

Validation Processes in Geometry Teaching: A Historical Analysis of Textbooks in Brazil (1930-1960)

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Abstract: This study investigates the approach to proofs and demonstrations in geometry in Brazilian regulations and textbooks for 13-year-old students between 1930 and 1970. Adopting a cultural history perspective, two moments of modernization in mathematics education are analyzed: the adoption of the recommendations of the International Commission on Mathematics Education since 1908 and the emergence of Modern Mathematics (MM) in the 1950s. Four representative textbooks were examined. The analyses identify the persistence of traditional deductive geometry, emphasizing the memorization of theorems until 1950, contrasting with the intuitive approaches proposed by curricular reforms. Conversely, the work of Osvaldo Sangiorgi (1960s) signals an attempt at a break with the past, introducing experimental methodologies influenced by MM. The study highlights the resistance and adaptations of Brazilian school culture in the face of international movements for pedagogical renewal.

Keywords: Proofs and Demonstrations. Intuitive Geometry. Deductive Geometry. Secondary Education. Modernizing Movements.

Procesos de Validación en la Enseñanza de Geometría: Un Análisis Histórico de Libros Didácticos en Brasil (1930-1960)

Resumen: Este estudio investiga el enfoque de pruebas y demostraciones en geometría en normativas y libros de texto brasileños para estudiantes de 13 años (1930-1970). Bajo la perspectiva de la historia cultural, se analizan dos momentos de modernización educativa: la apropiación de las recomendaciones de la Comisión Internacional de la Enseñanza Matemática desde 1908 y la emergencia de la Matemática Moderna (MM) en la década de 1950. Se examinaron cuatro libros de texto representativos. Los análisis identifican la persistencia de la geometría deductiva tradicional, enfatizando la memorización de teoremas hasta 1950, lo que divergía de los enfoques intuitivos de las reformas curriculares. Por el contrario, la obra de Osvaldo Sangiorgi (década de 1960) señala un intento de ruptura con ese modelo memorístico, introduciendo metodologías experimentales influenciadas por la MM. El estudio evidencia las resistencias y adaptaciones de la cultura escolar brasileña frente a los movimientos internacionales de renovación pedagógica.

Palabras clave: Pruebas e Demonstraciones. Geometría Intuitiva. Geometría Deductiva. Educación Secundaria. Movimientos Modernizadores.

Processos de Validação no Ensino de Geometria: Uma Análise Histórica de Livros Didáticos no Brasil (1930-1960)

Resumo: Este estudo investiga a abordagem de provas e demonstrações em geometria nas normativas e livros didáticos brasileiros para alunos de 13 anos, entre 1930 e 1970. Adotando uma perspectiva da história cultural, analisam-se dois momentos de modernização do ensino de matemática: a apropriação das recomendações da Comissão Internacional do Ensino de Matemática desde 1908 e a emergência da Matemática Moderna (MM) na década de 1950. Foram examinados quatro livros didáticos representativos. As análises permitem identificar a persistência da geometria dedutiva tradicional, enfatizando a memorização de teoremas até 1950, contrastando com as abordagens intuitivas propostas pelas reformas curriculares. Por outro lado, obra de Osvaldo Sangiorgi (década de 1960) sinaliza uma tentativa de ruptura com esse modelo baseado na memorização, introduzindo metodologias experimentais. O estudo evidencia as resistências e adaptações da cultura escolar brasileira frente aos movimentos internacionais de renovação pedagógica.

Palavras-chave: Provas e Demonstrações. Geometria Intuitiva. Geometria Dedutiva. Ensino Secundário. Movimentos Modernizadores.

1 Introduction

“Validation processes” in the context of geometry is the expression we have adopted in the title of a collective research project to study, from a historical perspective, how students were invited to learn, memorize, or perform tests, demonstrations, arguments, and justifications in geometry. The project in question aims to conduct a broad examination throughout the 20th century, involving different levels of education, from the early school years to the end of basic education and teacher training.

We believe that the expression “validation processes” can encompass various ways of producing justifications in geometry, considering different historical moments and different stages of learning. From this broad perspective, validation processes in school geometry are not restricted to formal demonstration in the strict sense of academic mathematics. They encompass a spectrum of discursive practices and actions that aim to confer credibility and acceptability on geometric statements. In the early years, this may involve justifying a property through the observation of multiple examples or the manipulation of objects. In later stages, validation can be achieved through experimental proofs based on particular cases or action resources – the so-called *pragmatic proofs* according to Balacheff (2000). In more advanced years, it can evolve into more structured arguments based on previously accepted properties, ideally culminating in the construction of deductive proofs. All these pedagogical actions are considered, within the scope of our research, as constituents of “validation processes.” We also note that the topic has been the subject of current research, especially in undergraduate courses, such as in the works of Silva (2023) and Correia (2018, 2024).

