s role: Mention at least 3 actions from a mathematical point of view	
Teacher´s role: Mention at least 3 actions from a mathematical point of view	
Assessments criteria: Develop indicators for the class goal	

Source: Own elaboration

3.3 Data Collection and Analysis

The data were collected through the final reports (class redesign) presented by each group, where they reflected on the simulated class, evaluating the components of the DSC and proposing improvement suggestions for the class redesign. Due to space limitations in this article, only reflections related to the Epistemic DSC were considered.

For data analysis, the content analysis technique was applied as a way of document review, which was of an external type, allowing to explain a written document in its context (Cáceres, 2003; López, 2002). To do this, according to Porta and Silva (2003), the procedure followed was the determination of objectives, which have been stated at the beginning of this document; the universe and the documents, which correspond to the reflections on the DSC and their improvement proposals; followed by the unit of analysis and how the DSC will be counted; subsequently, the categorization, classification, coding, and inventory. This is presented in the next section.

The analysis process focused on evaluating the reflections on the components of the Epistemic DSC proposed by each team, which was essential to understand each group's reflections on the simulated class and the redesign proposals. In this context, the analysis criteria proposed by Author (2022) were followed:

1. The DSC represents the theoretical stance with which the content was analyzed.
2. Content segments (units of analysis), whether phrases or paragraphs, were individualized for subsequent categorization.

In interpreting the data, the regulative criterion of dependency (Guba & Lincoln, 2002) was considered, which implies a process of controlling data interpretation through triangulation of the analysis carried out by the authors of this study.

3.4 Teaching of the Epistemic DSC for the case of the Pythagorean Theorem

During the DSC teaching sessions, the dimensions of the DSC specifically related to the content of the Pythagorean Theorem were taught. Below are the components of the epistemic DSC for the Pythagorean Theorem as taught to the students in the training module.

Component errors in the Pythagorean Theorem

Within the Epistemic DSC, the *errors* component focuses on the mistakes that can arise in the teaching of mathematics, particularly those made by the teacher (Breda *et al*., 2017). They can be classified as follows:

Classification by who commits the error:

1) Error committed by the teacher: Occurs when the teacher makes mistakes in explaining mathematical concepts or performing operations.

2) Error validated by the teacher: Happens when there are errors in the teaching material or in students' responses, and the teacher does not correct them, either by indicating that they are correct or by overlooking the mistakes.

Classification of the error committed by the teacher according to the cause:

1) Lack of mathematical knowledge by the teacher.

2) Teacher's distraction due to students' restlessness or some other aspect.

Classification according to content (Pythagorean Theorem):

1) Error in proposing problems: Involves presenting inconsistent problems, such as asking students to find other congruent Pythagorean triples to a given one, such as (4, 4, 5).

2) Representation error: Refers to errors in graphical representation, for example, incorrectly illustrating the Pythagorean Theorem.

3) Definition errors: This includes giving incorrect definitions of concepts such as triangle, hypotenuse, and leg.

4) Proposition errors: These occur when the Pythagorean Theorem is incorrectly stated.

5) Procedural error: Although the Pythagorean Theorem is not a procedure itself, the teacher may make mistakes when performing calculations related to it.

6) Error in proof: This involves considering the generalization of specific cases as a proof, even though in Elementary Education it is not strictly considered an error to perform less rigorous proofs.

These classifications provide a framework for identifying, understanding, and correcting the errors made by the teacher when teaching the Pythagorean Theorem, thus contributing to improving the quality of mathematical instruction.

Component ambiguities in the Pythagorean Theorem

The *ambiguities* component focuses on the importance of having clearly expressed definitions and procedures adapted to the corresponding educational level. It also emphasizes the need to adjust explanations, verifications, and demonstrations to that level, as well as to use metaphors in a controlled manner. These ambiguities can arise for several reasons:

1) Through the use of manipulative materials: Using manipulative materials in teaching the Pythagorean Theorem can create ambiguities if not employed with precision and clarity. For example, representing the geometric meaning of the Pythagorean Theorem with volumes of water in containers of different shapes or spherical balls can be ambiguous if students do not correctly understand the relationship between the areas of the squares of the sides of the right triangle. Therefore, it is crucial that the use of these materials is supported by a detailed and precise explanation that clarifies the relationship between the visual representation and the mathematical concept being taught.

