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Instrument-to-think-mathematics-with: intertwining mathematical concepts and schemes of use in the instrumental genesis of the Mathematics teacher

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Abstract: This article discusses the integration of digital technologies in the Mathematics classroom and the process of appropriation of these technologies by teachers and students. Based on theoretical studies and reflections on the actions of teacher-researchers, it is analyzed how the activities proposed by the Mathematics teacher reveal levels in a continuous process of appropriation of the artifact and impact the constitution of didactic instruments that enhance the learning of Mathematics. The research is anchored in the theoretical framework of instrumental genesis, especially for the role of usage schemes that are established in the development of the execution of the activity. The results of the study propose categories for different activities in a dynamic mathematics environment. In addition, they indicate that artifacts evolve into instruments as an instrument-for-thinking-mathematics-with as mathematical concepts and the constitution of usage schemes of the artifact intertwine in a coordinated whole.

Keywords: Instrumental Genesis. Digital Technologies. Dynamic Mathematics. Usage Schemes. Tools-for-thinking-mathematics-with.

Instrumento-para-pensar-las-matemáticas-con: entrelazando conceptos matemáticos y esquemas de utilización en la génesis instrumental del profesor de Matemática

Resumen: En este artículo se discute la integración de las tecnologías digitales en el aula de Matemáticas y el proceso de apropiación de estas tecnologías por parte de profesores y estudiantes. A partir de estudios teóricos y reflexiones sobre el accionar de docentes-investigadores, analizamos cómo las actividades propuestas por el docente de Matemática revelan niveles en un proceso continuo de apropiación del artefacto e impactan en la creación de instrumentos didácticos que potencian el aprendizaje de Matemática. La investigación se ancla en el marco teórico de la génesis instrumental, especialmente en lo que respecta al papel de los esquemas de uso que se establecen en el desarrollo de la actividad. Los resultados del estudio proponen categorías para diferentes actividades en un entorno matemático dinámico. Además, señalan que los artefactos evolucionan hasta convertirse en instrumentos como instrumentos para pensar matemáticamente a medida que los conceptos matemáticos y la constitución de esquemas para usar el artefacto se entrelazan en un todo coordinado.

Palabras clave: Génesis Instrumental. Tecnologías Digitales. Matemáticas Dinámicas. Esquemas de Uso. Instrumentos-para-pensar-las-matemáticas.

Instrumento-para-pensar-matemática-com: entrelaçando conceitos matemáticos e esquemas de utilização na gênese instrumental do professor de Matemática

Resumo: Neste artigo, discute-se a integração das tecnologias digitais na sala de aula de Matemática e o processo de apropriação destas tecnologias por professores e alunos. A partir de estudos teóricos e de reflexões sobre ações dos professores-pesquisadores, analisa-se como as atividades propostas pelo professor de Matemática revelam níveis em um processo contínuo de apropriação do artefato e impactam na constituição de instrumentos didáticos que potencializam a aprendizagem de Matemática. A investigação está ancorada no quadro teórico da gênese instrumental, especialmente para o papel dos esquemas de utilização que se estabelecem no desenvolver da realização da atividade. Os resultados do estudo propõem categorias para distintas atividades em ambiente de matemática dinâmica. Além disso, apontam que os artefatos evoluem para instrumentos como um instrumento-para-pensar-matemática-com na medida em que conceitos matemáticos e a constituição de esquemas de utilização do artefato se entrelaçam em um todo coordenado.

Palavras-chave: Gênese Instrumental. Tecnologias Digitais. Matemática Dinâmica. Esquemas de Utilização. Instrumentos-para-pensar-matemática-com.

1 Introduction

With the integration of digital technologies in mathematics education, a field of research has emerged to understand how students and teachers utilize these technologies and how their use is proposed in mathematics classrooms. This article, focused on the use of dynamic mathematics and supported by Rabardel's (1995) instrumental approach, analyzes how the activities proposed by the mathematics teacher reveal the process of appropriation of an artifact and impact the constitution of didactic instruments that enhance the learning of mathematical concepts.

The instrumental approach, detailed in Section 3 of this article, is a theoretical contribution that has helped researchers investigate and seek to understand the processes involved in incorporating digital technologies into the classroom. In this sense, we present recent works in the area that are grounded in this theoretical framework and highlight how the present research aligns with this context.

Buteau, Muller, Mgombelo, and Sacristám (2019) proposed a model to describe the four stages of the instrumental genesis process of undergraduate students in mathematical investigation situations within a programming environment. The authors describe these stages as: (1) instrumental initiation; (2) instrumental exploration; (3) instrumental reinforcement; and (4) instrumental symbiosis.

Cuéllar and Miraval (2018) analyzed the instrumental genesis process of the mathematical object, the exponential function, using GeoGebra. According to the authors, GeoGebra favored the processes of instrumentalization and instrumentation of the mathematical object addressed, based on the identification of schemes constructed by the participants.

Yao (2020) examined the relationship between the instrumental genesis of pre-service mathematics teachers and the development of geometric knowledge when solving geometry construction problems with Geometer's Sketchpad software. Data analysis revealed a coevolution between instrumental genesis and geometric knowledge. According to the author, this relationship can be summarized as follows: guided by his prior knowledge of geometry and dynamic geometry techniques, the participant first used particular dynamic geometry tools to obtain a geometric figure. By manipulating this geometric figure through various methods of dragging and measuring, as well as other actions mediated by instruments, the participant observed new geometric properties. The participant then used this newly developed knowledge to guide his use of dynamic geometry, which led him to create a dynamic figure or develop an alternative construction. The author also highlights the dynamic interactions between the student, the mathematical task, and the technological tool, through which new geometric knowledge and meaningful ways of using technology emerge.

