Sociedade Brasileira de Educação Matemática

RIPEM

The Induction Principle in Initial Teacher Training

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10.37001/ripem.v15i3.4594

Received • 04/05/2025
Approved • 31/07/2025
Published • 01/09/2025

Editor • Gilberto Januario

Abstract: The Principle of Induction is a method that allows generalisations in the theory of natural numbers and, based on it, it is possible to discuss the notion of infinity in Mathematics in basic education. Given the importance of this method, an empirical study was carried out with the aim of identifying specialised knowledge about the Principle of Induction, which was presented by mathematics licentiate undergraduates. In the light of the theories on the specialised knowledge of mathematics teachers, the data was produced from a qualitative study, which used group interviews and document analysis as its methodological procedure. The results show that the undergraduates have specialised knowledge about the method, as they are able to differentiate it from empirical induction. As for the Principle of Induction in basic education, the group defended the use of manipulable materials and empirical induction, emphasising the importance of considering the individual needs of students in the transition from Arithmetic to Algebra.

Keywords: Finite Induction. Mathematical Induction. Teacher Training. Basic Education.

Princípio da Indução na Formação Inicial de Professores

Resumo: O Princípio da Indução é um método que permite generalizações na teoria dos números naturais e, a partir dele, é possível discutir a noção de infinito em Matemática na educação básica. Dada a importância do referido método, foi desenvolvida uma pesquisa empírica, com o objetivo de identificar conhecimentos especializados sobre o Princípio da Indução, os quais foram apresentados por licenciandos em Matemática. À luz das teorizações sobre o conhecimento especializado do professor de Matemática, os dados foram produzidos a partir de um estudo qualitativo, que adotou como procedimento metodológico a entrevista em grupo e a análise documental. Os resultados apontam que os licenciandos apresentam conhecimentos especializados sobre o método, ao serem capazes de diferenciá-lo da indução empírica. Quanto ao Princípio da Indução na educação básica, o grupo defendeu o uso de materiais manipuláveis e indução empírica, salientando a importância de considerar as necessidades individuais dos estudantes na passagem da Aritmética para a Álgebra.

Palavras-chave: Indução Finita. Indução Matemática. Formação de Professores. Educação Básica.

Principio de Inducción en la Formación Inicial de Profesores

Resumen: El Principio de Inducción es un método que permite generalizaciones en la teoría de los números naturales y, a partir de él, es posible discutir la noción de infinito en Matemáticas en la Educación Básica. Dada la importancia de este método, desarrollamos esta investigación empírica con el objetivo de identificar conocimientos especializados sobre el Principio de

Inducción presentados por estudiantes de licenciatura en Matemáticas. A la luz de las teorizaciones sobre el Conocimiento Especializado del Profesor de Matemáticas, los datos se produjeron a partir de un estudio cualitativo que adoptó como procedimiento metodológico la entrevista grupal y el análisis documental. Los resultados indican que los licenciandos demuestran conocimientos especializados sobre el Principio de Inducción al ser capaces de diferenciarlo de la inducción empírica. En cuanto al Principio de Inducción en la Educación Básica, el grupo defendió el uso de materiales manipulativos e inducción empírica, destacando la importancia de considerar las necesidades individuales de los estudiantes en la transición de la Aritmética al Álgebra.

Palabras clave: Inducción Finita. Inducción Matemática. Formación de Profesores. Educación Básica.

1 Introduction

The Principle of Induction is a method that guarantees definitions and demonstrations in the theory of the set of natural numbers (*IN*). In other words, generalisations in the natural numbers are allowed, since the method can be stated in two equivalent ways:

Principle of Mathematical Induction - let $P(n)$ be a sentence defined on the set of natural numbers, if:

(i) $P(0)$ is a true sentence;

(ii) Given a natural n, if $P(n)$ is true, then $P(n+1)$ is true;

Therefore, the sentence $P(n)$ is true for all natural n.

Principle of Finite Induction – let X be a subset of *IN*, if:

(i) 0 belongs to X;

(ii) Given a natural n, if n belongs to X, then $n+1$ belongs to X.

Then $X = IN$.