2) Through the use of metaphors and gestures: This includes scenarios where the teacher employs metaphors, such as representing a fraction as *a pizza* or describing a function as *a machine that transforms*, or comparing the graph of a function to something *one-way*. In the context of the Pythagorean Theorem, an example of ambiguity could arise when using the metaphor of *a ladder* to explain the relationship between the sides of a right triangle. While these metaphors can be helpful, if not explained clearly or used superficially, they could lead to confusion rather than clarifying the concept. It is essential that the use of metaphors and gestures be precise and complemented by a clear and detailed explanation of the underlying mathematical concept.

3) Through the use of dynamic software programs: When resorting to interactive software programs to teach the Pythagorean Theorem, ambiguity may arise if it is not properly adapted to the educational level or if clear and precise explanations are not provided.

4) Other possible causes: This includes contextual factors or specific characteristics of classroom dynamics that could generate ambiguities and do not fit into the previous categories.

Recognizing and correcting these ambiguities is crucial to ensure effective and comprehensible mathematical instruction, tailored to the students' educational level, and utilizing pedagogical resources accurately.

Component richness of processes in the Pythagorean Theorem

The *richness of processes* component focuses on the importance of integrating relevant mathematical processes into the sequence of activities related to the Pythagorean Theorem. To achieve teaching rich in mathematical processes, it is essential to reflect on how to structure instruction to promote active student participation in mathematical activity. Here are some specific examples for this mathematical concept:

1. Selection of activities: It is fundamental to choose activities that encourage the performance of significant mathematical processes. For example, in the context of the Pythagorean Theorem, one could begin with an activity where students measure the sides of different right triangles and then formulate conjectures about the relationships between the lengths of the sides.

2. Didactic sequence: To enrich a didactic sequence with relevant mathematical processes for the Pythagorean Theorem, it is crucial to design stages that include manipulation, experimentation, and justification. For example, after students have formulated conjectures, they can conduct demonstrations using geometric models or diagrams to support their claims.

3. Richness of processes in a basic sequence: A well-structured sequence should address different mathematical processes at key stages. For example, after students have demonstrated the theorem, they can engage in group discussions where they communicate and justify their arguments using precise mathematical language.

4. Incorporation of other processes (megaprocesses): In addition to the processes mentioned above, it is essential to integrate megaprocesses such as problem-solving (Proença, Campelo & Oliveira, 2024) and mathematical modeling (Melo & Bisognin, 2021) into the study of the Pythagorean Theorem. For example, students can apply the theorem to solve real-world problems, such as calculating distances on a map or determining the dimensions of a rectangular room.

Planning activities that involve these processes will contribute to an enriching educational experience, where students not only acquire mathematical knowledge but also develop fundamental cognitive and problem-solving skills.

Component representative sample of the complexity of the Pythagorean Theorem

The component of *representative sample of the complexity of the mathematical object* in the context of the Pythagorean Theorem addresses the need to present a variety of interconnected partial meanings to address the inherent complexity of this notion. This component involves assessing whether the selected partial meanings for implementation are a faithful representation of the complexity of the mathematical notion intended to be taught.

Firstly, it is necessary to analyze whether the partial meanings, such as definitions, properties, and procedures, are a representative sample of the complexity of the mathematical notion of the Pythagorean Theorem. This involves considering various perspectives and approaches that allow students to address different types of tasks related to that mathematical object.

Secondly, when reviewing the curriculum, it is necessary to assess whether the sample of partial meanings implemented in the instructional process is consistent with those contemplated in the curriculum overall, whether at the national, stage, or educational cycle level. The *representative sample* refers to the contextual and metaphorical connection of meanings, depending on the mathematical object taught and the educational level. For example, in the case of teaching the Pythagorean Theorem in the 12-16 age stage, the inclusion of geometric and algebraic meanings would be considered a representative sample, while excluding some of these aspects would not provide a complete view.

Once the partial meanings for implementation are selected, it is necessary to assess whether a representative sample of representations of the mathematical object and tasks in which it is applied or emerges is presented. This approach will contribute to students developing a network of well-connected partial meanings, promoting a comprehensive understanding and competence in solving a variety of problems related to the Pythagorean Theorem.