Dijke-Droogers, Drijvers, and Bakker (2021) sought to study the intertwining of techniques for using digital tools and conceptual understanding, focusing the research on the theory of instrumental genesis for statistical modeling, investigating students' modeling processes in the digital environment TinkerPlots. In particular, the authors analyzed how emerging techniques and conceptual understanding were intertwined in the instrumentation schemes developed by 28 students. Through data analysis, the authors identified six typical instrumentation schemes and observed a bidirectional interweaving of emerging techniques and conceptual understanding.

Bozkurt and Uygan (2020) analyzed the classroom practices of a mathematics teacher considered a beginner in the use of digital technologies in light of the theory of instrumental genesis. The results of the case study indicated that although the teacher planned inquiry lessons and focused on students' active use of dynamic geometry, she was unable to use the resource in a way that was functional to her pedagogical approach. The authors point out that, in part, the problems occurred because the teacher planned the use of dynamic geometry shaped by her own instrumental genesis, neglecting the students' instrumental genesis processes. Thus, the problems faced by this teacher were related to the lack of technical planning of the activities and the instrumentation problems students experienced.

Buteau, Muller, Mgombelo, Sacristám, and Dreise (2020) sought to understand how university students learn to use programming as an instrument for mathematical investigations, employing the instrumental approach to analysis. The authors propose four instrumental stages of student development. The proposed approach takes into account not only the development of individual schemes, but also the development of a complex web of schemes, considering that students appropriate programming as an instrument.

Thus, we highlight that the survey carried out on research in the area of mathematics education and digital technologies reveals that the theoretical framework of instrumental genesis, used to analyze and understand the intertwined relationships between technological appropriation and mathematical understanding, is current, pertinent, and relevant.

In the study presented in this article, we identified that the activities proposed by the mathematics teacher, our research object, can reveal their level of technological appropriation of the artifact and indicate the activity's potential for learning mathematical concepts. In the research carried out, we aim to categorize different activities proposed in the GeoGebra environment, focusing on the theme of 'notable points' in the 'triangle', anchored by the theory of instrumental genesis and emphasizing the analysis of usage schemes underlying each activity. The results of this study, presented in Section 5, indicate that the category of an activity can reveal the level of understanding of the technological instrument and its potential as an instrument-to-think-mathematics-with. Thus, we present below the theoretical framework that supports our understanding of instruments-to-think-mathematics-with. Next, we address theoretical aspects of the theory of instrumental genesis and usage schemes, the methodological approach of the research, the presentation and analysis of the categories of activities, and the final considerations.

2 Instruments-to-think-mathematics-with

In this section, we discuss our understanding of the use of digital technologies (DTs) in mathematics education. Based on this understanding, we argue that technological artifacts must become true epistemological and cognitive instruments, capable of enabling “thought experiments” (Gravina & Basso, 2012), in which students can expand their ways of thinking and express their mathematical reasoning, making the technological instrument an extension of thought. We therefore understand that technological artifacts affect people and the way they structure their mathematical thinking. However, this relationship does not occur in a single direction; people also affect the artifact, transforming it in the process.

The technological artifacts available today to mathematics teachers, and we will focus in particular on software developed for thinking and doing mathematics, can be considered cognitive instruments that amplify our cognitive capabilities and abilities, establishing a dynamic and reciprocal relationship between thinking and DTs. Papert (1980) coined the expression “objects-to-think-with” four decades ago, referring to objects that could be used to think about formal systems. The author was referring to the turtle in the LOGO environment, but he was also interested in the invention of new objects-to-think-with. In the same direction, the authors Shaffer and Clinton (2006) proposed the expression ‘toolsforthoughts’, highlighting again the reciprocal relationship established between subject and tool. For these authors, the subject and technological tools are placed in a situation of symbiosis, so that, in the same way that the subject affects the tool, the tool also affects the subject.

To better highlight this reciprocal relationship between subject and artifact in the development of mathematical thinking, we highlight a cycle of important actions that drive this development: explore → conjecture → validate → explain → argue, and this cycle can be enhanced by DTs and, in particular, by a dynamic mathematical environment. Based on Papert (1980) and Shaffer and Clinton (2006), and based on instrumental genesis, we understand that DTs for learning mathematics must be conceived as instruments-for-thinking-mathematics-with, in which subjects think and express themselves through DTs. Dynamic mathematics is configured as a favorable environment for these reciprocal relationships to be established, in which subjects and technological artifacts place themselves in a situation of symbiosis, engaging in a mutual process of action and reaction.