It is possible to formalise the theory of natural numbers constructively using set theory (Halmos, 1970). In this way, the Principle of Induction can be demonstrated. Another way of formalising is axiomatically (Ferreira, 2010; Lima, 1976), which is commonly adopted in mathematics teacher training courses. In this case, the entire theory of natural numbers can be established on the basis of Peano's axioms, which, in addition to the Principle of Induction, have two other axioms: 1) if two natural numbers are distinct, then their successors are distinct; 2) zero is not the successor of any natural number. In Peano's axiomatics, the natural number and its successor are recognised as primitive terms.

Mathematical induction differs from empirical induction in the natural sciences. In the latter, we start from the particular to the general, so that the verification of formulae or statements is carried out by examining a few particular cases. In this context, it is worth pointing out that maths only uses empirical induction to develop conjectures, and uses the axiomatic method to define entities and demonstrate theorems through deduction and induction. In this way, mathematical induction is a method that makes it possible to define by recurrence and demonstrate theorems in the field of natural numbers.

Silva and Savioli (2012) investigated maths undergraduates' understanding of the difference between empirical induction and mathematical induction. By analysing written records, the authors identified that the undergraduates mistakenly understand empirical induction and mathematical induction as similar methods. As a result, the results found by Silva and Savioli (2012) indicate that undergraduates apply the Principle of Induction in a technical way, without reflecting on why the method validates generalisations in natural numbers.

In this context, through the application of a Didactic Engineering involving Fibonacci sequences for maths undergraduates, Vieira (2016) investigated, among other issues, the perception of undergraduates when defining the Fibonacci sequence inductively. Rodriguês, Costa and Custódio (2018) also analysed aspects of empirical induction and mathematical induction in textbooks for the ninth grade of primary school.

Pinto, Grilo and Grilo (2020), in turn, identified the presence of number theory topics in textbooks. The authors observed that Peano's axioms appear explicitly in the textbook, in a language that is different from that presented in Number Theory for maths undergraduates. For Pinto *et al*. (2020, p. 69), the two approaches, in basic education and higher education, "aim to build a theory of the Set of Natural Numbers that justifies the properties of its usual operations and the order relation".

In addition, Souza and Oliveira (2023) explained that it is possible to develop activities for secondary school students using mathematical modelling that involve demonstrations using the Principle of Induction. According to Borges (1995, p. 6), "the methods of induction and deduction reach their fulfillment in mathematics. The mental operations of analysis, synthesis, abstraction and generalisation appear naturally on every page of a mathematics textbook". For the author, this statement gives rise to the question: "What are the implications of all of this for the training of a mathematics teacher?" (Borges, 1995, p. 6).

With this in mind, we set out to identify specialised knowledge about the Principle of Induction presented by mathematics undergraduates. To do this, we used the notion of specialised knowledge from Carrillo, Climent, Contreras and Muñoz-Catalán (2013), as discussed in the next section.

2 What kind of Mathematics for teacher training?

For some time, researchers have been trying to answer the question posed in the title of this section, with the aim of arguing in favour of a necessary rapprochement between the mathematical training offered in mathematics degree courses and future professional practice in basic education (Fiorentini & Oliveira, 2013; Moreira & David, 2010). This rapprochement is justified, according to Santos and Lins (2016), by the realisation that mathematicians' working practices are not the same as those of primary school teachers, nor even those of teachers who work in teacher training.

Therefore, in an effort to identify the specific nature of the mathematics to be taught in undergraduate courses, studies have distanced themselves from political arguments to justify the existence or not existence of certain mathematical content in initial training, in order to rely on more conceptual issues (Almouloud, Figueroa & Fonseca, 2021; Dorantes & Vargas, 2019; Moreira & Viana, 2016; Resende & Machado, 2012). Therefore, as Santos and Lins (2016, p. 370) point out, "It is not a question of thinking that teachers need a less sophisticated and 'heavy' mathematical education than a bachelor's degree in maths, but an education [...] that offers some ways of dealing with the mathematical demands of their professional practice".

In this direction, different theorisations circulate in the area of Mathematics Education, dealing with a specific Mathematics for teaching, which differs from the mathematical knowledge needed by other professionals (Ball & Bass, 2003). From Ma's (1999) perspective, it is not enough for teachers to know the conceptual structure and basic attitudes inherent in elementary maths, as they need to teach it to students.

