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Algebraic thinking in the early years of elementary school: reflections necessary for teacher education from the perspective of the theory of objectification

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Abstract: This article aims to discuss the characterization of algebraic thinking from the perspective of the theory of objectification in order to raise reflections necessary for teacher education who teach K1 through K5 (the first years of elementary school). The theoretical discussion is based on the concepts of knowing, knowledge, learning, mathematical thinking, among others. In turn, some reflections are presented based on a section of the empirical data of a master's dissertation. In the context of the aforementioned qualitative formative research, a multimodal approach was adopted for the analysis of the videotaped data. Among the results, the emergence of the elements that characterize algebraic thinking (indeterminacy, denotation and analyticity) in the collective engagement between the teachers and the researcher-educator is highlighted through the discursive and non-discursive elements. In short, it is argued that the articulations of different semiotic means with the use of cultural artifacts, digital or not, can move and resignify algebraic thinking, both in teacher education and in basic education.

Keywords: School Algebra. Multimodal Approach. Cultural Artifacts. Teacher Education. Mathematics Education.

Pensamiento algebraico en los primeros años de la educación primaria: reflexiones necesarias para la formación docente desde la perspectiva de la Teoría de la Objetivación

Resumen: Este artículo tiene como objetivo discutir la caracterización del pensamiento algebraico desde la perspectiva de la Teoría de la Objetivación con el fin de plantear reflexiones necesarias para la formación de docentes en los años iniciales de Educación Primaria. La discusión teórica se basa en los conceptos de conocimiento, aprendizaje, pensamiento matemático, entre otros. A su vez, se presentan algunas reflexiones basadas en una sección de datos empíricos de una disertación de maestría. En el contexto de la investigación-formación cualitativa mencionada, se adoptó un enfoque multimodal para el análisis de los datos videograbados. Entre los resultados, se destaca la emergencia de los elementos que caracterizan el pensamiento algebraico (indeterminación, denotación y analiticidad) en la interacción colectiva entre los docentes y el investigador-formador a través de elementos discursivos y no discursivos. En resumen, se argumenta que las articulaciones de diferentes medios semióticos con el uso de artefactos culturales, digitales o no, pueden mover y resignificar el pensamiento algebraico, tanto en la formación docente como en la Educación Básica.

Palabras chave: Álgebra Escolar. Enfoque Multimodal. Artefactos Culturales. Formación

Docente. Educación Matemática.

Pensamento algébrico nos anos iniciais do Ensino Fundamental: reflexões necessárias à formação docente na perspectiva da Teoria da Objetivação

Resumo: Neste artigo, objetiva-se discutir sobre a caracterização de pensamento algébrico na perspectiva da Teoria da Objetivação a fim de suscitar reflexões necessárias à formação de professores dos anos iniciais do Ensino Fundamental. Para a discussão teórica, alicerça-se nos conceitos de saber, conhecimento, aprendizagem, pensamento matemático, entre outros. Por sua vez, são apresentadas algumas reflexões a partir de um recorte dos dados empíricos de uma dissertação de mestrado. No contexto da referida pesquisa-formação de natureza qualitativa, assumiu-se uma abordagem multimodal para a análise dos dados videogravados. Dentre os resultados, sublinha-se, por meio dos elementos discursivos e não discursivos, a emergência dos elementos caracterizadores do pensamento algébrico (indeterminação, denotação e analiticidade) no engajamento coletivo entre as professoras e o pesquisador-formador. Em suma, defende-se que as articulações de diferentes meios semióticos com o uso de artefatos culturais, digitais ou não, podem movimentar e ressignificar o pensamento algébrico, tanto na formação docente, como na Educação Básica.

Palavras-chave: Álgebra Escolar. Abordagem Multimodal. Artefatos Culturais. Formação de Professores. Educação Matemática.

1 Introduction

In mathematics classes, specifically when the topic is algebra, many teachers must have heard this question: who put letters in mathematics? Such concern can trigger other questions, namely: what does it mean, why, when, and how are letters introduced in the teaching of mathematics?

The history of mathematics, particularly algebra, reveals that undetermined quantities (unknowns, variables, parameters, etc.) have not always been denoted by letters. According to Fiorentini, Miorim, and Miguel (1993), the algebraic language went through three major stages: rhetorical, syncopated, and symbolic.

At the stage of *rhetorical language*, algebraic ideas were expressed through everyday language, such as cuneiform writing recorded on clay tablets and oral recitations by the Babylonians (Radford, 2011b). This phase refers to ancient civilizations, such as the Egyptians (circa 2000 BC) and the Babylonians (circa 1700 BC).

At the stage of *syncopated language*, algebraic ideas ceased to be expressed only through words and began to be represented also through concise and abbreviated expressions referring to unknown quantities (Radford, 2021a). This phase refers to Diophantus, who introduced the term *arithmos* to indicate an unknown in his equations.

At the stage of *symbolic language*, algebraic ideas began to be expressed through a symbology that represented undetermined quantities. This phase was marked by the introduction of letters to denote unknowns by the French mathematician François Viète (1540-1603).

By describing these stages,

> We thus return to the understanding that both the field of algebra and that of other mathematical knowledge were historically developed by individuals from different civilizations, at different times, to meet the needs posed by practical experience and

> their own development as a science, with their symbols being the possibility of representation and concretization for the communication of their concepts, as well as their processes of generalization and abstraction (Sousa, Panossian & Cedro, 2014, p. 31).

On the other hand, it is worth mentioning Radford's (2011a) criticism that, in the sociocultural context of algebra development, algebraic languages emerged from historical needs rather than as a linear progression of "mathematical evolution" leading to the "pure abstraction" of mathematical objects. As an example of this statement, the author highlights that

> syncopated algebra was not an intermediate stage of maturation in which knowledge took a kind of rest in its march towards symbolism. Rather, it was a mere technical strategy imposed by the limitations of writing and the lack of a press in the epoch of diligent scribes who had to copy manuscripts by hand (Radford, 2011a, p. 77).[1]

Such reflections place us before the need to consider the historical-cultural movement of concepts in the organization of mathematics teaching practices, in particular school algebra – the one worked in basic education– and not merely focus on the manipulation of alphanumeric symbolism devoid of meanings (Moretti & Radford, 2023, 2015; Panossian, Sousa & Moura, 2017; Almeida, 2017; Sousa, Panossian, & Cedro, 2014).

According to Almeida (2017), in contemporary times, there are two broad approaches to school algebra, which understand it as (a) a *specific language* used to represent essentially unknown values and (b) a *peculiar way of thinking* about mathematical situations. In this context, we believe that understanding school algebra restricted to alphanumeric language implies disregarding other languages and, consequently, the logical movement of algebraic entities in other historical and cultural contexts. In other words, we need to focus on the type of reasoning that emerges in problem-solving in order to identify which strategies and justifications are presented when working with unknown quantities.

Furthermore, research by Radford (2022), Carraher, Schliemann, and Schwartz (2017), Kieran, Pang, Schifter, and Ng (2016), and Blanton (2010) indicates that the international movement in Early Algebra has identified new needs related to mathematics teaching and learning. "Early algebra builds on the background contexts of the problems, only gradually introduces formal notation, and tightly weaves together existing topics from early mathematics" (Carraher, Schliemann, & Schwartz, 2017, p. 262).