For the specific case of the Pythagorean Theorem, some partial meanings that can be worked on in the eighth grade of basic education are:

1) Geometric meaning: The squared signs that appear represent areas or the symbols represent the length of the sides or the lengths squared. In these two aspects (which we call geometric meaning), numbers or letters substitute lengths of segments or areas of squares and their function is to represent them. From this perspective, the symbols represent magnitudes or relationships between them, but although operations can be performed with them, they are not considered independent objects from the magnitude they represent. The values that the symbols can take are those that allow the magnitudes and the situation they represent (for example, when calculating square roots, only the positive value is obtained). Although a lot of algebra is used in the solution, we speak of geometric meaning if the symbols are considered to represent magnitudes (length and area).

2) Arithmetic-algebraic meaning: When the values that the symbols can take are those that you want to consider and are not conditioned by the situation they initially represented. The symbols are now considered objects on which actions can be performed and even the objects, relationships, and situations they represent can be dispensable. This interpretation leads to understanding a, b, c as numbers or letters that do not represent geometric quantities and that $a^2 = b^2 + c^2$. First as Pythagorean triples of whole numbers, passing through Pythagorean fractions and reaching triples (a, b, c) that may not be integers but that fulfill the relationship $a^2 = b^2 + c^2$.

3) Other meanings: The Pythagorean Theorem also has interesting connections with other problems and theories that can expand the partial meanings mentioned above, but only

some will be considered in this work focused on basic education, where this theorem is taught for the first time. These connections include: the notion of a vector modulus, prism diagonal, distance, golden section, dynamic symmetry, logarithmic spirals, angle trisection, cube duplication, circle squaring, determination of the value of π, concept of irrational number, star regular polygons and polyhedra, number theory, construction of angles and polygons, continued fractions, trigonometry, analytical geometry, Hilbert spaces, etc.

4 Evaluation of the errors component and proposal for class improvement

According to point 3.4. regarding the classification of errors based on content related to the Pythagorean Theorem, Table 3 summarizes the results of the reflection on simulated classes by the students regarding the *errors* component of the Epistemic DSC. We have added to this classification cases where groups have stated the absence of errors in their simulation as *no presence of errors*, and when they have expressed errors that are not mathematical, they were categorized as *other types of errors*.

Table 3: Results of the categorization of the errors component

Errors	G1	G2	G3	G4	G5	G6
In proposing problems	X					
Representation	X					
Definition						
Proposition	X					
Procedure						
Proof						
No presence of errors			X	X		
Other types of errors*	X	X			X	X

Source: Own elaboration [* Corresponds to errors declared by the groups, which are incorrect didactic decisions]

In Table 3, the types of errors are presented, as explained in the course for the Epistemic DSC component of content related to the Pythagorean Theorem, manifested in the reflections of the students following the simulation of their classes. The identified types of mathematical errors varied between problems with formulating questions (error in proposing problems), incorrect representations (representation error), and conceptual errors (error in proposition). No errors related to the definition of the concept, procedure, or demonstration were observed in the reflections.

Additionally, it can be observed in Table 3 that groups G1, G2, G5, and G6 acknowledged the presence of errors in their simulated classes, either in the instructions, in formulating questions, in graphical representation, or in mathematical conceptualization.

Group 1 demonstrated an ability to identify errors in the design and simulation of their didactic proposal, offering substantial improvements in the redesign of the class. However, confusion was observed in categorizing mathematical errors as poor didactic choices, indicating a partial understanding of the error component of the Epistemic DSC. Despite this challenge, the group demonstrated a conscious and reflective approach in using the error component when reviewing their simulation experience. The connection between these reflections and the redesign proposal demonstrates an active effort to improve the didactic quality of the simulated class, as evidenced: "*[...] errors were made when discovering the Pythagorean Theorem concretely. This was due to errors in the instructions and guidance given by the teachers when* *developing the activity [...]"* (G1).

Additionally, this group acknowledges errors that have been categorized in this paper as "other types of errors" because they do not correspond to the mathematical realm. As stated in the following evidence: *"[...] there was no closure in the activity where the mathematical concept addressed was institutionalized"* (G1).

In their reflection, Group 2, when pointing out the errors evidenced in the class, presents confusion in identifying a mathematical error and highlights, instead, a cognitive aspect related to the students' prior knowledge, as evidenced: *"Yes, problem number 2 was poorly posed and there was confusion when explaining it, which caused the students to work with roots with a value of 50, which meant an error, because, for the corresponding grade, the students do not have the knowledge to address these roots"* (G2).