Supported by Shulman (1987), efforts to conceptualise Mathematical Knowledge for Teaching (MKT) have enabled a wide range of research aimed at mapping the mathematical knowledge needed to carry out tasks related to teaching mathematics and providing information to support the teacher training process. On the other hand, Carrillo *et al*. (2013) propose that teachers' knowledge is specialised, in order to configure the Mathematics Teacher’s Specialized Knowledge (MTSK), eliminating the reference to Common Content Knowledge (CCK) proposed by Ball *et al*. (2018). With this in mind, Table 1 summarises the subdomains corresponding to MTSK.

Table 1: CEPM subdomains

Domains	**Subdomains**	
Mathematical Knowledge	Knowledge of Topics	Includes knowledge of mathematical concepts and procedures, together with their theoretical foundations, including a certain degree of formalisation.
	Knowledge of the Structure of Mathematics	Includes knowledge of the ideas and main structures relating to specific items being covered, or knowledge of the connections between current and previous topics and upcoming items.
	Knowledge of the Practice of Mathematics	Includes how to proceed in mathematics, knowledge of ways of knowing and creating or producing in mathematics, involves the use of demonstrations and proofs, knowing how to define and use definitions, argue, generalise or explore aspects of mathematical communication.
Pedagogical Content Knowledge	Knowledge of Mathematics Teaching	Enables the teacher to choose a particular representation or material for teaching a concept or procedure, to be able to select suitable examples, tasks and teaching resources for learning the content and includes knowledge of teaching theories.
	Knowledge of Features of Learning Mathematics	Allows the teacher to know the way students think about mathematical tasks, to identify the most frequent difficulties students have, as well as to know how to detect wrong answers.
	Knolwedge of Mathematics Learning Standards	This includes knowledge of the objectives, contents, procedures and materials proposed by the official curriculum regulations, conventional support materials and forms of assessment.

Source: Adapted from Carrillo *et al*. (2013).

A CCK is one that anyone with maths training has, but uses it as a tool, without necessarily being able to explain why or where it comes from (Ribeiro, 2009). Anyone with an education should know that, in the set of integers, it is not possible to divide by zero, but they may not be able to explain in which situations this division is indefinite or indeterminate. In this context, maths teachers are expected to possess a type of Specialised Content Knowledge (SCK) that enables them to explain each of these situations in a way that is comprehensible to their students.

Unlike the CCK perspective, SCK differs from general mathematical ability and, according to Ball *et al*. (2008), requires further study in order to understand the most important dimensions of teachers' professional knowledge. In view of this, Grilo, Barbosa and Maknamara (2020) systematised some of these dimensions when they identified the supposed skills and abilities that maths teachers need to have in order to be able to teach adequately. Among the skills, Grilo *et al*. (2020, p. 12) point out that teachers who teach maths must be able to "unpack, connect, anticipate, articulate, understand and prove mathematical ideas in a way that is associated with the specific demands of teaching".

With this in mind, these skills can be exemplified in the context of the Principle of Induction by means of a problem that frequently appears in basic education: showing that the number of diagonals of an *n-sided* convex polygon is given by. The skill of unpacking requires the teacher to present the statement making explicit the ideas and mathematical procedures to be adopted. Thus, such a problem could be "unpacked" as follows, according to Box 2:

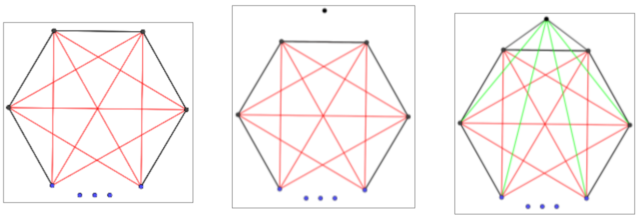
Box 2: Proposed "unpacked" activity to prove, by induction, the number of diagonals of a convex polygon

Answer each of the following questions to show by induction that the number of diagonals of an *n*-sided convex polygon is given by $d_n = \frac{n(n-3)}{2}$, for $n \geq 3$. i) What is the concept of a diagonal of a convex polygon? ii) What is the minimum number of sides of a convex polygon? What is the name of this polygon? iii) What can you say about the number of diagonals of a convex polygon that has exactly 3 sides? iv) Considering $n = 3$, is the equality $d_n = \frac{n(n-3)}{2}$ true or false? v) If we add a new vertex to an *n-sided* convex polygon, how many sides will be added to the "original" polygon? And how many diagonals? vi) Considering that, for a given $n \geq 3$, $d_n = \frac{n(n-3)}{2}$, is true for an *n*-sided convex polygon, what can you say about the number of diagonals in an $n + 1$-sided convex polygon?