In the Brazilian context, some concerns have surrounded teachers who teach mathematics, such as: What is "algebra"? What is "thinking algebraically"? What characterizes "algebraic thinking"? These questions arise in light of the current curriculum demands proposed by the National Common Curriculum Base (BNCC) (Brasil, 2018), which advocates for the introduction of algebra from the initial years of elementary school, with an emphasis on developing algebraic thinking. We emphasize that although this type of mathematical thinking has emerged as a curriculum necessity, there is no precise definition in the normative document above, nor does it define practical ways to work with initial algebra. Given this gap, we ask ourselves: How can the perspective of algebraic thinking, as advocated in the theory of objectification, raise necessary reflections to educate teachers who will teach K1 through K5?

[1] Considered a matter of coherence in the translation into Brazilian Portuguese, the sentence was modified without changing the meaning.

The theory of objectification (TO) is a teaching and learning theory proposed by Professor Luis Radford, with philosophical inspirations in dialectical materialism, Freirean pedagogy, and the historical-cultural school of Vygotsky and his collaborators. In the field of mathematics education, Radford (2021a) advocates for incorporating political, social, historical, cultural, critical, reflective, and ethical perspectives into teaching-learning processes. He chooses the algebraic field as a target of research in many of his studies, aiming to integrate the concepts of TO concepts in the basic education classroom. In this scenario, we emphasize that, according to the author, teaching and learning occur dialectically based on the relationships between teachers and students; that is, even though they perform different functions, they work side by side, assuming co-responsibility. Thus, by recognizing the dialectical relationship in teaching-learning, we guide the possibilities of teacher learning without losing focus on the implications for student learning.

In the middle of such discussions emerges the proposition of this article, as an excerpt from the author's master's dissertation under the co-author's supervision, developed within the scope of the Programa de Pós-Graduação em Educação Matemática e Tecnológica [Postgraduate Program in Mathematical and Technological Education] of the Universidade Federal de Pernambuco [Federal University of Pernambuco], which aimed to "characterize the algebraic thinking that emerges from the collective engagement of teachers who teach K1 through K5 in formative activities involving the introductory study of equations with digital interactive simulations" (Almeida, 2024, p. 31).

In this particular text, we aim to discuss the characterization of algebraic thinking from the perspective of the theory of objectification in order to raise reflections necessary for the education of teachers who teach K1 through K5 (the first years of elementary school). To this end, we structured our argument by inviting readers to encounter some conceptual elements of TO; subsequently, we discussed about the methodological aspects to focus on the results and discussion eventually. Finally, we revisit and refine some reflections about initial algebra that are necessary for teacher education.

2 Algebraic thinking from the perspective of the theory of objectification

One of the founding ideas of TO is the notion of *collective learning* in which subjects engage procedurally with each other through the development of an activity[2] for non-alienating production (Radford, 2021a). In this context, there is a distinction between knowing and knowledge. While *knowing* is defined as a general entity available historically and culturally, *knowledge* refers to the particular encounter with knowing. Dialectically, knowing is constituted as knowledge in a continuous and unfinished process called *objectification*. For example, in the excerpts from the formative activity presented below, the teachers collectively come across specific ways of solving problems involving the introduction to equations. Therefore, the strategies mobilized, the arguments presented, and the understanding of the organization of algebra teaching are configured as particular manifestations of more general algebraic knowledge: first-degree polynomial equations.

Another important point is the understanding of *thinking* proposed in the TO. According to Radford (2011c), *thinking* has a multimodal nature, i.e., it encompasses both material and ideational components. As examples of material components, we have gestures, speech, writing, rhythms, and signs, among others. As for the ideational aspects, we have the subject's

2 The concept of activity advocated in TO has a strong social meaning, linked to the union of subjects not only to perform a certain task, but mainly to seek to understand what, how, and why they are making certain choices and taking certain actions. In this sense, a joint activity takes place between teachers and students in the classroom: the teaching-learning activity.

inner speech and imagination. In this way, the connections between these components express the unity of human thinking and its emergence and evolution.

Based on Vygotsky's historical-cultural current and Marx's materialist-dialectical philosophy, Radford (2021a, p. 147) points out that: "Sensory cognition emphasizes the idea that our thoughts, feelings, actions, and all our relationships with the world (hearing, perceiving, smelling, feeling, etc.) are *historical intertwinings* of our body and material and ideational culture." Therefore, to analyze the movement of thinking, both in teacher education and in basic education, we must consider the conception of *sensory cognition* proposed by Radford (2021a), in which the mind, body, and the world are interrelated entities.

> Like all thinking systems, mathematical thinking originates from the convergence of various processes in society, which, in their interaction, produce and modify one another. As a result, thinking, particularly mathematical thinking, incorporates and refracts the various processes of society, expressing intrinsic social tensions and contradictions (Radford, 2021a, p. 216).

From this perspective, we emphasize that the cultural nature of mathematical thinking in the theory of objectification advocates that we cannot understand a particular form of thought without understanding other forms of thought. According to Radford (2021a, p. 232), from a materialist-dialectical perspective, this position can be assumed when we investigate "the economic, political, and ideological dimensions of the social processes that such forms of thinking incorporate and refract.".

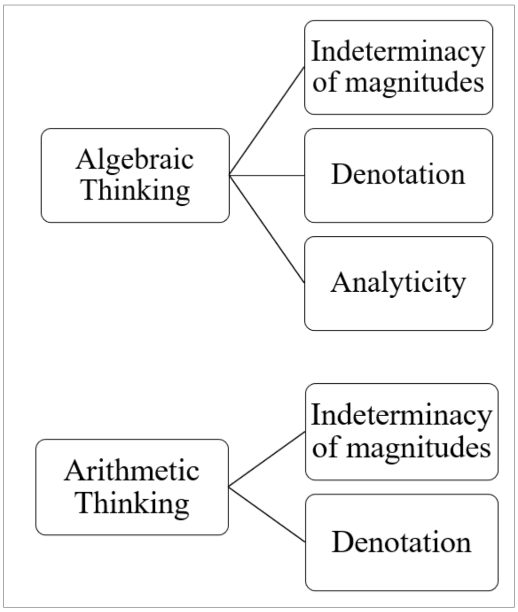
Regarding the material components of mathematical thinking, the author argues that, to make the historical-cultural dimension evident, teachers and students employ various signs and artifacts, among other linguistic devices. In this context, the author defines the *semiotic means of objectification* as

> Objects, tools, linguistic devices, and signs that individuals intentionally use in the processes of social meaning-making to achieve a stable form of consciousness, to make their intentions clear, and to carry out their actions to achieve the object of their activities are called semiotic means of objectification. These are semiotic insofar as they are key pieces in the production of meanings embedded in the processes of objectification (Radford, 2021a, p. 136).

We also highlight that TO has other theoretical elements, but they are not the central focus of the discussion that we seek to undertake in this text. That said, based on the above concepts, we move on to emphasize the debate in the field of algebraic thinking.

Unlike the view that defends a continuity between arithmetic and algebra, Radford (2014) argues that there is a rupture between these two areas. In this scenario, taking into account that teachers who teach mathematics may confuse the teaching of algebra with the teaching of arithmetic or vice versa (Radford, 2008, 2014), we elucidate the differences and specific aspects of algebraic and arithmetic thinking in Figure 1.

Figure 1: Characterizing vectors of arithmetic and algebraic thinking



Source: Almeida (2024).