Furthermore, despite not noticing it, Group 2 also made a *representation error* in the class simulation, which was not highlighted in their reflection. In their class redesign proposal, although not declared, they propose to make a change in an image presented in the problem intended to clearly observe the shape of a right triangle (the previous one represented another type of triangle), as seen in the following evidence: *"First, we adjusted the exercise by recognizing that the trajectory of a swimming person can vary. When examining it from the point where we must conclude the exercise, we observe a clear change in the perspective of the image accompanying the exercise"* (G2).

Group 5 highlighted an aspect related to the necessary prior knowledge for understanding the worked mathematical activity. However, this is not a mathematical error but rather an aspect related to the cognitive adequacy of the class. In their class improvement proposal, the group proposes a change regarding this aspect: *"In the design, we noticed that during the beginning of classes, there is no appropriate activity to address prior knowledge [...]"* (G5); *"To improve the beginning of the class, we propose to carry out a question roulette that allows introducing prior knowledge [...]"* (G5).

Group 6 did not identify mathematical errors in the class. However, they confuse the *errors* component of the Epistemic CID with the possibility of using error as a teaching strategy, as evidenced in the following evidence: *"In the presented sequence, [...] even though they contained errors, it was not investigated whether they were correct or could be resolved"* (G6).

In the class redesign proposal, this group reinforces such an idea, which is reflected in the writing: *"In the explanations provided by the teachers, mathematical errors are observed that the students perceive as improbable and understand why [...]"* (G6). On the other hand, Groups G3 and G4 claimed not to have identified errors in their proposals.

All groups, even those that did not identify mathematical errors, proposed improvements for their simulated classes. These improvements include adjustments to the instructions, changes in representation, and strategies for addressing cognitive misunderstandings.

Despite the differences in identifying errors, all groups that declared them demonstrated a reflective awareness in the analysis of their simulated classes. The connection between reflections and redesign proposals indicates an active effort to improve the quality of the instructional process.

The above results suggest the need for greater clarity in differentiating between mathematical aspects and pedagogical decisions during the study of the *errors* component of the Epistemic DSC.

5 Assessment of the ambiguities component and improvement proposal for the class

According to what was stated in section 3.4 regarding possible ambiguities in the content related to the Pythagorean Theorem, Table 4 presents the results of the reflection on the simulated classes by the students concerning this component of the Epistemic CID; it has been added to this classification the case in which groups have reported the presence of ambiguities, however, these did not correspond to it, and were categorized as *does not apply*.

Table 4: Results of the Categorization of the Ambiguities Component

Ambiguities	G1	G2	G3	G4	G5	G6
Due to the use of manipulative materials						
Due to the use of metaphors and gestures						
Due to the use of dynamic computer programs						X
Other possible causes		X				
Does not apply	X		X	X	X	

Source: Own elaboration

In Table 4, the ambiguities declared by student groups are presented, according to the reasons stated in section 3.4.2 all groups reported ambiguities identified in their simulations, which varied between the use of computer programs, other possible causes, and, mostly, declared ambiguities that did not apply.

Group 1, for example, categorized as ambiguity what really corresponds to errors in explanation, which had been highlighted in the previously analyzed component. The following is an excerpt from their reflection on this component: *"The explanations given were not coherent, as there was a conceptual confusion on the part of the teachers between comparing the area of each triangle with the length of each triangle's side"* (G1).

In their class improvement proposal, the group emphasized the importance of deepening the study of the content to avoid explanations that contain errors or are ambiguous. This confusion indicates the need for greater clarity in differentiating between mathematical errors and pedagogical decisions during the analysis of the quality of the simulated class. The evidence is presented: *"The explanations must be precise and clear; to achieve this, we will prepare better conceptually to understand the Pythagorean Theorem"* (G1).

Group 2 recognized that the representations used could generate confusing understandings in students. This is evident in the following account: *"The image that represented problem 2 was not consistent with what the problem stated. We as teachers [...] misrepresented the dimensions and position of the right triangle"* (G2).

In their improvement proposal, they advocate for modifying the problem and ensuring results to optimize the quality of the class, which is presented: *"To begin with, we need to change the problem to one that is easier and more understandable for the students. Before giving the class, we must be sure of the results of each problem we are going to pose, since if the teacher shows insecurity, the student will feel unsure of what is being taught"* (G2).

Group 6 pointed out an ambiguity derived from the use of dynamic computer programs, indicating problems in the implementation of the resource, as can be seen in the following account: *"In the sequence presented, ambiguities were identified where students began their* *learning construction erroneously, as the ICT resource was poorly implemented and yielded results that were not relevant"* (G6).