With dynamic mathematics, new possibilities emerge for accessing mathematical objects, opening up space for new ways of thinking-mathematics-with. This is because dynamic mathematics environments enhance access to and manipulation of mathematical objects. Furthermore, the fact that we relate to mathematical objects in new ways also imposes new ways of looking at the activities proposed in the classroom. Therefore, it is not enough for mathematics teachers to make a good choice of technological artifacts to use with their students. The activities proposed with these artifacts can make a difference in the mathematics classroom, as they promote ‘thinking in mathematics,’ transforming the artifacts into instruments-to-think-mathematics-with. Thus, for DTs to be instruments-to-think-mathematics-with, it is crucial to choose activities that require students to construct mental models that trigger the cycle of exploring → conjecturing → validating → explaining → arguing and favor the development of abstraction, generalization, and flexible thinking skills, provoking the thought experiments pointed out by Gravina and Basso (2012). Experiences in which one can access, manipulate, and modify mathematical objects through technological artifacts enable students to learn how to perform the same types of experiments in their minds, even in the absence of these artifacts. Thus, it becomes possible to incorporate these artifacts into students' cognitive activity, enabling new ways of thinking-mathematically-with, transforming the artifacts into true cognitive instruments.

From this perspective, DTs can be used as instruments-to-think-mathematics-with, engaging students in thinking about a problem and trying to solve it, rather than being used to solve the problem by the student. Thus, the proposed activities can be developed to promote mathematical thinking, rather than facilitating or accelerating problem-solving procedures. In this sense, the approach to proposing a problem must consider the possibilities of exploration or construction, requiring and/or developing mathematical knowledge. The teacher who uses DTs, having the clarity that a classic problem presented in books can become elementary or outdated in a dynamic mathematics environment, will be able to recognize that these environments open up possibilities for exploring problems that were previously beyond the scope of their classes. We conclude this section with some provocations: What instruments do we, as mathematics teachers, want to develop with our students? What usage schemes do we want to trigger? What cognitive activities do we want to provoke? What learnings and understanding gains do we want to achieve? These questions can guide the mathematics teacher in using digital artifacts as instruments-to-think-mathematics-with.

In the following section, we address theoretical aspects of instrumental genesis.

3 The Instrumental Genesis

Instrumental genesis is the process of transforming an artifact into an instrument. Rabardel (1995) states that an artifact consists of the material or symbolic object itself, or part of a more complex object. We can consider GeoGebra as an artifact, but also its resources, such as the tools it provides, such as points, segments, and polygons, among others. The instrument, in turn, is defined by Rabardel (1995) as a mixed entity, comprising the artifact and the usage schemes that the subject develops to use it for a specific purpose. Therefore, the instrument arises from the relationship established between the subject and the artifact and is an individual construction of the subject, triggered by the need to perform specific tasks.

Thus, we can state that the instrument is not provided to the subject; rather, it results from the interaction between the subject and the artifact, and depends on the objective and task that the subject needs to perform. Therefore, artifacts and schemes are associated, but are independent, since the same scheme can be applied to different artifacts, just as the same artifact can be associated with different schemes, which can give rise to a diversity of instruments. Likewise, the same artifact constitutes different instruments for each subject who uses it.

Therefore, we consider it important to advance the notion of an instrument by understanding the concept of a scheme.

3.1 Concept of Scheme in Piaget's Theory

For Piaget (1970), who studied the birth of intelligence in its sensorimotor dimension, schemes constitute means by which the subject can assimilate situations and objects with which they are confronted. Thus, schemes are structures that extend biological organization and share with it an assimilative capacity to incorporate external reality into the subject's organizational cycle: everything that responds to a need is susceptible to assimilation.

Piaget believes the scheme is itself a product of assimilative activity. Reproductive assimilation constitutes the schemes, which acquire their existence as soon as a behavior, however uncomplicated it may be, gives rise to an effort of repetition and is, therefore, schematized. The scheme of an action is, therefore, the structured set of generalizable characteristics of the action, that is, which allow the same action to be repeated or applied to new contents (Rabardel, 1995).

Every scheme constitutes a totality, i.e., a set of mutually dependent elements that cannot function without each other. Indeed, they imply each other. It is the global meaning of the act that guarantees the simultaneous existence of the constitutive relations of the schemes as a totality (Piaget, 1970). However, even though they originally constitute isolated totalities, the schemata coordinate, through mutual assimilation, into new and original totalities that are broader and possess general properties as well. Thus, in young children, the coordination of several schemes in a single act results from the need to achieve a goal that is not directly accessible through an isolated schemes. This implies the mobilization of schemes previously related to other situations and their coordination, resulting in the formation of a central scheme of action that incorporates a series of subordinate schemes.

Based on the concept of scheme in Piaget's theory, we will now address the concept of scheme in Rabardel's instrumental approach.

3.2 The Concept of Scheme in the Instrumental Approach

In Rabardel's instrumental approach, the author defines the concept of usage scheme, which is associated with schemes related to the use of an artifact. Various types of schemes comprise the class of usage schemes. They refer to two dimensions of activity: activities related to secondary tasks, that is, those related to the management of particular characteristics and properties of the artifact; and primary, main activities, directed towards the object of the activity, and for which the artifact is a means of realization (Rabardel, 1995).

These two dimensions of activity also allow us to distinguish two levels of schemes within the usage schemes: (1) usage schemes, which are related to secondary tasks (Rabardel, 1995). These schemes belong to the level of elementary schemes, in the sense that they are not decomposable into smaller units capable of responding to an identifiable sub-objective, but this is not necessary: they can be constituted into wholes by articulating a set of elementary schemes. What characterizes them is their orientation towards secondary tasks corresponding to specific actions and activities directly related to the artifact; (2) instrumented action schemes, which consist of totalities whose meaning is given by the global act intended to effect transformations in the object of the activity. These schemes incorporate, as constituents, the schemes of the first level (usage schemes). What characterizes them is that they are related to primary tasks (Rabardel, 1995). Usage schemes coordinate with each other and with other schemes to constitute instrumented action schemes.