We reiterate that our theoretical positioning is based on the perspective of algebraic thinking presented in the theory of objectification, which recognizes the existence of a relevant relationship between arithmetic and algebraic thinking, but highlights the *epistemological ruptures* by assuming it is impossible to extract all school algebra from arithmetic. In this sense, Radford (2021b, p. 173) proposes that

> From the perspective of the theory of objectification, the characteristic of algebraic thinking is not only found in the nature of the magnitude (that is, in the nature of the object about which one reasons), but also in the type of reasoning that is done with magnitudes. More precisely, from our perspective, three conditions would characterize algebraic thinking: the first has to do with the objects of reasoning; the second with how objects are symbolized (i.e., this is, therefore, a semiotic problem); and the third, with how one reasons about the objects of reasoning.

According to Radford (2021b), the three elements that characterize algebraic thinking are *indeterminacy of magnitudes*, *denotation*, and *analyticity*. *Analyticity* is configured as the primary element characterizing algebraic thinking, as the other two elements also characterize arithmetic thinking (Radford, 2021b).

Analytical reasoning is constituted by two fundamental characteristics: (a) the establishment of relations between determined and undetermined quantities, operating with unknown quantities as if they were known, and (b) the performance of operations in a deductive manner, deducing propositions. In general, thinking analytically requires considering the undetermined as if it were determined and deducing from the premises.

The *indeterminacy of magnitudes* refers to work involving undetermined or unknown quantities, which are designated by unknowns, variables, and parameters, among others. However, based on Gomes and Noronha (2020), we point out that algebra is not restricted to the use of indeterminacies in the elaboration and resolution of problems since to characterize algebraic thinking, it is necessary to understand the meaning of the indeterminate, provided with meaning and not merely the use of mechanical techniques and procedures for the manipulation of letters and numbers. In other words, work with unknown quantities occurs as a known part of the problems and is not necessarily denoted through alphanumeric symbolism (Radford, 2021b).

Finally, we have *denotation*, which concerns the different ways of naming and symbolizing the undetermined quantities involved in the problem. Some of the ways of denoting are drawings, gestures, speech, writing, alphanumeric symbolism, unusual signs, or even a combination of them.

To better understand the three elements that characterize algebraic thinking, we use the example of a first-degree polynomial equation, "7n + 2 = 6n + 8," discussed by Radford (2021a). In order to find the value of the unknown term "n" (the undetermined), students can work with the equality sign "=" from a relational perspective, that is, with the notion of equivalence between the operations of the left and right members of the equality relation. In this sense, to solve the problem, it would be necessary to subtract "2" and "6n" on both sides of the equation to conclude that "n = 6". Such an example requires logical-deductive reasoning, i.e., assuming the premises as true and operating with the unknown as if it were known to determine the value of the unknown.

On the other hand, if the method used were trial and error, in which students could replace "n" with "1", "2", and so on, until reaching "6", the vector of *analyticity* would not be present. In this case, from the TO perspective, even with the presence of *indeterminacy of magnitudes* and *denotation*, the students would be thinking arithmetically. However, the interesting question here is: How can we invite students and teachers to think analytically about problems involving equations of the Ax + B = Cx + D type?

Filloy and Rojano (1989) suggest that in equation problems of the Ax + B = C type, students generally use arithmetic methods. This is because, adopting the operational perspective of the equal sign "=," students understand what is on the right side as a result of the operations on the left side; therefore, they subtract B from C and divide by A. However, in equations with unknowns on both sides, such as Ax + B = Cx + D, this resolution method is no longer effective. In this case, students can resort to truly algebraic reasoning: operating deductively with the unknown quantity as if it were known. It is in this type of equation that we seek to deepen our studies.

To expand the repertoire regarding school algebra, according to Radford (2021b), one possible path is to understand that *the use of alphanumeric symbolism does not characterize algebraic thinking*. Although some problems explicitly use letters to represent unknown quantities (unknowns, parameters, variables, etc.), the type of reasoning may not be analytical, i.e., when no meaning is attributed to the letters and much less a deduction is made from the hypotheses. For example, to solve the equation 4x + 2 = 2x + 6, it is possible to assign known values (1, 2, ...) to "x" until concluding that the result is 4. In this context, even when working with alphanumeric symbolism, the type of reasoning mobilized would be in the field of arithmetic, i.e., arithmetic thinking would emerge in the process of solving the equation (Radford, 2022a, 2022b, 2021b). In the field of algebra, we focus on presenting results and discussing them.

Another important path, defended by the theorist, is related to the understanding that *algebra is not generalized arithmetic*. For Radford (2021b), as in the study by Filloy and Rojano (1989), there are epistemological ruptures between these fields of mathematics. In turn, although the author considers that there is a relationship between arithmetic and algebraic thoughts, as we can see in the results of Almeida's dissertation (2025), it is impossible to extract all school algebra from arithmetic.

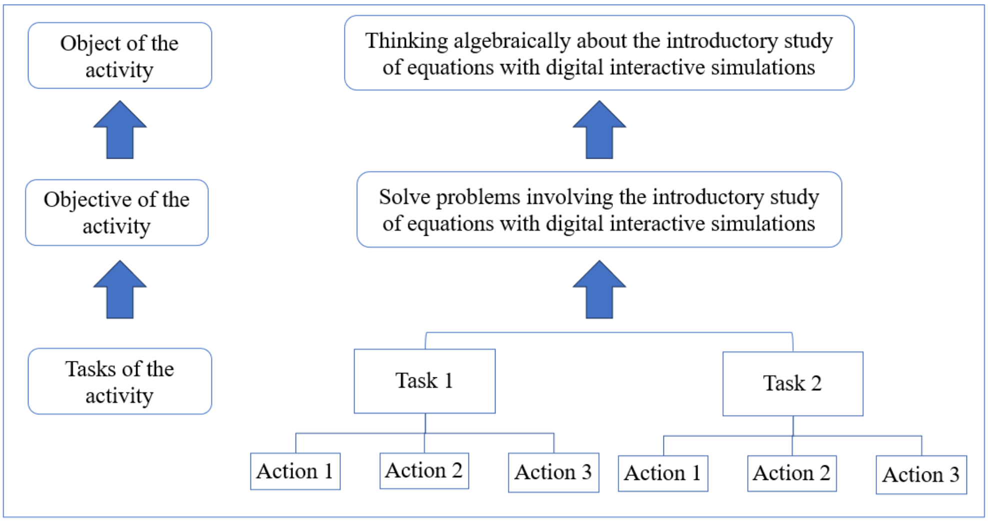
3 Methodological aspects

To discuss the characterization of algebraic thinking from the perspective of the theory of objectification and to raise reflections necessary for the education of teachers who teach the initial years of elementary education, we present an excerpt from Almeida's dissertation (2025).

To produce data for our master's degree research, we conducted a continuing education course with teachers of the initial years at a municipal elementary school in Pernambuco. This in-person course consisted of two four-hour meetings, totaling eight hours of workload. In this article, we specifically emphasize the analysis of an experience with a formative activity involving a group of three women teachers, identified by codenames.

In the theory at stake, the notion of activity assumes a strong social sense of involvement, where subjects engage in resolving a specific *task* with a particular *objective*, aiming to achieve a specific *object*. Thus, Radford (2021a) proposes that every teaching-learning activity has this structure: object-objective-task. In our context, the formative activities (see Figure 2) around the statement problems were structured as follows.