This group suggests improving the situation by including appropriate representations. Moreover, they propose correcting both the activity and the resources, questioning their suitability for students. The evidence of their improvement proposal is presented: *"To address this situation, it would be beneficial to add appropriate representations. Additionally, it is proposed to correct both the activity and the resources, and to question whether these are appropriate for the students"* (G6).

Groups G3, G4, and G5 expose ambiguities that do not apply, which relate to aspects of class management. Group 3 refers to modeling the activity after the instructions, which they also specify in their improvement proposal. Group 4 declares unclear instructions as an ambiguity, so in the improvement, they emphasize improving the start and end of the class. Meanwhile, Group 5 points out the absence of explanations to prevent confusion in students, so they propose adding questions during the activity development.

In various groups, there was some confusion in identifying the ambiguities component. Despite the mentioned confusions, the groups presented improvement proposals that address both aspects related to mathematical errors and clarity and pedagogical efficacy. These suggestions reflect an effort by future teachers to improve the quality of the simulated classes and demonstrate an understanding of the importance of reflection and adaptation in teaching.

Several groups highlight the importance of clarity in explanations and the need for deeper conceptual preparation. However, the research could benefit from greater clarity in conceptualizing the ambiguity component and ongoing attention to differentiating between ambiguity and error.

6 Assessment of the process richness component and improvement proposal for the class

According to what was outlined in section 3.4.3 regarding the diversity of enriching processes for content related to the Pythagorean Theorem, Table 5 presents the results summarizing the skills expressed in the reflections on the simulated classes by students concerning this aspect of the Epistemic DSC. It is important to note that the Chilean curriculum establishes four skills (processes) that must be developed integrally in all mathematical content of the plan: problem-solving, arguing and communicating, representing and modeling. In addition to these skills, other processes identified by the groups that enrich the mathematical activity have been included in this classification, such as trial and error, verification, demonstration, and conjecture formulation, among others.

Table 5: Results of the Categorization of the Process Richness Component

Process Richness	G1	G2	G3	G4	G5	G6
Problem-solving	X	X	X	X	X	X
Arguing and communicating	X	X	X	X		X
Representing			X	X		
Modeling		X	X			
Other processes						X

Source: Own elaboration

As seen in Table 5 and after the analysis of the six groups, it was evident that all recognized the importance of fostering mathematical skills among students. Problem-solving and arguing and communication were considered fundamental mathematical skills by most of the groups. Meanwhile, the skills of modeling and representing were the least mentioned in the groups' reflections.

It is important to note that Group 6 highlighted the presence of another process related to the verification of results, as indicated in the following testimony: *"[...] Throughout this process, they are given the opportunity to review and confirm the results"* (G6).

Moreover, there is a consensus on the importance of integrating arguing and communication continuously throughout the entire teaching process, as can be seen in the following evidence: *"Although the development of the ability to argue and communicate is evident in the last questions of the guide, unfortunately, this dimension was not present continuously throughout the activity"* (G1); "*While it was not developed solidly, arguing among students and with their peers was briefly promoted. However, it is an aspect that could be improved in future classes to stimulate reasoning and mathematical communication"* (G2).

While some groups highlight the presence of important processes such as problem-solving, representation, and arguing, they also identify areas for improvement in the application of these processes, as seen in: *"We believe it is necessary to add and give more emphasis to the ability to argue and communicate through general questions and learning guide questions. This would provoke students to discuss and share their observations"* (G1).

The need to balance the use of different mathematical skills is recognized, ensuring that none are relegated or used disproportionately: *"The argumentative skill was used much more; therefore, it will be balanced with the rest of the skills used"* (G3).

Specific improvements are suggested, such as placing greater emphasis on representation and fostering the communication of ideas among students: *"[...] ask more peers to present and explain their representation to contrast the results obtained [...]"* (G4).

All participating groups, in their redesign proposals, indicated integrating the presence of skills that were highlighted to a lesser degree in their simulations; some giving more emphasis to modeling and others to representing, as evidenced in the following stories: *"[...] Another modification would be to provide a problematic situation about the building and not just its measurements. This would give more meaning to the activity to be carried out"* (G4); "*To modify this, [...] add suitable representations where students can verify and teachers clarify without any concern"* (G6).