We can take, for example, the construction of a square in GeoGebra. For an individual who already has experience with constructions in a dynamic geometry environment, the action of constructing a square constitutes a scheme of instrumented action, which incorporates and coordinates usage schemes subordinate to the general construction, such as drawing a straight line segment, a circle, or a perpendicular line. On the other hand, for a less experienced individual, the simple construction of a straight line perpendicular to a segment passing through one of its ends can constitute an instrumented action scheme, which incorporates more elementary usage schemes. Thus, it becomes evident that the criterion used to distinguish the schemes, that is, whether it is related to a secondary or primary task, does not refer to a property of the scheme itself, but to its status in the subject's final activity. The same scheme can therefore, depending on the situation, have the status of a usage scheme (for example, the construction of a perpendicular line in the process of constructing a square) or the status of an instrumented action scheme (for example, for an inexperienced subject, learning to use the basic tools of GeoGebra to construct a line perpendicular to a segment passing through one of its ends).

The set of usage schemes, instrumented action schemes, and instrumented collective activity schemes belongs to the class of schemes that Rabardel (1995) refers to as utilization schemes. The author observes that these different types of schemes are in relationships of mutual dependence. Utilization schemes are related, on the one hand, to artifacts that likely have the status of means, and on the other hand, to the objects on which these artifacts enable action. They are the organizers of the action, the utilization, and the use of the artifact. However, the usage schemes do not apply directly; they must be instantiated according to the specific context of each situation. They are then updated according to a procedure tailored to the specific circumstances.

The implementation of usage schemes in new but similar situations, therefore, a process of assimilation, leads to the generalization of schemes by extending the classes of situations, artifacts and objects for which they are relevant. Likewise, it leads to their differentiation, as they often have to adapt to different and new specific aspects of situations. In situations that are new to the subject, the accommodation process becomes, for some time, dominant. It leads to the transformation of available schemes, their reorganization, fragmentation, and recomposition, as well as reciprocal assimilation and coordination, which progressively produce new schematic compositions, allowing for the renewed and reproducible mastery of the new class of situations (Rabardel, 1995). Such mechanisms arise, for example, when new artifacts must be used as means of action or when they must point to new objects or new transformations in these objects.

The usage schemes can then be considered familiar schemes, easily mobilized, contributing to an automated operation, characteristic of usual situations, and well controlled by the subject. The implementation of usage schemes in different situations is in a relationship of relative independence. The same usage scheme can be applied to a multitude of artifacts belonging to the same class or similar classes. On the other hand, an artifact will probably fit into a multiplicity of usage schemes that will give it different meanings and, sometimes, different functions, that is, a multiplicity of instruments. Thus, the same artifact can have a different instrumental status for each subject.

3.3 The Notion of Instrument and Instrumental Genesis

Once the concept of scheme is understood, we can proceed to the concept of instrument and the definition of instrumental genesis, as outlined by Rabardel (1995). As we have already stated, the notion of instrument cannot be reduced to artifact, since the instrument includes an artifact component and a usage scheme component, which results from the subject's own and autonomous construction. It is not only the artifact that is associated by the subject with their action to perform the task, but also the usage schemes that will allow an instrument to be inserted as a functional component of the subject's action. In other words, the constitution of the instrumental entity is a product of the subject's activity.

Both components of the instrument, artifact and scheme, are closely associated with each other, and the boundaries between them are difficult to determine. Thus, an instrument that becomes permanent, that is, capable of conservation and, therefore, of reuse, consists of the combination of these two stable invariants (artifact + schemes), which, together, constitute a potential means of solution, treatment, and action in a situation. Artifacts are, most of the time, pre-existing, but are still instrumentalized by the subject. The schemes are, for the most part, part of the subject's repertoire and can be generalized or adapted to the new artifact. However, it is possible that, in certain situations, new schemes may be entirely constructed or developed. Rabardel characterizes these processes as two: instrumentation and instrumentalization.

Instrumentation processes are related to the emergence and evolution of usage and instrumented action schemes; that is, the genesis of schemes, the assimilation of new artifacts into the schemes, providing new meanings to the artifacts, and the accommodation of schemes that contribute to changes in the meaning of the instrument. In other words, the progressive discovery of the artifact's properties by the subject is accompanied by the accommodation of their schemes and by changes in the meaning of the instrument, resulting from the association of the artifact with new schemes, through instrumentation processes (Rabardel, 1995).

Instrumentalization can be defined as a process of enriching the properties of the artifact by the subject. A process that relies on intrinsic features and properties of the artifact and gives them a status based on the current action. They constitute, for the subject, a characteristic, a permanent property of the artifact, or more specifically of the artifact component of the instrument. Thus, instrumentalization processes concern the emergence and evolution of artifactual components of the instrument.

4 Methodological Path

This article proposes a study based on the authors' theoretical reflections. Based on studies, research, and classroom experiences as mathematics teachers and mathematics teacher educators in the use of DTs, we recognize that incorporating DTs into mathematics classes is a complex process that can occur in different ways and at varying levels of understanding for mathematics learning. These different approaches, which were identified throughout our experiences as teachers and researchers, are related to the instrumental genesis of the mathematics teacher. Furthermore, this study identified that the potential of an activity proposed by the teacher is related to the usage schemes and instrumental action schemes that must be constituted and utilized for its realization.