Source: Prepared by the authors.

Also, according to the skills cited by Grilo *et al*. (2020, p. 12), this example also shows us the teacher's ability to: connect different areas of Maths - Arithmetic and Geometry; articulate different methodological strategies, since this situation allows the teacher to resort to drawings or the use of *software,* as shown in Figure 1, and not just algebraic manipulation; prove the mathematical ideas involved.

Figure 1: Using *software* to show the number of diagonals in a convex polygon



Source: Prepared by the authors.

With regard to the last skill, Steele and Rogers (2012) discussed *Mathematical Knowledge for Teaching Proof* (MKT-P), one of the fundamental components of which is the ability to know whether or not a mathematical argument is a proof. According to the authors, it is common for teachers to prefer empirical arguments to deductive proofs, as they consider them to be more convincing or easier to use in the classroom, as well as classifying proof as a topic reserved for high-achieving students.

This position conflicts with the role of proof in maths, as a way of thinking and reasoning about its very nature. In view of this, for Grilo, Barbosa and Luna (2016), the absence of discussions about different teaching strategies for the use of proofs and demonstrations in initial teacher training does not favour their use in basic education classrooms. Therefore, in an attempt to bring the initial training process closer to discussions that relate the Principle of Induction to topics studied in basic education, the aim was to identify specialised knowledge about mathematical induction presented by a group of undergraduate students.

3 Methodological aspects

In accordance with the proposed objective, the research was developed based on qualitative research, as it seeks to understand the meanings attributed by the subjects participating in the research to the situations investigated in an inductive way, so that the interpretations presented start from particular situations to systematise general issues (Creswell, 2016). With this in mind, it should be noted that in order to answer the study's guiding question - what specialised knowledge about the Principle of Induction do mathematics undergraduates present? -, group interviews and document analysis were used as data production strategies.

According to Lichtman (2010), interviews are widely used in qualitative research and can be carried out individually or in groups (also called focus groups). In the case of a focus group, the researcher enables interaction between each member of the group, stimulating reflection by all the members, without worrying about reaching a consensus (Lichtman, 2010).

In this process, 12 students enrolled in the Numerical Sets subject, which is part of the 5th semester of the Maths degree course at the Universidade Estadual de Feira de Santana (UEFS), took part in the focus group. The documentary analysis focused on the students' answers in the written assessment of the subject, which covered the topic of natural numbers and integers. According to Gil (2008), these assessments can be characterised as personal documents and contribute to understanding the research problem when used to supplement the data obtained by other procedures.

Therefore, when considering the syllabuses and teaching plans available on the course website (www.matematica.uefs.br), the Principle of Induction appears explicitly in the subjects Mathematical Logic and Set Theory M (1st semester), Number Theory (2nd semester) and Number Sets (5th semester), as specialised mathematical knowledge to be taught. This justifies the choice of student group.

It is worth mentioning that before the focus group began, the students were informed about the objectives and procedures adopted in the research. Then, after signing the Informed Consent Form, they were given a questionnaire containing open and closed questions about their contact with the Principle of Induction in the course subjects.

The focus group lasted 1 hour 40 minutes and was recorded using two audio capture devices and one audio and video device. At this point, the motivating question was: what is the Principle of Induction? Thus, after depleting the interactions between the participants, the researchers showed them images taken from some basic education textbooks, with the aim of having the participants identify, or not, the Principle of Induction.

The data was analysed using the steps of the Textual Discourse Analysis proposed by Moraes (2003). To unify the data, the answers were grouped according to the subdomains of the maths teacher's specialised knowledge. This procedure allowed for a thorough reading of the data and separation into meaningful units that generated the excerpts presented in each subdomain.

The excerpts, in turn, were presented on behalf of the focus group, since the excerpts presented are the result of consensus opinions. In this way, the new emergent was captured as a result of the discursive constructions elaborated by the participants and constructed by the researchers, in dialogue with the theories that anchor the research.

4 Results and discussion

To present the data, two analytical categories were established. In the first, the data that explains the knowledge of the content was gathered, so that the following were identified: the students' knowledge of Induction topics (definition), its structure (connections) and mathematical practice (demonstration). In the second category, the data focuses on Pedagogical Content Knowledge, and didactic strategies were listed for the use of mathematical induction in basic education, as well as difficulties to be faced in teaching and knowledge about curriculum organisation.