Overall, the groups show a reflective and proactive attitude towards improving the quality of the simulated classes, recognizing the importance of developing mathematical processes integrally for effective teaching.

In this component, the students explicitly mention the skills (processes) declared in the Chilean curriculum; however, it should be noted that, from the teacher's observation, the second author of this article, other processes manifested in the activities, such as trial-and-error, manipulation, and experimentation. This is justified because, prior to the activity related to the comprehensive study of the Pythagorean Theorem, the skills from the Chilean curriculum were analyzed, as the subject program indicated. This leads us to reflect that, in future instances, emphasis should also be given to other mathematical processes, as mentioned, which are part of the teaching activity.

7 Assessment of the representativeness component and improvement proposal for the class

The analysis reveals that all participating groups reflected on the various modes of expression (verbal, concrete, pictorial, symbolic) present in the simulations. However, none of the groups considered whether the types of experiences carried out were sufficient to achieve the EI provided (partial meanings of the Pythagorean Theorem).

Group 1 was tasked with addressing EI1. This indicator required the development of an activity where students could verify or demonstrate the Pythagorean Theorem as a relationship between the areas of the squares built on the sides of the right triangle, which constitutes the geometric meaning of the Pythagorean Theorem.

Group 2 took on the responsibility of designing and simulating an activity related to EI2. This indicator explores another geometric meaning of the Pythagorean Theorem, specifically the relationship between the lengths of the sides of the triangle.

Groups G3 and G5 were assigned to simulate a class experience related to EI3. This indicator addresses the arithmetic-algebraic meaning of the Pythagorean Theorem, specifically the reciprocal of the theorem.

Group 4 was responsible for developing a class experience related to EI5. This indicator focuses on the geometric meaning of the Pythagorean Theorem, which establishes a relationship between the lengths of the sides of a right triangle.

Group 6 was asked to perform a class simulation on EI4. This indicator involves a connection between one of the geometric meanings of the Pythagorean Theorem and the Theorem of Thales.

Although all groups used a variety of representations in their simulated classes, areas for improvement were identified in relation to understanding the assigned EIs and the need to diversify activities to fully address the meanings of the Pythagorean Theorem. This is reflected in the following testimonies: *"[...] perform the conversion from concrete representation to a verbal representation, through intentional questions about the relationship between areas of the figures"* (G1); *"To strengthen this area, the use of concrete figures could be implemented, allowing students to manipulate them [...]"* (G2); *"To present a partial meaning through different modes of expression, it is necessary to incorporate concrete, pictorial, and symbolic elements in the class, avoiding relying exclusively on learning guides [...]"* (G6).

The results show that the students failed to apply what they learned in the training module (section 3.4.4) about the representativeness of the Pythagorean Theorem in reflecting on their class simulations. This indicates a need for a deeper integration of the theoretical insights into practical classroom settings, ensuring that all forms of representation effectively contribute to a comprehensive understanding of the mathematical concepts involved.

8 Discussion and Conclusions

Effective mathematics teaching requires reflective and well-prepared teachers capable of analyzing and understanding various factors influencing teaching and learning processes (Godino, Batanero & Font, 2003). This study explored integrating class simulations and the Epistemic DSC in training future mathematics teachers in Chile, aiming to foster reflective competence from the early stages of their professional preparation.

This research focused on two questions: How do future teachers apply the components of the Epistemic DSC when reflecting on simulated classes? and What connections can be identified between the reflections of future teachers and their proposals to improve these classes?

The first question addressed future teachers' competence in handling the components of the Epistemic DSC during their reflections on simulated classes. Results reflect varied approaches and skills among the groups. Some students showed a solid understanding and precise application of the components, such as identifying mathematical errors and evaluating ambiguities, while others were confused, especially in distinguishing between mathematical aspects and pedagogical decisions. All groups faced challenges in analyzing the representativeness of the Pythagorean Theorem in their class simulations, indicating a need to strengthen training in conceptualizing and applying the components of *errors*, *ambiguities*, and *representativeness*.

These findings are supported by recent research (Hummes *et al*. 2023; Hummes & Seckel, 2024) and international findings indicating that teachers face challenges in understanding the epistemic aspects of tasks (Stahnke, Schueler & Roesken-Winter, 2016).