Figure 2: Structure of formative activities involving statement problems



Source: Almeida (2024).

The structures of the formative activities, related to tasks 1 and 2, were organized based on the guidelines for the mathematical and social dimensions proposed by Radford (2021a). In particular, inspired by the statement problems presented in the studies by Radford (2021a, 2021b), we designed tasks related to the process of introducing equations in the teaching and learning of initial algebra. In this article, we will focus on task 2.

As regards *mathematical dimension*, it is worth highlighting that, according to Radford (2021b), the set of problems that can be presented in natural language –as a statement– and translated semiotically by concrete and/or iconic means is quite limited. However, it is sufficient to move the first encounter with algebraic thinking in the context of initial algebra. Linked to this assumption, we also assume when planning tasks that:

> Statement problems are neither cognitively nor culturally neutral. (...) Inevitably, the problems ostensibly show some aspect of the nature of the world as it is mathematized, and provide the basis for illustrating how truth can be established. (Radford, 2021b, p. 183)

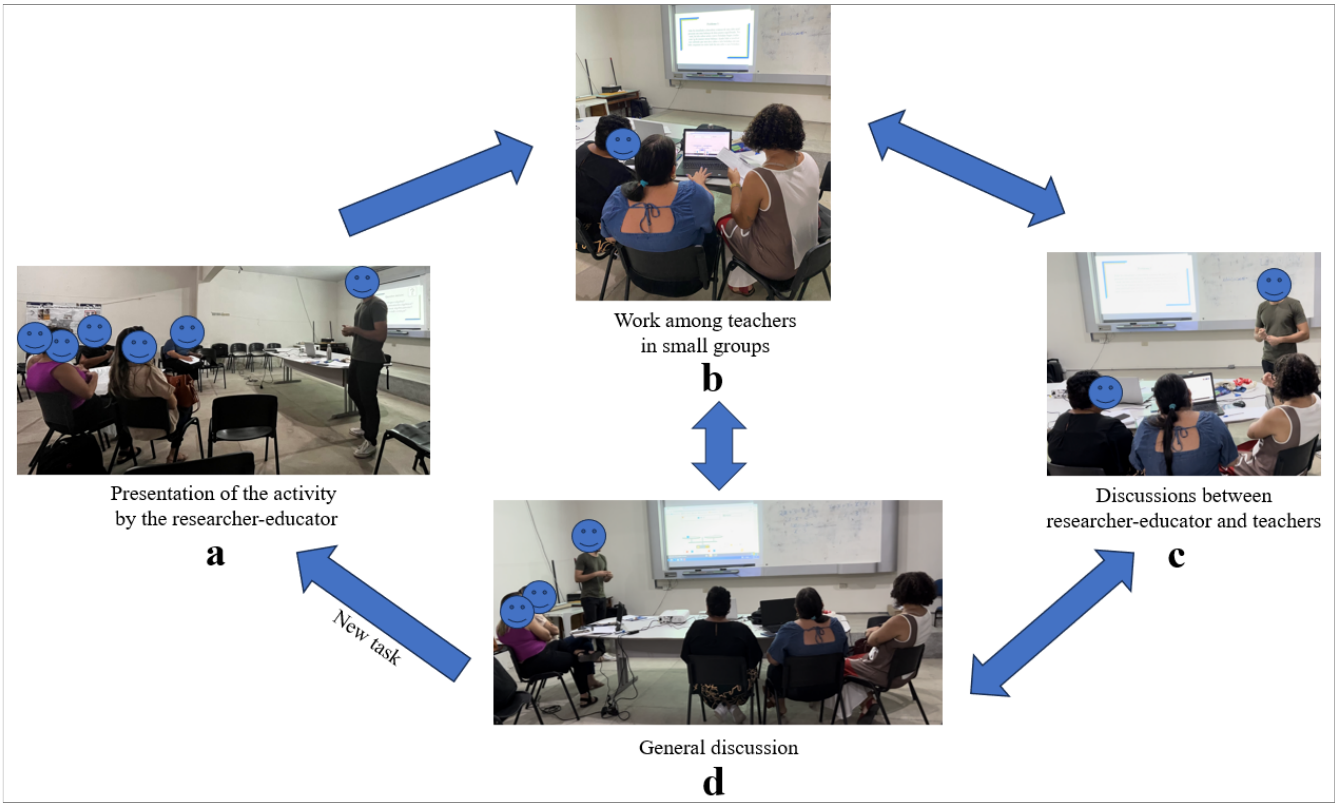
From the point of view of the *social dimension*, the actions related to the tasks (*Action 1* – Problem solving using the simulator, *Action 2* – Writing the problem resolution in natural language, and *Action 3* – Writing the problem resolution in alphanumeric language) were fundamental to fostering collective engagement between the teachers and the researcher-educator. However, considering the research objective, Almeida (2024) emphasized Action 1 in the data analysis, specifically solving statement problems involving equations using the digital simulator Equality Explorer: Basic[3].

In addition to the aforementioned actions, aiming to address the third issue mentioned above regarding the social dimension, "we promote collective forms of knowledge production and modes of human collaboration of a non-alienating nature" (Radford, 2021a, p. 178). This fact occurred throughout the formative process, as the teachers reevaluated their positions in their relationships with one another.

In general, we emphasize that there is a biunivocal relationship between the mathematical and social dimensions in the proposed tasks, meaning that we believe the social organization affects the mathematical organization and vice versa. With this, we reiterate that collective work influences problem-solving, particularly through the diversity of strategies, points of view, and ways of arguing and positioning oneself in the presence of others, among other aspects.

As for the forms of collective engagement, we experience the formative activities through the different stages proposed by Radford (2021a):

Figure 3: Main moments of the formative activities



Source: Almeida (2024) based on Radford (2021a).

When the researcher-educator presented the activity, the tasks were explained so that

3 For more information, see: https://phet.colorado.edu/en/simulations/equality-explorer-basics.

the teachers could understand what was required. Therefore, when working in small groups, the teachers discussed among themselves and with the researcher-educator. Finally, there was a general discussion of the proposals from the small groups. As illustrated in Figure 3, after a moment *a*, there was no hierarchy between moments *b*, *c*, and *d*. In this text, the data were categorized according to the moments *b* and *c*.

Furthermore, the data were registered through (a) recordings of the meetings, in video and audio format, to analyze the gestures and speeches of the small and large groups; (b) recordings of the notebook screens; (c) written productions of the teachers to resolve the tasks; and (d) notes of the researcher-educator in the logbook, with some observations about the formative research.

To record the data, an educator from the municipality and an external person collaborated with the researcher. Each became responsible for recording a group. Furthermore, the last person recorded the discussions of the large group.

The diversity of data records is justified by the multimodal analysis approach (Moretti & Radford, 2023a; Radford, Arzarello, Edwards & Sabena, 2017; Radford, 2015; Arzarello, 2006) of algebraic thinking that comprises different semiotic means to represent it (Radford, 2011c), in addition to our concern in achieving not only the research objectives but also contributing to teachers' education.

To analyze the data, we transcribed the participants' speeches and movements, listing them on numbered lines in the format "N-n," where "N" represents the number of the analysis episode and "n" is the order of the line within that episode. For example, in "2.10", "2" represents the second episode, and "10" is the position of the speech line in that episode.