4.1 Mathematical Knowledge

The three subdomains related to content knowledge were identified in the focus group. In relation to Topic Knowledge, when asked what the Principle of Induction is, the undergraduates were able to state it, according to the excerpt that summarises the discussion that took place in the group in response to this question.

> *I think it would go back to an exercise I did in the [Number] Sets class. We tested for one, as [quotes colleague's name] said, that if we want to prove, generalise, as [quotes another colleague's name] said, I already have a starting point. I assume that it holds for one quantity, so I'll make sure it holds for that quantity plus one. I think it would be this idea, generalising, taking a minimum element as a starting point.*

The group showed clarity about the purpose of the Principle of Induction, which seeks to guarantee generalisations in the set of natural numbers. In addition, the students showed that they were aware of the existence of two conditions that must be verified for the Principle of Induction. In explaining the need to verify a starting point or minimum element, the group refers to the first condition of the Principle of Induction, known as the basis of induction. The second condition, known as the hypothesis of induction, is described by the group with the assumption that if it is valid for "a quantity", then it must be guaranteed that it is also valid for "that quantity plus one".

It is also worth emphasising that the students assumed the notion of cardinality for a natural number, serving as a counting model. However, it can be seen that their conceptual understanding is still linked to the use of formulae, so that they use the Principle of Induction more as a demonstration technique than as a method that guarantees definitions in the theory of the set of natural numbers. For example, even though the researchers instigated them based on the image in the textbook of the factoring process using successive divisions, the group was unable to establish an association with the Well-Ordering Principle, which is equivalent to the Induction Principle.

> *When we see the formula we already think of Induction (...) If the formula generalises, then it's induction.*

So when you see it there, you have a little formulation and you generalise, so it's an induction. In the other case, there wasn't [referring to an image in the textbook showing a process of successive divisions].

We saw this in Principles [Methodology Applied to Education]. (...) Then we did that process of open questions where we always tried to generalise. In this process of investigation, we first tried to find some kind of formula to generalise what we had first left open. And always this idea: I had the sequences and then I went back and did it... It was exactly induction, it was the same, the same. In the end we arrived. It was induction! You started from that point and in the end you generalised for everyone.

In this context, they recounted their experience in another subject, which they thought involved induction. When asked if they had had the same perception at the time, the students said they had not. In this way, it was analysed that the experience reported by the group brings mathematical induction closer to empirical induction, since it was an activity that involved the participation of basic education students.

Furthermore, when dealing with the process of investigation to try to find a formula, the group did not initially distinguish between identifying patterns and validating a generalisation. In other words, they don't distinguish between empirical induction and mathematical induction.

Regarding the Knowledge of the Structure of Mathematics subdomain, the group established relationships between the Principle of Induction and ideas associated with sequences, arithmetic progression (AP), geometric progression (GP), geometry and differential and integral calculus. The latter was related by the group when they were shown a page from a textbook that deals with the region bounded by a regular polygon.

Group: There's no way round it, is there? If there's a formula, whatever the size of this figure, we'll be able to tell what the ...

Researchers: You're talking so much about formulas, so can I understand that you're looking at the Principle of Induction as a method of demonstrating that formula [points to slide]? The area of a geometric figure, in general, we define. If we take this as a definition, can we still say that we see the Principle of Induction in the definition?

Group: But we define it in a general way, right?

Researchers: Yes, in a general way. Because in primary education, demonstrations aren't very popular. This is a definition, right? So, the book is defining that the area of the region of a regular polygon is like this [pointing to the slide]. Can we still use or apply the Principle of Induction to this definition?

Group: I think so, because it all depends on the number of sides.

Researchers: We can define the area of a polygon with 3, 4, 5, 6 sides, but the generalisation will occur from the moment there is some strong instrument in mathematics that guarantees validity for all n. Would this instrument be induction? Is that what you're trying to say?

Group: I think so.

Group: But if it tends to infinity it will be a circle, right?

[A buzz arises in the group.]

Group: But if n goes to infinity, doesn't it become a circle?

Researchers: [Quotes student's name] asked an important question: there are two different views. (...) The first is: you choose a random n, then you define it. The second, you make n tend to *infinity, which is to grow without stopping. By making n tend to infinity, in other words, by making n vary, do we have the Principle of Induction?*

Group: I don't think so, because the Principle of Induction is... Let's say you have an infinite number line, then I take a number from that line and it applies to that number, and infinity is not a number. Then the Principle of Induction wouldn't apply.