The investigation, which took place within the scope of the research project 'Instrumental genesis processes of the use of digital technologies by mathematics teachers' [Processos de gênese instrumental de uso das tecnologias digitais por professores de Matemática], counts on teachers-researchers who not only investigate the incorporation of DTs in the basic education mathematics classroom, but also turn to their own practices as mathematics teacher educators, reflecting on them, as they recognize teaching practice as a rich source of research and knowledge. To develop this knowledge, it is necessary to be in contact with one's own practice, which Schön (2000) refers to as knowledge-in-action. To the author, in professional practice, uncertain or unique situations arise that require new solutions and the development of new strategies, a process known as reflection-in-action.

Through reflection-in-action, the teacher begins to build a repertoire of experiences that are mobilized in similar situations. Thus, we understand that reflecting on reflection in action characterizes a new level of teacher reflection, allowing for the review and planning of future actions.

From this perspective, reflecting on our own practice in the education of mathematics teachers for the use of DTs enabled us to problematize, analyze, and understand the complexity inherent in the process of instrumental genesis, which develops in different dimensions: first, the teacher must experience their own process of instrumental genesis to appropriate the artifact presented to them, transforming it into an instrument for thinking about mathematics; then, they must experience the process of professional instrumental genesis, understanding the potential of the resource as a didactic tool to promote mathematical learning, which involves understanding this tool in its cognitive and epistemological dimensions; finally, the teacher must recognize that their students will also experience a process of instrumental genesis, until they transform the artifact into an instrument-for-thinking-mathematics-with.

Thus, this research was developed from theoretical studies and analyses of activities proposed in GeoGebra by in-service or pre-service mathematics teachers who took Mathematics Education and Digital Technologies courses at the university (omitted). Based on the studies carried out, this article presents the categorization of activities developed for the use of dynamic mathematics. We identified four categories of activities, which will be presented and discussed in the following section, in light of the theoretical frameworks of the instrumental approach and the instruments-to-think-mathematics-with.

5 Categories of Activities with Dynamic Mathematics: a look at the usage schemes

As discussed in Section 3, the instrument is a unique construction for each subject, arising from the relationship between the subject and the artifact. In the context of using DTs for learning mathematics, the various software and applications currently available constitute, at first, artifacts for the mathematics teacher. The teaching instruments into which these artifacts can be transformed are not provided to the teacher; they must construct them themselves. In other words, such artifacts could become teaching instruments with potential for learning mathematics, which are configured as instruments-to-think-mathematics-with based on the personal and professional instrumental origins of the teacher. The instrumental entity, i.e., the teaching instrument as a functional component of the teacher's action, is the result of the teacher's activity, who develops, mobilizes, and coordinates schemes while revealing the properties and functionalities of the artifact.

In this section, based on the articulation with examples of dynamic mathematics activities developed in GeoGebra, we discuss the existing relationships between categories of activities and their potential, or not, to develop instrumental action schemes, which culminate in the development of instruments for students to think about mathematics, towards the reciprocal relationship that is established between student and GeoGebra as a potentializer of thoughts based on specific activities. We were guided by questions such as: What schemes should be mobilized to carry out the activity? What mathematical concepts should be used to carry out the activity? Is there coordination between artifact usage schemes and mathematical concepts?

The categories are associated with the possibilities of advances in the development of usage schemes for the artifact, which begin with proposals that require the student to employ elementary and non-decomposable usage schemes, whose central orientation is more associated with activities oriented towards the use of the artifact, and continue towards the development of proposals that require instrumented action schemes, which constitute more global actions oriented towards the coordination of mathematical concepts that underpin the use of software resources to carry out the activity or solve a problem. We understand that the process of constituting GeoGebra as an instrument-to-think-mathematics-with is complex and continuous, developin in an ascending spiral, incorporating new usage schemes to constitute the instrument. The categories presented here do not reveal the full complexity of the process, nor the concept of continuity, which involves comings and goings, intersections, but they provide indications of this evolution.

The activities were categorized and defined based on the usage schemes that must be developed, mobilized, transformed, reorganized, or coordinated. We propose the following categories:

- Category I: guided construction activities, which require the use of immediate, elementary, and non-decomposable usage schemes. In this category, specific actions are directed towards the use of GeoGebra resources.
- Category II: visual exploration and property identification activities, which require the use of basic manipulations, mobilizing point movement schemes to explore the dynamic object, to identify evident mathematical properties.
- Category III: unguided construction activities that require the use of instrumented action schemes, whose meaning is given by the global action, which incorporates elementary usage schemes that are coordinated with mathematical concepts to compose the solution of the activity. This category is subdivided into two subcategories:
- Subcategory IIIA: unguided activities of elementary mathematical constructions that present explicit properties.
- Subcategory IIIB: unguided activities of elementary mathematical constructions that present explicit and implicit properties.
- Category IV: unguided activities of complex mathematical constructions and implicit properties. They require the use of instrumented action schemes, whose meaning is given by the global action, which incorporates elementary usage schemes that are coordinated with mathematical concepts to compose the solution to the activity.

To illustrate this classification and guide the discussion and analysis, we provide examples of activities within the theme ‘Notable Points in the Triangle.’