Based on the above, we characterize the nature of the investigation as qualitative and of the formative research type (Longarezi & Silva, 2008). According to the authors, in formative research, the researcher immerses themselves in the study environment, either as an educator developing a pedagogical practice for the critical education of teachers in updating their knowledge or as a researcher, systematizing, analyzing, and understanding how this formative process occurs with teachers. In this sense, we consider that "research has social practice as its beginning and end" (Longarezi & Silva, 2008, p. 4059). Therefore, dialectically, research participants are formed through research, just as the productions of/in education constitute research.

4 Results and discussion

In this section, we organize the data into two categories of analysis: *4.1 The emergence of algebraic thinking in collective engagement among teachers*, and *4.2 The emergence of algebraic thinking in the collective engagement between teachers and the researcher-educator*. These categories refer to the experience of a formative activity with teachers who teach the initial years of elementary school. In particular, we present excerpts from moments that demonstrate our defense of the claim that collective work can reshape the resolution of algebraic problems, particularly in the introductory study of equations.

The second statement problem addressed involves people (Ana, Maria, and Fernanda) who have spheres of beige and red colors. The beige spheres (determined sizes) weigh 1g each. The grams of the red spheres are unknown (undetermined quantities). Therefore, the problem revolves around the quantities, known and unknown, present on the two-pan scale so that it is balanced (equality), aiming to reduce the equation and find the value of the gram of a red sphere (indetermination), as we can see in Chart 1.

Chart 1: Second statement problem proposed in the research

Problem 2: Ana, Maria, and Fernanda were tasked with determining the mass of the red sphere using a balanced two-pan scale. Knowing that, on one side of the scale, there are three red spheres and one beige sphere and, on the other side of the scale, there is one red sphere and five beige spheres, what is the mass of the red sphere found by them? Consider the beige sphere with a mass of 1 g.

Source: Almeida (2024).

The above-mentioned problem can be expressed in alphanumeric language as follows: $3x + 1 = x + 5$. By mobilizing analytical reasoning (Radford 2022a, 2022b, 2021b), one can subtract "x" and "1" and then divide the terms and coefficients in both sides of the equality by "2", concluding, through a logical-deductive process, that $x = 2$. Indeed, as the teachers spent energy solving the previous problem (Task 1)[4], overcoming arithmetic strategies through collective work with the researcher-educator and revealing signs of algebraic thinking, at the end of the discussion, the process of solving the problem became more synthetic.

Regarding the resolution of Task 1, Almeida (2024) points out that the fact that Rosália suggested an equation ($3x = 9$) equivalent to the initial equation ($2x + 1 = x + 3$) allowed the teachers to work around a mathematical operation: "separating" the blue cubes (undetermined quantities) and the beige spheres (determined quantities) into proportional parts. This made it possible to conclude that, for each blue cube, there were three beige spheres and, therefore, $x = 3$. This operation is also known, in other equation-solving contexts, as division (Radford, 2022b). Although we planned the strategy of separating the independent term by the coefficient of the unknown to be introduced in solving the second problem, we believed that the prior encounter with this algebraic knowledge was indispensable for the progress of the formative activity analyzed in the following subsections.

4.1 The emergence of algebraic thinking in collective engagement among teachers

In this category of analysis, we focus on the moment *b* of the formative activity, specifically on the work between teachers in small groups (see Figure 3 before). Below, we follow the small group discussion on translating the problem into the digital simulator:

2.112 Rosália: Off we...

2.113 Andréia: Off we go!

2.114 Rosália: I hope this [problem 2] is easier!

2.115 Andréia: [started reading the statement]. Ana, Maria, and Fernanda were challenged to discover the mass of the red sphere present on a balanced two-pan scale. Knowing that there are three red spheres on a plate... [she began to simulate the problem]. One, two, three [she resumed reading]. And a beige sphere.

2.116 Andréia and Rosália: Okay!

2.118 Andréia: And on the other side of the scale, there is a red sphere...

2.119 Rosália: Right!

2.120 Andréia: And five beige spheres [she starts to simulate the problem]. One, two, three, four, five. Okay, right?! Let's do that thing [she used the stacking function] that they [the teachers from the other group] put in, right?

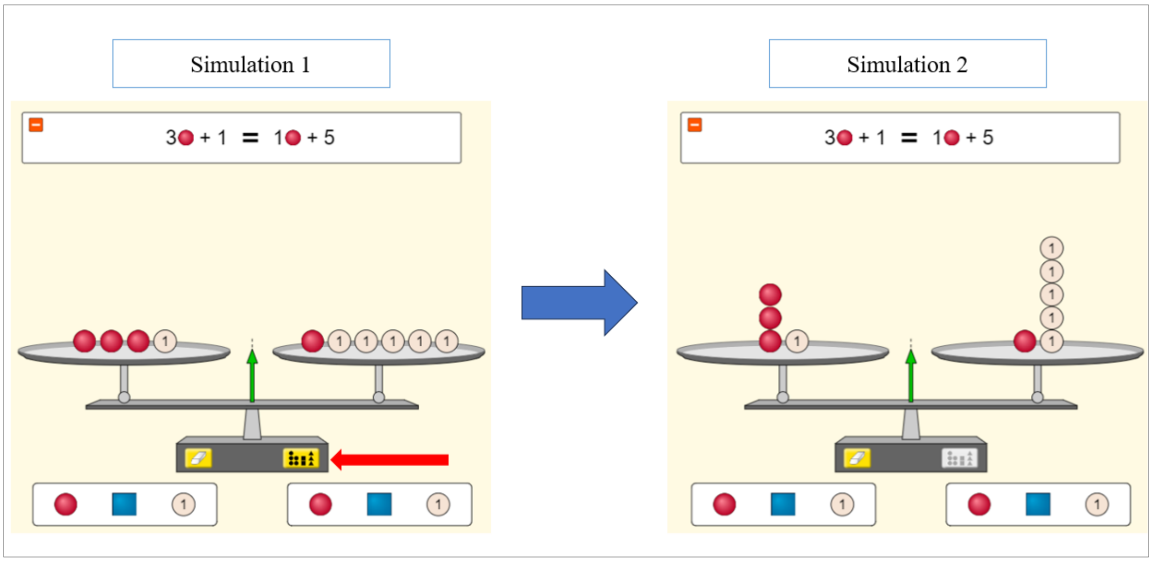
4 For a deeper understanding of this discussion, we recommend reading the work by Almeida (2024).

2.121 Rosália: That's it!

According to Radford (2021a), when performing a task, individuals do not always perceive the object of knowledge of the activity; that is, there is no linearity in the movement of consciousness. But, as we can see in the dialogue above (2.112 to 2.121), the teachers had no difficulty in translating the second problem in natural language to the digital simulator. In other words, they went straight to the object of the task: thinking algebraically about the introductory study of equations with digital interactive simulations.

In this process of solving the equation, the teachers used a function of the digital technological artifact to stack the red and beige spheres separately. In other words, they organized the determined and undetermined quantities to aid in visualization. In Figure 4, we illustrate such movement:

Figure 4: Stacking of determined and undetermined quantities



Source: Almeida (2024).