Throughout the history of geometry teaching in Brazil, different conceptions of what it means to ‘do geometry’ and what the objectives of geometric learning are have shaped the ‘validation processes’ favored in the classroom. Curriculum reforms, influenced by pedagogical movements and different views on the nature of geometric knowledge, have also left their mark on the ways in which students have been invited to justify their ideas and construct regimes of truth in geometry.

However, during the initial presentations of the project, we were questioned about the expression “validation processes”: what do we mean by validation in geometry? Perhaps the expression “validation” is precisely our object of investigation, especially when we consider, from a historical perspective, how understandings of what it means to validate have been constructed and modified in contexts of school geometry. In other words, the analytical challenge of this research lies in understanding how, over time, the methods, procedures, languages, and representations used to offer or develop justifications in geometry tasks proposed in the school setting – synthesized by validation processes – have changed. This investigation therefore seeks to analyze transformations in the ways of justifying in geometry, from intuitive and experimental explorations to formal demonstrations, considering how different historical, curricular, and pedagogical factors influenced what was considered acceptable validation at different moments in the teaching of geometry in Brazil.

This paper¹ covers a specific time period (1930-1960) referring to a specific school year – 13-year-old students – a segment that received different names: 3rd year of the 1st Cycle of Secondary Education (Francisco Campos Reform of 1930) and 3rd year of Junior High School (from the Capanema Reform of 1942). Adopting a historical perspective, we investigate how the validation processes were proposed in dialogue with the ideas of two international movements that circulated in Brazil, regarding proposals for the introduction of intuitive geometry as a preparatory stage for deductive geometry. The research uses educational regulations and textbooks representative of the periods analyzed as sources of investigation. In the first period (1930 to 1950), the analysis contextualizes the Brazilian scenario in general and, subsequently, after the enactment of the first Law of Guidelines and Bases for National Education (LDB 4.024/61) in 1961, when the states acquired autonomy for curricular organization, the focus of the analysis turns to the state of São Paulo.

The time frame from 1930 to 1960 is justified because it marks the beginning of the regularization of secondary education in Brazil. Prior to 1930, access to higher education occurred through a mixed system, which combined serial education offered by secondary schools – with Colégio Pedro II, founded in 1838, as a paradigmatic institution – with installment (preparatory) exams in arithmetic, algebra, geometry, and trigonometry. The 1960s, in turn, represent another significant moment of transformation in both the Brazilian educational system and mathematics teaching.

Before 1930, the books adopted at Colégio Pedro II can be considered references for secondary education. Among them, during the second half of the 19th century, were the works compiled by Cristiano Benedito Ottoni, senator of the Empire and professor of mathematics at the Royal Academy of Midshipmen. Ottoni's book *Elementos de Geometria* (Elements of Geometry) was a central reference at Colégio Pedro II, being recommended in the 1856, 1876, 1879, and 1881 curricula (Valente, 2004). A brief analysis of Ottoni's book (1857) indicates that, starting on the tenth page, the author introduces the concepts of axiom, theorem, reciprocal of a theorem, corollary, lemma, and problem. Next, three methods of demonstration are presented: direct demonstration, demonstration by absurdity, and demonstration by superposition. The entire work is developed using a deductive approach, characterized by the presence of numerous theorems and their respective demonstrations.

Valente (2000), when analyzing the positivist movement in Brazilian education during the Republic, takes as an example the book *Os Elementos de Geometria* de Clairaut (The Elements of Geometry by Clairaut), translated into Portuguese in 1892. This work developed a geometry without any concern for mathematical rigor, using an intuitive approach based on the

¹ This paper is a revised and expanded version of a study presented at *the Congress of the European Society for Research in Mathematics Education* (CERME 14), held in Bozen-Bolzano, Italy, in February 2025.

practical need to measure land. However, the researcher concludes that “considering the teaching of geometry in an intuitive and practical way, as Clairaut did, was something that did not catch on. School geometry had been following its course since Ottoni with an emphasis on geometric rigor and thus continued” (Valente, 2000, p. 210).

In summary, we can infer that geometry intended for Brazilian secondary education throughout the 19th century and into the early decades of the 20th century was characterized as hypothetical-deductive geometry, with an emphasis on the demonstration of theorems.

From 1930 onwards, two international ideas circulated in Brazil during the period under review: the first, represented by the creation of the *International Commission for Mathematical Instruction* (CIEM) at the Fourth International Congress of Mathematicians, held in Italy in 1908; and the second promoted by the European Organization for Economic Cooperation (OECE) at the *Royaumont* Seminar in France in 1959. In the Brazilian context, these movements were characterized as initiatives to modernize mathematics education, bringing together representatives from various countries, producing curriculum guidelines, and being appropriate to the national educational scenario, both in regulations and in textbooks. This research seeks to answer the following questions: – How were the proposals of international movements adapted and reinterpreted in the Brazilian context for the introduction of proof-based teaching and demonstrations in geometry in secondary education? What changes can be observed in the approaches and introductory exercises for proofs and demonstrations for 13-year-old students in Brazilian textbooks during the period analyzed, in relation to the appropriations of international movements?