In this process, the group observed that if the number *n* sides of a polygon inscribed in a circle tends to infinity, the polygon tends to a circle. With this, despite having a generalisation, in the sense that by increasing the number of sides of the polygon, it approaches a circle, the students pointed out that this is not the Principle of Induction. In this way, it seems that the group was able to explicitly demarcate the difference between the exhaustion method, considered to be the precursor of differential and integral calculus, and the method of demonstration by induction.

They also made connections between the Principle of Induction and other topics seen in basic education, such as AP and GP. This is especially true when it involves the sum of the terms of a finite AP, the number of diagonals of a convex polygon and sequences.

In primary school I saw it, but I wasn't supposed to generalise about AP and PG using induction. We would test the first, then add the second, the third, until we found a pattern in the successor. Understand? The successor in the sequence. It fits, that's what I see.

Once again, the excerpt above shows the presence of empirical induction, but this time it is demarcated as a principle that differs from induction. This is because the intention was not to generalise.

Another thing we discussed a lot in the [Numerical Sets] classes was conserving the methodology we use. Even though the Principle of Mathematical Induction and the Principle of Finite Induction are a little different, in the questions you retain the methodology of solving. So you have this logical power of resolution.

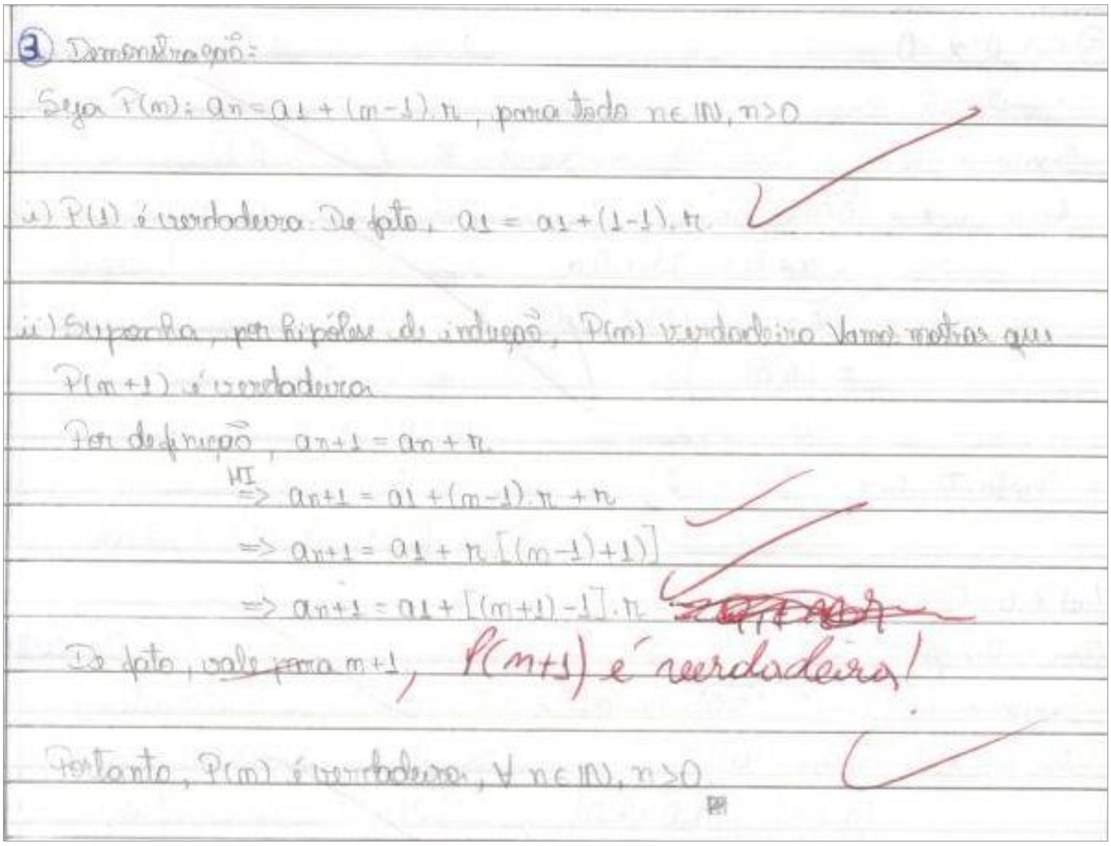
It can also be seen above that, as well as demonstrating knowledge of mathematical structure, by relating the Principle of Mathematical Induction to the Principle of Finite Induction, the group demonstrates knowledge of how to produce maths, also demonstrating a Knowledge of the Practice of Mathematics. That said, the students realised the existence of a methodological unit when it came to demonstrating using the Principle of Mathematical Induction or the Principle of Finite Induction. This knowledge was also verified when the students were required to carry out demonstrations in the assessments, as shown in Figure 2.