We believe that the teachers' decision to stack the determined and undetermined quantities contributed to the reorganization of thinking, making the denotation of the equation more synthetic. Here, they begin to visualize the equation in the Iconic Semiotic System (SSI)[5], not only as "x + x + x + 1 = x + 1 + 1 + 1 + 1 + 1" but also as "3x + 1 = x + 5." According to Radford (2022b, p. 10), "what previously required many words and actions is reorganized and contracted." Thus, from this action, we realized that the teachers began to make an effort to separate the necessary from the unnecessary, as well as to refine their semiotic activities; which later becomes a *semiotic contraction* (Radford, 2022b, 2021a), as we can see in the following discussion:

2.122 Andréia: So what is the mass of the red sphere they found? Consider the beige sphere with...

2.123 Rosália: A gram of mass!

2.124 Andréia: It's already changed here, right?! Because look...

2.125 Rosália: I always like to look here [pointed to the equality relationship in the simulation].

[5] In the introductory study of equations, Radford (2021b) proposes different semiotic systems –concrete, iconic, and alphanumeric. In this text, we emphasize the iconic semiotic system through the iconic images in the simulator, although we sometimes resort to alphanumeric language –involving letters, numbers and symbols– to summarize our analyses.

You always have to balance both sides... Because if here [right side] it's five plus one, six, three plus one, four... [she said to herself.]

2.126 Andréia: She [the question] wants to know the value of the red sphere.

2.127 Rosália: Right!

2.128 Andréia: There you go... It's balanced, isn't it?

2.129 Rosália: Yes, it is...

2.130 Sirlene: So let's take one out from one side and then from the other.

2.131 Andréia: And can't we do it as a proportion?

2.132 Rosália: I agree...

2.133 Andréia: So, take a look... Here...

2.134 Rosália: So, let's take one out!

2.135 Andréia: I'll take it out, then! If I take one from here... Shall we take out the red spheres?

2.136 Rosália: No! Shall we take out the [beige] balls right away?

2.137 Sirlene: One here and one there, right?

2.138 Andréia: If I take one from here [right] and one from here [left], then there will be three spheres equal to...

2.139 Rosália and Andréia: Right!

2.140 Andréia: So if I take out a little ball [red] from here [left] and one from here [right]

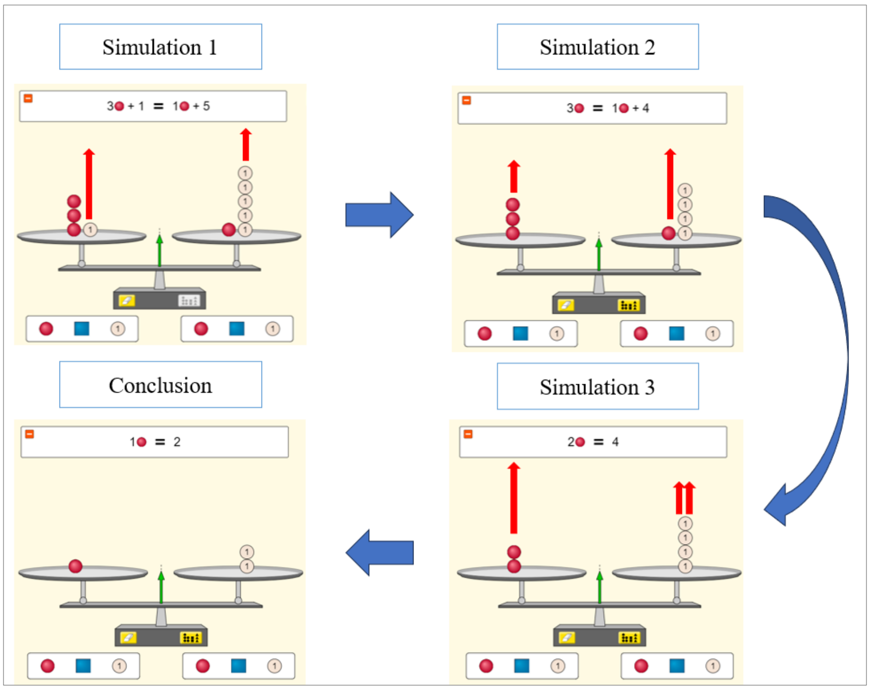
2.141 Rosália: Each [red sphere] is worth two grams!

[Andréia removed two beige spheres from the right side and one red sphere from the left side.]

From Rosália's speech (line 2.125), we notice that, while viewing the simulation, she initially directed her gaze to the relationship of equality. On the other hand, Andréia focused on the balance of the two-pan scale (line 2.128). In this scenario, Sirlene suggested operating on both sides (line 2.130), and Andréia asked if she could start by subtracting the unknown from both sides of the equation (line 2.135). This collective engagement "is an encounter that offers the possibility of coming into contact with other voices and perspectives, not for personal benefit, but for the creation of a common work (an idea)" (Radford, 2021b, p 192).

In the excerpt from the dialogue containing lines 2.122 to 2.141, we observe signs of algebraic thinking in solving the equation. The *indeterminacy was denoted through external discourse and the simulator, respectively, in everyday language, as the* "mass of the red sphere" (line 2.122) and iconic elements (Figure 4). In turn, when they assumed the premise of "taking out from one side and the other" in equal amounts (line 2.130), the teachers eliminated one beige sphere (line 2.138) and one red sphere (line 2.140) from each pan of the scale, deductively obtaining that two red spheres were equivalent to four beige spheres. Therefore, they concluded that one red sphere was equivalent to two beige ones (line 2.141). When we observe that the small group *operated with the undetermined,* considering the equivalence on both sides, we infer the presence of *analyticity* and, consequently, the emergence of *algebraic thinking*. We illustrate this movement in Figure 5.

Figure 5: Algebraic resolution of the second problem statement in SSI



Source: Almeida (2024).

In the resolution illustrated in Figure 5, one point that intrigued us was the change from equation “2x = 4” (simulation 3) to “x = 2” (conclusion). Unlike the simplification of equation “3x + 1 = x + 5” (simulation 2) to “2x = 4” (simulation 3), which made clear the type of reasoning emerging in the aforementioned external discourses (lines 2.138 and 2.140), it was not clear how the small group thought from simulation 3 to reach a conclusion. Although Andréia simulated the final answer presented by Rosália, there was a predominance of cultivation of internal discourse, so we were unable to understand the final strategy used. According to Radford (2021a), we do not access the ideational aspects of thinking without the support of material aspects, that is, the semiotic means of objectification. Faced with this impasse, we refined our inference by analyzing the data produced in the second category proposed here.

4.2 The emergence of algebraic thinking in the collective engagement between teachers and the researcher-educator

In this category of analysis, we focus on moment *c* of the formative activity, i.e., on the discussions between the researcher-educator and the teachers (see Figure 3, previously exposed). Below, we follow their dialogue.

2.142 Researcher-educator: How did you reason? Now I want to know...

2.143 Rosália: Yeah... Andréia took the small spheres out...

2.144 Sirlene: Andréia took them out...

[The researcher resumed a comment about problem solving.]

2.145 Researcher-educator: One thing I didn’t mention [in the general discussion of problem 1] is the issue of problem interpretation. We need to read the statement, understand the structure... Often the student memorizes the way it is written and then looks for the data. However, we also need to work on this issue of interpretation, in addition to mathematical calculations.