5.1 Category I Activities

A common characteristic of Category I activities is the preparation of a script for using the artifact, which usually omits the student’s exploration and discovery stage, focusing instead on the use of elementary usage schemes that constitute basic construction units. Each stage of the script prepared by the teacher corresponds to a basic and non-decomposable usage scheme, which the student must reproduce to achieve the greater objective: the total construction of the mathematical object under study, that is, the solution of the activity.

In activities in this category, the student’s actions are more directed towards using the artifact, and even without understanding the mathematical concepts involved in the proposed construction, it is possible to arrive at a solution. In other words, they contribute little to the transformation of the artifact into an instrument and do not mobilize the ‘making one think’ effect, since the student is focused on repeating the procedures presented to them.

Figure 1 provides an example of a Category I activity, which leads to the construction of a circle circumscribed within a triangle.

Figure 1: Example of Category I Activity

Construction of a circumscribed circle of a triangle

Follow the steps below in GeoGebra:

1. Using the tool  (polygon), construct any triangle ABC.
2. Using the tool  (perpendicular bisector), construct the perpendicular bisectors of the three sides of triangle ABC.
3. Using the tool  (intersection of two objects), mark the intersection point of the

perpendicular bisectors and name it M.

4. Using the tool (circle given center and one of its points), construct a circle with center at M and passing through point A.
5. Move points A, B, and C and observe the behavior of the circle.

Source: Research Data

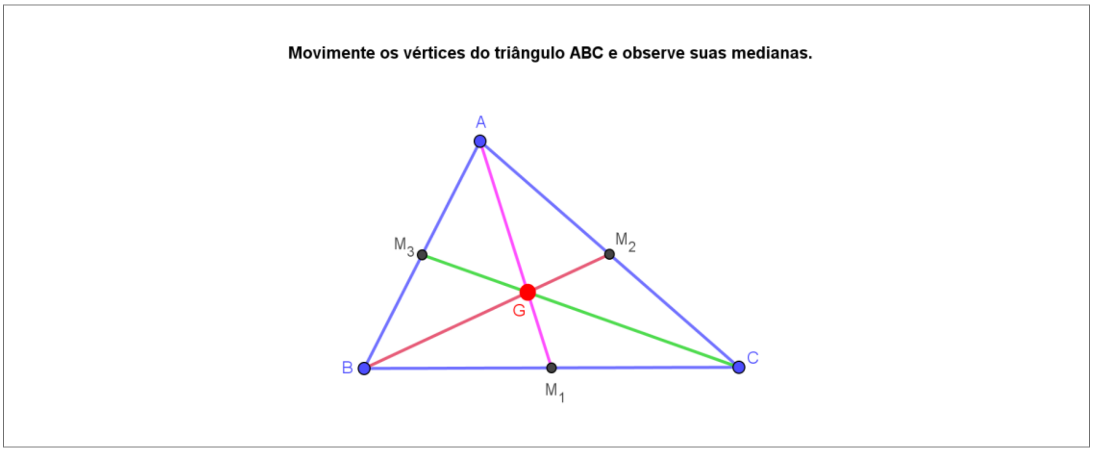
Note that the student does not evoke the mathematical concepts that are crucial to carrying out the construction, nor do they need to be in their repertoire. If the student is not invited to reflect on the construction carried out and reorganize the usage schemes as a whole, coordinated with the mathematical concepts inherent to the tools used, he/she may not understand that the point of intersection of the perpendicular bisectors of a triangle, called the circumcenter, corresponds to the center of the circumference circumscribed to that triangle. In other words, it is possible to be successful in carrying out the activity without understanding its outcome.

Activities in this category deviate from the way we typically conceive of using DTs in mathematics classes, in the sense of instruments-to-think-mathematics-with. This happens because the student simply repeats the procedures presented, without necessarily reflecting on the software tools that are mobilized and thus does not develop instrumented action schemes (Rabardel, 1995). Likewise, there is no need for the student to consider the mathematical properties and concepts present; they simply need to repeat the procedures to complete the activity.

5.2 Category II Activities

Category II activities are characterized by finished constructions, in which students must move points of the construction to observe their behavior and, based on their analysis, identify properties and regularities that characterize the construction. Figure 2 provides an example of an activity characteristic of this category. The schemes used by the student correspond to schemes for moving the vertices of the triangle to identify one of the properties of the barycenter, which states that all the medians of a triangle intersect at the same point.

Figure 2: Example of Category II Activity



Source: Research Data

Although it does not require geometric constructions from the student, activities in this category provide a symbiotic relationship between the student and GeoGebra, in which the student uses the artifact as an extension of thought. This is because it is necessary to explore and analyze the presented construction, identifying present regularities, as well as the underlying mathematical properties. In other words, GeoGebra becomes an instrument-to-think-mathematics-with, as it leads the student to mobilize the mathematical knowledge they possess and articulate it with new mathematical properties present in the construction.

5.3 Category III Activities

Category III is characterized by unguided construction activities and are categorized into two subcategories, which are distinguished by how mathematical properties are revealed to students in the proposed construction.

Activities in this category require students to coordinate mathematical concepts (which must be imposed in construction) with the use of software resources. It is necessary to plan the construction, identify the crucial mathematical properties, and list the GeoGebra tools that will be used based on intentions supported by mathematical concepts, which make it possible to reach the solution. Understanding the activity in its entirety and organizing ideas for construction implies the development of instrumental action schemes, consisting of elementary usage schemes already known to the student, which can be applied in the situation at hand (Rabardel, 1995).