Figure 2: Demonstration of the general term of an arithmetic progression

3. Uma progressão aritmética (PA) é uma sequência de números reais (a_n) tal que a_1 é dado e, para todo $n \in \mathbb{N}$, $n > 0$, tem-se que

$$a_{n+1} = a_n + r$$

onde r é um número real fixo chamado razão. Mostre que $a_n = a_1 + (n-1)r$ para todo $n \in \mathbb{N}$, $n > 0$.


3) Demonstração:
Seja P(n): $a_n = a_1 + (n-1).r$, para todo $n \in \mathbb{N}, n>0$
i) P(1) é verdadeira. De fato, $a_1 = a_1 + (1-1).r$
ii) Suponha, por hipótese de indução, P(n) verdadeiro. Vamos mostrar que P(n+1) é verdadeira.
Por definição, $a_{n+1} = a_n + r$
$\overset{HI}{\Rightarrow} a_{n+1} = a_1 + (n-1).r + r$
$\Rightarrow a_{n+1} = a_1 + r[(n-1)+1]$
$\Rightarrow a_{n+1} = a_1 + [(n+1)-1].r$
De fato, vale para n+1, P(n+1) é verdadeira!
Portanto, P(n) é verdadeira, $\forall n \in \mathbb{N}, n>0$

Source: Research data.

Figure 2 shows the demonstration, using the Principle of Mathematical Induction, of the general term of an AP in an assessment. The demonstration clearly shows the basis of induction in item (i) and the use of the induction hypothesis in the implication in which the acronym HI appears in item (ii). In item (ii), we see the use of the expression "holds for $n + 1$", as well as a manipulation involving successors, without carrying out the subtraction.

In this case, it is analysed that since subtraction is not an operation defined in the set of natural numbers, the demonstration uses the concept of predecessor in addition to the concept of successor: the natural number $n - 1$ is the predecessor of n, because n is the successor of $n - 1$. Therefore, considering that the concept of predecessor derives from Peano's axioms, in the demonstration, the number $(n - 1) + 1$ is the successor of $n - 1$; the number $(n + 1) - 1$ is the predecessor of $n + 1$; both $(n - 1) + 1$ and $(n + 1) - 1$ are equal.

5 Pedagogical Content Knowledge

With regard to the Knowledge of Mathematics Teaching subdomain, the group considered the use of manipulable materials, empirical induction and the use of images to be relevant in facilitating the understanding of basic education students. This was especially true when it came to sequences and the possibility of inferences with a view to generalisation.

In the subject [Instrumentalisation for Mathematics Teaching] INEM 5, we are working with some applications of the subjects and we are seeing some realisations of these subjects in the textbooks. The author uses various realisations to develop the student's algebraic thinking. For example, he takes that question about the toothpicks and says: I can create two squares with seven toothpicks and asks: with 16 toothpicks, how many can I make? According to this, he asks the student to create an algebraic form to generalise to as many toothpicks as they want. And we've also seen this in a Geometry question, which says: it applies to such and such a geometric figure, let's say. Then it says: create, according to induction, a larger figure and now imagine that it's larger in a given proportion. And then: could you generalise a formula that could be done? So you can see it in primary education.

I think students generally have a lot of resistance to this kind of example [referring to the example of AP and GP]. Although the two examples follow the same principle, the same reasoning, when we go to this more formal part, which involves letters, which are different concepts, I think they have more resistance [referring to primary school students]. But when it goes the other way, using manipulative materials or even other contextualisations, even with images, they are more receptive. But I think it's also another way of showing, at least I've seen it this way and learnt, but it doesn't involve all the students.