2.146 Sirlene: Yeah!

2.147 Rosália and Andréia: That's it!

In the excerpt above, the researcher-educator asked the teachers what they had thought. However, before actually starting the discussion on the resolution of problem 2, he made a statement about the interpretation process, particularly regarding the structure of the writing, which must be taken into consideration in teaching and learning. In this scenario, we see what research in the field of teacher education based on TO has shown: formative activities extend beyond algebraic knowledge, also contributing to the emergence of knowledge related to the organization of teaching (Romeiro, Moretti, & Radford, 2024; Moretti & Radford, 2023).

Continuing the dialogue, Andréia began explaining the resolution.

2.148 Andréia: We know that on one side, there are three red spheres and one beige sphere. We were putting it, right, as it is there... [pointed to the simulation already proposed by them and resumed the question of the problem.] What is the mass of the red sphere they found?

2.149 Researcher-educator: So, what would we do in this case?

2.150 Rosália: We must remove the grams [beige spheres] right away. That's what Andréia did. Take out equal grams [he made the gesture of removing from one side and the other.]

2.151 Researcher-educator: Right! I wanted to listen to you a little... [pointing to Sirlene.]

2.152 Sirlene: Yes, precisely that. We took them out. Then we took out a [red] sphere [waited for Andréia to handle the simulator.] We need to take it out on the other [right] side. There, we identify that each [red] sphere is worth two [beige] ones. So, we withdrew another sphere, two more beige ones, and we came to the conclusion that one sphere is worth two [grams].

2.153 Rosália: Perfect!

From the excerpt above, we observe that, although Rosália began presenting the argument (line 2.150), the discussion took a different direction. Concerned about the positions the teachers were assuming up to that point – essentially the same functions as in the previous activity – the researcher-educator invited Sirlene to speak (line 1.152). This inviting stance shows that the educational project of that formative process was not based solely on the axis of knowledge production but also on the axis of subjectivity production. According to Radford (2021a), for a collective to be constituted as non-alienating, it requires the active participation of everyone, guided by ethical principles that seek to open up possibilities for others to express themselves, as well as to welcome their point of view.

In turn, Sirlene continued describing the resolution (line 2.152) but did not explain why she removed a beige sphere from one side and two beige spheres from the other. Until that moment, we had no discursive or gestural elements that would allow us to infer the justification for them eliminating equivalent but distinct quantities on both sides of the scale, that is, dividing both sides by two.

Following the discussion, Andréia made the following statement.

2.154 Andréia: Now, this relationship, I was also thinking... There are two [red] spheres there, okay?! But then, from the moment I look at the other side, I can make this connection: there I have four [beige spheres], and here I have two [red spheres]. I multiply... To myself. [...] And the result is two!

2.155 Sirlene: I used division... I divided it, because... look [pointed to the notebook screen]... Two is divided by two and four is divided by two.

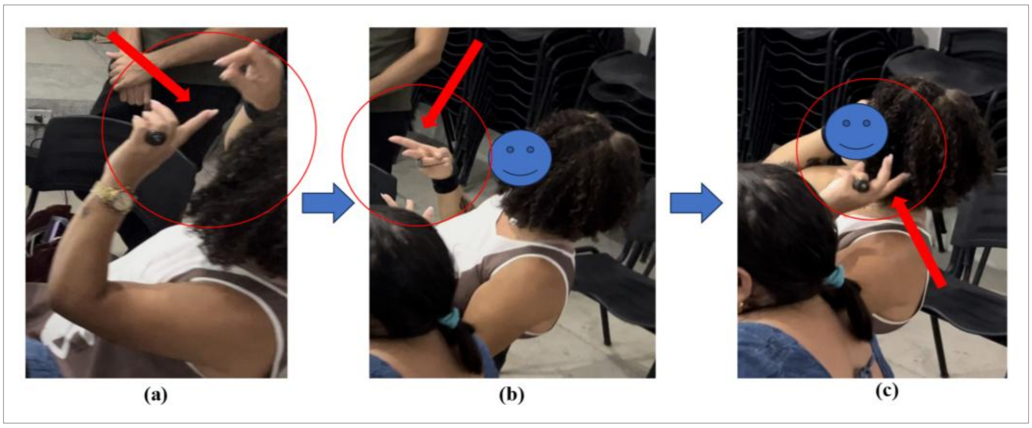
2.156 Researcher-educator: That's right, exactly!

2.157 Sirlene: That's what I did.

As we can see, in this part of the dialogue (lines 2.153 to 2.157), without needing to simulate the equation again, the teachers detached themselves from the digital technological artifact and presented two arguments to support their reasoning in concluding the problem.

The first argument was related to the multiplication operation. In this context, Andréia resorted to the idea of proportionality between both sides of the equation (line 2.156). Considering that "2x = 4" (Figure 6 (a)) and "2 . 2 = 4" (Figure 6 (b)), she concluded that "x = 2" (Figure 6 (c)). It was not evident that the multiplication performed mentally by Andréia was, in fact, "2 . 2 = 4", but by external speech and gestures (*denotation* of the *indeterminacy* in Figure 6), we infer that. In this sense, in mathematical terms, we would have a line of reasoning based on the transitive property of the equality relationship: by assuming two premises –the equation in the scene and mental multiplication– Andréia deductively inferred that the mass of the red sphere was two grams. Therefore, we found evidence of *analytical reasoning* and, consequently, *algebraic thinking* from the TO perspective (Radford, 2022a; 2022b; 2022c).

Figure 6: Andréia's gestures in solving problem 2



Source: Almeida (2024).

Therefore, the second argument referred to the operation of division. In this scenario, Sirlene, starting from equation "2x = 4", divided both sides by two and concluded that "x = 2" (line 2.166). Therefore, in resolving the second problem, through the speeches set out in lines 2.130, 2.138, 2.140, and 2.155 (*denotation of known and unknown quantities*), we explicitly found that, by assuming as premises that they could perform the mathematical operations of "eliminating" (subtracting) and "separating" (dividing) on both sides of the balanced two-pan scale (equation), the small group: (i) operated *deductively* and (ii) work with the *indeterminacy* ("the mass of the red sphere," line 2.122) as if it were determined. At these points, we observe the presence of *analyticity*.

Based on Radford's studies (2022a, 2022b, 2021a, 2021b), we conclude that the way of thinking and proceeding corroborates the three conditions of the elements that characterize algebraic thought, given that: a) the unknown mass of the red sphere (the unknown) was identified as one *indetermination*; b) the unknown or the unknown term was *denoted* through external speeches and gestures; and c) the result (the mass of the red sphere is 2g) was *deduced* from equations that are equivalent to each other.