Subcategory IIIB activities differ from those in subcategory IIIA in that they present known mathematical properties in unusual configurations, in which these properties are less evident (implicit properties). Implicit properties require a more in-depth investigation of the construction in the elaboration and verification of hypotheses, as well as articulation with other mathematical concepts already known to the student. On the other hand, Subcategory IIIA activities exhibit explicit properties, which are presented in configurations that are considered more common and which students can more easily identify.

Figure 3 provides an example of a Subcategory IIIB activity, which proposes the construction of a circle inscribed in a given triangle.

Figure 3: Example of Subcategory IIIB Activity

In GeoGebra, construct a circle inscribed in a given triangle.

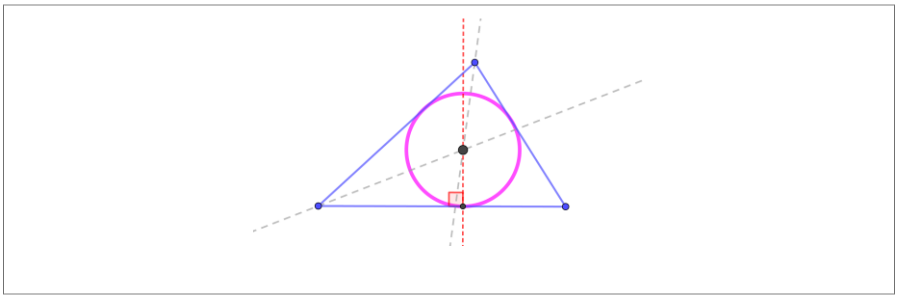
Source: Research Data

It is clear that there are few instructions in the problem statement (the student must develop the construction steps). The first step is to correctly interpret the statement, in order to identify that the initial construction should be the triangle and not the circumference (reversing the order leads to a different problem than the one presented, with different dependency relationships between the triangle and the circumference, and a different resolution strategy). The next step is to construct a given triangle using the Polygon tool or three straight line segments. The use of these tools corresponds to student usage schemes, focusing on the instrument, which enables the execution of secondary-order tasks (Rabardel, 1995). Once the triangle has been constructed, it is necessary to find the center of the circle inscribed in that triangle. At this point, mathematical concepts need to come into play: How to determine the center of the circle? Recognizing that the incenter (intersection point of the internal bisectors of a triangle) corresponds to the sought center is crucial to continue solving the problem and is revealed here as an implicit property (it is not declared, and cannot be immediately identified), which characterizes this activity as Subcategory IIIB. Based on this mathematical concept, usage schemes for constructing bisectors must be used.

It is worth noting that the construction of the incenter can be considered a more general (primary) problem to be solved, which coordinates more specific (secondary) problems, in the same sense that Rabardel (1995) defines two dimensions of activity, which encompass two levels of schemes: usage schemes, related to secondary tasks; and instrumented action schemes, related to totalities, determined by the global act of solving the activity. Thus, the construction of the incenter involves: (1) construction of the triangle (usage scheme); (2) construction of at least two bisectors (usage schemes); (3) construction of the intersection point of the bisectors (usage scheme). The totality coordinated in a set of elementary schemes constitutes an instrumented action scheme (Rabardel, 1995), namely, the construction of the incenter of any triangle.

However, the problem is not yet finished, as it is necessary to construct the circle inscribed in the triangle. The points of tangency of the circumference at the sides of the triangle are not provided by the construction carried out so far; they need to be determined by the student. Again, mathematical concepts must come into play; that is, recognizing that tangent lines to a circle are perpendicular to the radius of the circle at the point of tangency (again, implicit concepts not stated in the problem). Subject and GeoGebra are involved in a reciprocal relationship, placing themselves in symbiosis, i.e., a dynamic relationship between thought and DTs. Once the mathematical idea guiding this stage of construction has been identified, GeoGebra usage diagrams are employed to construct a line perpendicular to at least one side of the triangle that passes through the incenter. Finally, the intersection point of the perpendicular line with the respective side is marked (usage scheme), and the circumference (usage scheme) inscribed in the triangle is constructed, solving the problem (instrumented action scheme), as illustrated in Figure 4. Here, the subject is identified as acting, thinking, and expressing him/herself through GeoGebra, configuring a mutual process of action and reaction (Shaffer & Clinton, 2006).

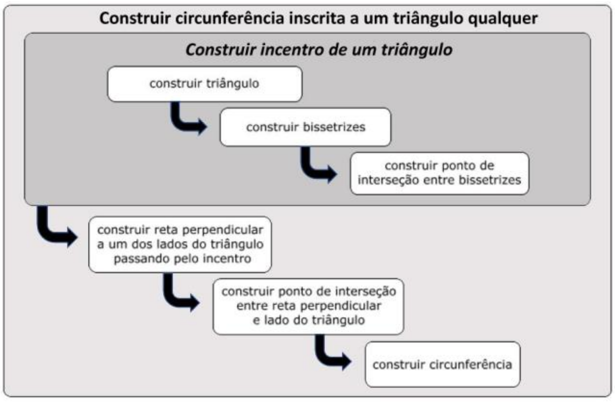
Figure 4: Construction of the circle inscribed in a given triangle



Source: Research Data

Figure 5 presents possible usage schemes and instrumented action schemes used in carrying out the activity, in which the gray regions represent instrumented action schemes and the white regions represent usage schemes.