The excerpts above show that the group does not consider the possibility of using technological resources, nor does it choose the use of manipulable materials as more suitable for working with the Principle of Induction in basic education, without disregarding the possibility of involving empirical induction processes. These choices were imbricated in the Knowledge of Features of Learning Mathematics, when the students considered the importance of the teacher observing the individual characteristics of their students and of each class, as shown below:

Sometimes you want to work on everything, you want to follow the timetable, but one class is not the same as another. So it's very important to observe our students. If I explain it one way and my student doesn't learn, I can explain it ten times in the same way and my student still won't understand. So I think it's important that we talk about the same thing, but in different ways.

This concern with the learning process in basic education is highlighted once again when the group presents situations that refer to the Knolwedge of Mathematics Learning Standards. This is when they emphasise that one of the greatest difficulties faced in basic education is the transition from arithmetic to algebra.

I don't think it's so much the students. It's the way it's taught in the classroom. I'm not talking so much about the teachers, but when you analyse school life, there comes a certain moment when everything changes and the student has to reinvent themselves: what they used to use as a letter, now maths is starting to come into it. So the student creates this resistance because they haven't been prepared beforehand, hearing: Maths is this, it's exact, you can only give one value, you can only do it one way. That's it, they use numbers in one way. Then a letter appears and says that this letter can take on such and such a value. So it ends up that the student always experiences this shock, and this shock generates resistance: Oh, I don't like maths. But why don't you like it? Because I don't understand it! Then when you ask them about a subject that requires generalisation, they say that it's too difficult, that they can't understand it. There's the issue that up until 5th grade, it's Pedagogy, and when we get to 6th grade, we start taking these classes ourselves, and they start to realise this shock and when we bring something different, they say it's too difficult.

Just to add to what she said, this is very common in the transition from arithmetic to algebra, which *is what we're seeing at INEM now. When they work with numbers, the students find it easy, but when they put in a letter, they find it very difficult. So I think this process depends a lot on the transition from arithmetic to algebra.*

In this vein, it should be noted that, according to the National Common Curriculum Base (in portuguese: Base Nacional Comum Curricular - BNCC), work with ideas related to regularity, generalisation of patterns and the property of equality should begin in the early years of primary school and be deepened in the final years. The document also emphasises the obvious relationship between the thematic units "Numbers" and "Algebra", especially when working with sequences in the early years. The BNCC also suggests that other fields should be explored throughout school, such as the different meanings of numerical variables in an expression, the generalisation of properties and patterns and associations with the development of computational thinking (Brazil, 2018).

6 Final considerations

In order to identify the specialised knowledge of mathematical induction presented by mathematics undergraduates, a qualitative study was carried out using a focus group. The group, made up of 12 undergraduates, was clear about one of the purposes of the Principle of Induction - to prove theorems in the theory of the set of natural numbers.

However, the group did not recognise that the Principle of Induction guarantees definitions by recurrence. In addition, their written records from the assessment questions showed a mastery of the demonstration technique, both in verifying the step called "basis of induction" and, in the second step, in correctly applying the induction hypothesis.

It should also be noted that the group had a precise conception that the Principle of Induction is a way of guaranteeing generalisations in the set of natural numbers, which differs from empirical induction. The undergraduates understood that empirical induction is a way of identifying patterns, relevant behaviour for producing mathematical knowledge, but it is not a method of demonstration that guarantees mathematical truths. For basic education, the group advocated the use of manipulable materials in conjunction with empirical induction, taking care when moving from arithmetic to algebra.

Finally, it was evident that the group recurrently cited subjects on the course that make up the Practice as a Curricular Component axis (e.g. INEM), when reporting on previous experiences with the Principle of Induction. With the exception of the Number Sets subject, the *locus* of the data production, no experience reports were identified in subjects from the Mathematical Knowledge axis already taken by the participants.

This fact leads us to consider the need to broaden the discussion on the Mathematics Teacher's Specialized Knowledge, in order to develop studies that deal with the Trainers Teacher's Specialised Knowledge who work on mathematics degree courses. Therefore, investigating the Trainers Teacher's Specialised Knowledge, based on the subdomains of the MTSK, could reveal the gaps in initial training, with regard to the mastery of Mathematical Knowledge and Pedagogical Content Knowledge, and consequently the consequences for the practice of future mathematics teachers in basic education.

Acknowledgements

The research was funded by the State University of Feira de Santana through FINAPESQ.

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