At the end of this moment, we had the following statements:

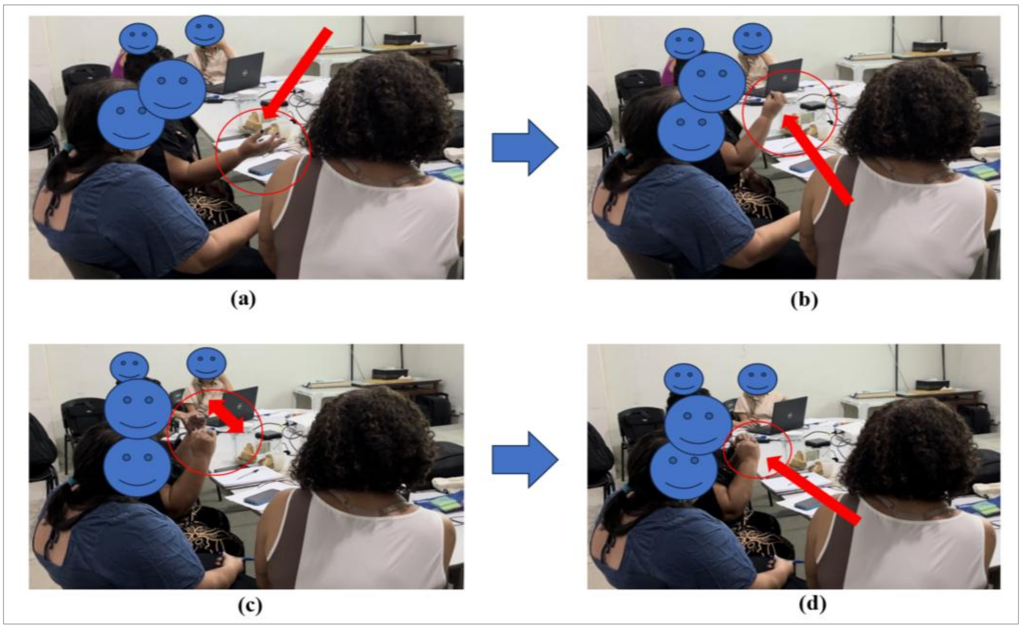
> **2.158 Researcher-educator:** Please, note that this problem is similar to the other, but some aspects change. And mainly that you mobilized other strategies.
>
> **2.159 Sirlene:** I also realize that we will be better teachers. That's what I've noticed... Because it makes us think about possibilities. I used multiplication here and she [Andréia] used division. And we know that taking from one side means taking from the other [she made the withdrawal gestures, moving her hand from top to bottom] so that there can be equivalence, equality [she made the gesture by moving both hands horizontally, first closed, and then opening them].
>
> **2.160 Rosália:** For sure!

In this dialogue, the researcher-educator made a general comment about problem solving. Then, teacher Sirlene reiterated the strategies mobilized and corroborated our observation that they thought *analytically* when stating: "We know that taking away from one side we have to take away from the other in order to have equivalence, equality" (line 2.159).

Furthermore, Sirlene highlighted the importance of the formative process for teachers when she stated in line 2.159: "We will be better teachers," mainly because the formative process contributed to "thinking about possibilities" regarding the introduction to school algebra. This statement illustrates that learning, from the perspective of TO, occurs through a process that encompasses both the encounter with new knowledge and the process of becoming, in this case, teachers who teach algebra in the early years based on alternative possibilities.

Linked to the discursive elements in line 2.159, we observe that Sirlene used the body to denote the subtraction operation (Figure 7, (a) and (b)) and the equivalence in the equation (Figure 7, (c) and (d)).

Figure 7: Sirlene's gestures in solving problem 2



Source: Almeida (2024).

Figure 7 illustrates precisely the fact that we can observe the materialization of thought through bodily movement (Radford, 2021a). In this case, the hand gestures denoting the removal of an object and the equality between both sides reinforce that algebraic thinking is a tangible social practice in which the body materializes the ideational component (Radford, 2011c).

As a summary of the analyses of the small group discussions, we consider that, initially, the operation "eliminate" (subtraction) on both sides of the equation was clear, while the operation "separate" (divide) was not. In this way, to characterize algebraic thinking, it was essential to focus on the collective engagement in which teachers and researcher-educators "are involved in comparisons, distinctions, and taking positions regarding knowledge, which generates new ideas along the way, while everyone constitutes themselves as subjectivities" (Radford, 2022b, p. 192).

Finally, we consider that, despite not being the focus of this text, the theory of objectification reveals an intertwining between the production of knowledge and subjectivities in the area of mathematics education. With this, we express here our desire to delve deeper into these aspects in future investigations.

5 Final considerations

This article presents excerpts from a master's research project that illustrate the movement of raising reflections necessary for the education of teachers who teach the initial years of elementary school, based on the objective of discussing the characterization of algebraic thinking from the perspective of the theory of objectification

In the context of the formative activity under analysis, referring to the problem of a statement that can be translated into the equation $3x + 1 = x + 5$, we found evidence that the teachers found a way of thinking about the equations algebraically, mobilizing the strategy of neutralizing terms or coefficients, i.e., operating with determined quantities (beige spheres) and undetermined quantities (red spheres) in both members of the equality. In short, we summarized their discussions in alphanumeric language, in which they obtained the following equations that were equivalent to each other: "$3x + 1 = x + 5$" (subtracted one from both sides); "$3x = x + 4$" (subtracted x from both sides); "$2x = 4$" (divided both sides by 2); (and concluded that) "$x = 2$". In Radford's terms (2022a, 2022b, 2021b), the small group encountered the algebraic procedures of eliminating equal objects (subtraction) and separating objects into proportional parts (division) on both sides of the equation, assuming a relational perspective of equality. Therefore, *analytical reasoning* emerged through deductive work with the *unknown* in the foreground, which was *denoted* through speech and gestures.

Throughout the analyses, we questioned ourselves about the reconceptualization of the meanings attributed by the research participants regarding (i) the indeterminacy – Is it worked on in the foreground? Is it deduced from the premises? –; (ii) the notion of equality – Is it in a relational perspective? Is the notion of equivalence used? –; and (ii) mathematical operations – Are the determined and undetermined quantities on both sides of the equation operated on? –. Thus, we leave these guiding questions to help future research in the area of mathematics education, which focuses on algebraic thinking in the TO aspect, based on activities involving the solving of equations.

Furthermore, it is worth highlighting that the types of reasoning manifested in the resolutions of the proposed problems were identified through the recognition and crossing of various semiotic means of objectification, such as hand gestures that refer to the notions of balance and equality, pointing with the fingers to indicate and count determined and undetermined quantities, hand gestures that refer to the movement of removing and separating (subtraction and division operations), and external discourses to socialize, argue, and substantiate reasoning, among others. These elements reinforce the relevance and legitimacy of an analysis of algebraic thinking in teacher education through a multimodal approach (Romeiro, Moretti & Radford, 2024; Moretti & Radford, 2023a).

We also reiterate that, despite the short period and the small sample space of participants in the formative research, the results corroborate Radford's (2022a, 2022b, 2021a, 2021b, 2011c) defense that the characterization of algebraic thinking goes beyond the identification of vectors (indeterminacy, denotation, and analyticity) and the recognition of semiotic means of objectification; because it is necessary to observe how these aspects emerge in the experiences of the activities – in our context, the activities of the teachers and the researcher-educator, the side-by-side work between teachers-researchers. Therefore, it is essential to consider that algebraic thinking from the perspective of TO is not something abstract. Rather, it occurs in the concreteness of human activity; i.e., beyond proving (a), (b), and (c), it is essential to focus on the nuances of collective engagement.

In short, in addition to reflections in the field of education of teachers who teach mathematics, we believe that this study has implications for the basic education classroom, mainly by calling for problematizations around the curriculum prescriptions proposed by the BNCC concerning the organization of the teaching of school algebra, in particular the knowledge of equations, from the initial years of elementary school. In this sense, one can consider the diversity of semiotic means and cultural artifacts, characteristic of each stage of schooling, for working with algebraic thinking in a dynamic, multimodal, dialectical, and collective perspective.

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