The second question focused on the connection between future teachers' reflections on simulated classes and improvement proposals derived from these reflections. Regardless of how well groups used the components of the Epistemic DSC, all demonstrated a commitment to improving didactic quality through adjustments in instructions, changes in representations, and strategies to address cognitive misunderstandings. This suggests that reflective awareness is linked to efforts to elevate instructional quality (Hill, Rowan, & Ball, 2005).

The findings highlight the relevance of class simulation as a formative strategy, providing future teachers with practical experiences and meaningful learning opportunities. Reflective competence extends beyond simulated practice, requiring specific conceptual and methodological frameworks (Hatton & Smith, 1995). The combination of class simulation with the Epistemic DSC offered a structured approach for reflection, emerging as a valuable strategy in the initial training of mathematics teachers by promoting structured reflection on pedagogical practice.

Addressing variabilities in future teachers' ability to apply these criteria suggests the need for improvements in teacher training programs. This study contributes to teacher training knowledge by highlighting the importance of cultivating reflective competence early in professional preparation.

Despite valuable findings, this study has limitations that may have influenced result interpretation, presenting opportunities for future research to expand knowledge in teacher training and reflective competence. Conducted at a Chilean university with Basic Education Pedagogy students specializing in Mathematics, the applicability to other contexts or educational levels might be limited. The small sample size limits generalization and understanding of individual variabilities in applying the Epistemic DSC. The focus on early-stage students might not reflect patterns in more experienced teachers. Finally, evaluations were based on self-reflections, and incorporating expert assessments could validate these findings.

Future research should include expert analysis of errors, ambiguities, process richness, and representativeness in class simulations. Contrasting these assessments with future teachers' reflections would provide a comprehensive view of perceptions, contributing to understanding areas for improvement in pedagogical preparation. Longitudinal research could offer a complete view of reflective competence development over time. Replicating the study in various contexts and with larger samples would allow examination of how contextual specifics influence the application of the Epistemic DSC. Considering external variables, such as prior teaching experience or attitudes towards mathematics, could enrich understanding factors impacting reflective competence. Addressing these limitations and suggestions can significantly advance knowledge in teacher training and strengthen formative strategies promoting reflective competence in aspiring mathematics educators.

References

Aghakhani, S., Lewitzky, R. A., & Majeed, A. (2023). Developing reflective practice among teachers of mathematics. *International Electronic Journal of Mathematics Education, 18*(4), 1-10.

Bradley, E. G. & Kendall, B. (2014). A Review of Computer Simulations in Teacher Education. *Journal of Educational Technology Systems, 43*(1), 3-12.

Breda, A., Font, V. & Pino-Fan, L. (2018). Evaluative and normative criteria in Didactics of Mathematics: the case of didactical suitability construct. *Bolema*, *32*(60), 255-278.

Breda, A., Pino-Fan, L. R. & Font, V. (2017). Meta Didactic-Mathematical Knowledge of Teachers: Criteria for The Reflection and Assessment on Teaching Practice. *Eurasia Journal of Mathematics, Science and Technology Education*, 13(6), 1893-1918.

Cáceres, P. (2003). Análisis cualitativo de contenido: una alternativa metodológica alcanzable. *Psicoperspectivas. Individuo y sociedad*, *2*(1), 53-82.

Corbin, J. & Strauss, A. (2014). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks, CA: Sage publications.

Davis, B. (2008). Is 1 a prime number? Developing teacher knowledge through concept study. *Mathematics Teaching in the Middle School, 14*(2), 86-91.

Dyer, E. B. & Sherin, M. G. (2016). Instructional reasoning about interpretations of student thinking that supports responsive teaching in secondary mathematics. *ZDM Mathematics Education, 48*(1), 69-82.

Font, V., Planas, N. & Godino, J. D. (2010). A model for the study of mathematics teaching and learning processes. *Infancia y Aprendizaje*, *33*(1), 89-105.

Giacomone, B., Godino, J. D. & Beltrán-Pellicer, P. (2018). Developing the prospective mathematics teachers' didactical suitability analysis competence. *Educação e Pesquisa, 44*, 1-21.

Gibson, D. C., Knezek, G., Redmond, P. & Bradley, E. (2014). *Handbook of games and simulations in teacher education*. Chesapeake, VA: Association for the Advancement of Computing in Education.

Godino, J. D. (2013). Indicators of didactical suitability for mathematics teaching and learning processes. *Cuadernos de Investigación y Formación en Educación Matemática*, (11), 111-132.