Figure 5: Usage schemes in the example of Subcategory IIIB



Source: Research Data

We highlight, as an example, that the activity of constructing the incenter of a triangle constitutes a Subcategory IIIA activity, presenting an unguided construction situation with explicit properties.

Activities in this category are presented as instruments-to-think-mathematics-with, as they both mobilize schemes of use and instrumented action and articulate mathematical concepts in the development and verification of hypotheses during resolution.

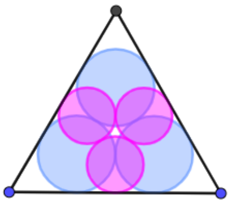
5.4 Category IV Activities

Category IV activities aim to break with the relationship between mathematical properties and prototypical and particular configurations, requiring students to perceive simple configurations within more complex ones.

A common characteristic of activities in this category is the proposal of challenging constructions without a guided script, which implies properties that do not have the same immediate recognition as in activities at the previous level. Figure 6 provides an example of an activity characteristic of Category IV.

Figure 6: Example of Category IV Activity

Construct the following dynamic figure, which must remain stable when moved, preserving the tangencies suggested in the figure.



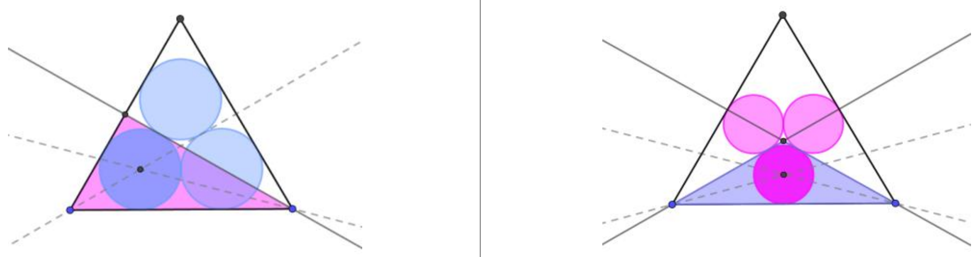
Source: Research Data

The solution to the problem, which suggests the construction of six circles, tangent to each other three by three, and tangent to the sides of an equilateral triangle, involves identifying other triangles, not evident in the figure presented, and recognizing the determination of the incenters of these triangles, which correspond to the centers of the circles inscribed in them.

To construct the three blue circles, the student must visualize three triangles inside the given equilateral triangle (one of these triangles is highlighted in Figure 7 (a)). Once the triangles have been identified, it is necessary to implement elementary and global usage schemes that respond to previously identified sub-objectives, in order to construct the circles inscribed within each of them.

Similarly, to construct the three pink circles, it is necessary to identify three other triangles, one of which is highlighted in Figure 7(b). Again, elementary and global usage schemes, made up of more elementary usage schemes, must be coordinated with mathematical concepts to construct the three circles.

Figure 7: Triangles implicit in the construction



(a)	(b)

Source: Research Data

For many students, the difficulty in engaging with activities at this level lies in the fact that the fundamental mathematical configurations and properties required to solve the problem are often hidden in new configurations, making it challenging to recognize and/or identify them. It is necessary to articulate the analysis of these properties in relation to other mathematical concepts already known to the student. As stated above, these activities require the identification and application of properties presented in unusual situations, which necessitates students operating at higher levels of cognition and understanding of mathematical concepts. GeoGebra, in this case, enables thinking experiments (Gravina & Basso, 2012), and the subject acts, thinks, and expresses themselves through it, evidencing the cycle of mathematical thinking: explore → conjecture → validate → explain → argue.

From the study presented, we identified that the continuous advancement in the process of appropriation of the instrument, as illustrated in these categories of activities proposed by the teacher, occurs in two dimensions: in the complexity and use of mathematical concepts, and in the constitution of schemes for using the artifact. As these two dimensions intertwine and become a coordinated whole, we increasingly evolve towards understanding the instrument as an instrument-to-think-mathematics-with.

6 Final Considerations

The instrumental genesis process of the mathematics teacher is a fundamental factor for the implementation of the integration of DTs in mathematics classrooms. The appropriation of DTs by the mathematics teacher is a non-trivial, time-consuming (Drijvers et al., 2010, p. 108) and complex process (Fugçestad, Kynigos, & Monaghan, 2010, p. 296). The understanding that the didactic instrument developed by the teacher can constitute a powerful instrument-to-think-mathematics-with is the central pillar of this article. Based on the study presented in this article, we propose categories of activities that can establish a conceptual basis for understanding the different ways in which DTs are being proposed, in the sense of approaching or distancing themselves from thinking in mathematics.

We demonstrate that understanding that mathematical concepts and usage schemes can be developed in a coordinated and joint manner during activities with DTs is fundamental to the success of such proposals. If new possibilities for learning-mathematics-with are made possible by DTs, it is increasingly relevant and current that mathematics classes take into account the universe that opens up with instruments-to think-mathematics-with.

Thus, activities with characteristics from categories II, III, and IV must be prioritized, in which the student is led to think about mathematics with DTs (think-mathematics-with).

Moreover, more than that, teacher education for using DTs in the classroom must include reflections on the transformation of DTs into instruments-for-thinking-mathematics-with, and the planning of activities that promote thinking in mathematics through the cycle explore → conjecture → validate → explain → argue.

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