Godino, J. D., Batanero, C. & Burgos, M. (2023). Theory of didactical suitability: An enlarged view of the quality of mathematics instruction. *Eurasia Journal of Mathematics, Science and Technology Education, 19*(6), 1-20.

Godino, J. D., Batanero, C., & Font, V. (2003). *Fundamentos de la enseñanza y el aprendizaje de las matemáticas para maestros*. Granada, Universidad de Granada.

Godino, J. D., Batanero, C. & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM Mathematics Education*, *39*(1), 127-135.

Godino, J. D., Batanero, C. & Font, V. (2019). The Onto-semiotic Approach: implications for the prescriptive character of didactics. *For the Learning of Mathematics, 39*(1), 37-42.

Godino, J. D., Font, V., Wilhelmi, M. R. & De Castro, C. (2009). Aproximación a la dimensión normativa en didáctica de las matemáticas desde un enfoque ontosemiótico. *Enseñanza de las Ciencias*, *27*(1), 59-76.

Godino, J. D., Giacomone, B., Batanero, C. & Font, V. (2017). Onto-Semiotic Approach to Mathematics Teacher's Knowledge and Competences. *Bolema*, *31*(57), 90-113.

Guba, E. & Lincoln, Y. (2002). Paradigmas en competencia en la investigación cualitativa. In C. Denman & J. Haro (Ed.). *Por los rincones. Antología de métodos cualitativos en la investigación social* (pp. 113-145). Hermosillo Sonora: Editorial El colegio de Sonora.

Hatton, N., & Smith, D. (1995). Reflection in teacher education: Towards definition and implementation. *Teaching and Teacher Education, 11*(1), 33-49.

Hill, H. C., Rowan, B. & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Huang, R., Takahashi, A. & Ponte, J. P. (2019). *Theory and practice of lesson study in mathematics*. Cham, Switzerland: Springer International Publishing.

Hummes, V., Breda, A. & Font, V. (2022). El desarrollo de la reflexión sobre la práctica en la formación de profesores de matemáticas: una mirada desde el Lesson Study y los Criterios de Idoneidad Didáctica. In J. Lugo-Armenta, L. Pino-Fan, M. Pochulu & W. Castro (Ed.). *Enfoque ontosemiótico del conocimiento y la instrucción matemáticos: Investigaciones y desarrollos en América Latina* (pp. 221-241). Osorno, Chile: Universidad de los Lagos.

Hummes, V., Breda, A., Font, V. & Seckel, M. J. (2023). Improvement of Reflection on Teaching Practice in a Training Course That Integrates the Lesson Study and Criteria of Didactical Suitability. *Journal of Higher Education Theory and Practice, 23*(14), 208-224.

Hummes, V., & Seckel, M. J. (2024). Advancing teacher reflective competence: Integrating lesson study and didactic suitability criteria in training. *Frontiers in Education, 9*, 1-9.

López, F. (2002). El análisis de contenido como método de investigación. *En-clave Pedagógica*, *4*, 167-169.

Mason, J. (2002). *Researching your own practice. The discipline of noticing.* London: Routledge-Falmer.

Melo, C. B. S. & Bisognin, E. (2021). Modelagem Matemática como proposta de itinerário formativo no Novo Ensino Médio: uma possibilidade para o desenvolvimento de habilidades e competências. *Revista Internacional de Pesquisa Em Educação Matemática*, *11*(1), 24-36.

Ortíz, F. J. R. (2023). Los paradigmas epistémicos en la investigación educativa. *Revista Educativa Avanza, 1*(1), 29-36.

Porta, L. & Silva, M. (2003). La investigación cualitativa: El Análisis de Contenido en la investigación educativa. *Anuario digital de investigación educativa*, *14*: 1-18.

Proença, M. C., Campelo, C. S. A. & Oliveira, A. B. (2024). Prospective mathematics teachers' reflections on their strategies for solving a simple combination problem. *Revista Internacional de Pesquisa em Educação Matemática, 14*(1), 1-13.

Speed, S. A., Bradley, E. & Garland, K. V. (2015). Teaching Adult Learner Characteristics and Facilitation Strategies Through Simulation-Based Practice. *Journal of Educational*

Technology Systems, 44(2), 203-229.

Stahnke, R., Schueler, S. & Roesken-Winter, B. (2016). Teachers' perception, interpretation, anddecision-making: a systematic review of empirical mathematics education research. *ZDM Mathematics Education, 48*, 1-27.