The theoretical and methodological basis of this research is founded on studies of the international circulation of pedagogical ideas which, since the 19th century, have established and maintained an extensive system of exchanges and mutual observations, structured by pedagogical missions (Matasci, 2016) and, since the 20th century, international scientific congresses and the creation of commissions dedicated to discussing new programs for teaching mathematics enabled the continuity of these cultural transfers, encompassing the circulation of educators, experiences, books, and knowledge between different countries.

Although the centrality and relevance of European culture in the genesis of these two international movements, led by influential European mathematicians, is recognized, the debates and recommendations formulated in these forums achieved global visibility, crossing the Atlantic and becoming important references for Brazilian educational reforms. It should be noted, however, that these guidelines were adapted to local contexts, considering that, as Chartier (2009, pp. 46-47) points out,

the strength of dominant cultural models does not nullify the space for their reception [...], the imposition of new disciplines, the insinuation of new submissions, the definition of new rules of conduct must always yield or negotiate with entrenched representations and shared traditions.

The present study aims to analyze the process of appropriation² (Chartier, 1990) of two important modernizing ideas by Brazilian school culture. Covering a period of approximately 50 years, the research focuses on the processes of validation in geometry and, specifically, on the way in which geometric proofs and demonstrations were introduced in secondary education for students aged 13 and above. The aim is to identify the dynamics of permanence, innovation,

² Appropriation, as we understand it, aims to provide a social history of interpretations, referring to their fundamental determinants (which are social, institutional, and cultural) and inscribed in the specific practices that produce them (Chartier, 1990, p. 26).

and reinterpretation of international proposals in the national context.

The contribution of this study lies not only in inventorying geometric content, but also in analyzing the dynamics of school validation practices in Brazil during a period of redefinition of curriculum guidelines. By focusing on the introduction of proofs and demonstrations for 13-year-old students, the research seeks to broaden understanding of how school culture negotiated, resisted, or incorporated deductive rigor and intuitive experimentation. This analysis of the paper is structured in four main sections: initially, the theoretical-methodological framework based on cultural history and the circulation of pedagogical ideas is presented. Next, the curricular norms resulting from the two modernization movements and their repercussions on the Brazilian scenario are examined. The third section is devoted to a detailed analysis of four representative textbooks, comparing official prescriptions with the validation practices proposed by authors such as Euclides Roxo and Osvaldo Sangiorgi. Finally, the concluding remarks summarize the transformations observed in the ways of justifying geometry, highlighting the tensions between modernizing ideas and Brazilian school tradition.

2 First modernizing movement

Schubring (2003) points out that the official guidelines of the CIEM, under the leadership of Felix Klein, advocated the inclusion of basic notions of variable quantities and functional dependence in mathematical concepts, along with a methodological reorientation with an emphasis on intuition and practical applications. However, forces opposed to the official rhetoric were identified, especially on the part of classical secondary schools in Italy, which rejected intuitive approaches in favor of the pursuit of maximum rigor through the axiomatic teaching of geometry. This scenario suggests that the proposals for teaching geometry were not consensual, reflecting a significant debate between a deductive perspective and an experimental approach.

Although Brazil did not participate in the IV International Congress of Mathematics in Rome (1908), Professor Eugênio de Barros Raja Gabaglia (1862-1919), from Colégio Pedro II – a leading institution in the establishment of Brazilian secondary education – represented the country at the Fifth Congress in England in 1912 (Schubring, 2003). However, international discussions and proposals were not effectively incorporated into mathematics education in Brazil, as there were no significant changes in the programs at Colégio Pedro II during the 1910s and 1920s (Valente, 2004).

It was not until the 1930s that the proposals of the first modernization movement in mathematics education were incorporated into the official secondary school curriculum, with the Francisco Campos Reform in 1931. This process took place through another teacher at Colégio Pedro II, Euclides Roxo (1890-1950), who introduced mathematics as an integrated subject for the first time, combining arithmetic, algebra, and geometry, in line with the recommendations that:

Starting from lively and concrete intuition, logical thinking will grow little by little until it gradually reaches formal expression; [...] knowledge will be acquired, at first through experimentation and sensory perception, and then slowly through analytical reasoning. Thus, with regard to geometry, formal demonstrative study should be preceded by a preparatory course aimed at intuitive teaching of an experimental and constructive nature. (Bicudo, 1942, p. 157)

The preceding quotation highlights the persistence of deductive geometry, originating from practices in place since the 19th century, while also emphasizing the innovation introduced in the recommendations: the inclusion of intuitive, lively, and concrete geometry. This novelty

represented a challenge to be implemented by the new regulations for secondary education.

The 1931 program introduced several innovations that brought about changes in the conception of teaching at the time, sparking intense debate. Euclides Roxo was the target of criticism from a conservative audience, notably from his colleague at Colégio Pedro II, Almeida Lisboa. The *Jornal do Commercio newspaper* published several papers documenting the controversies between the two teachers. The following excerpt is part of one of the criticisms made by Lisboa in the aforementioned newspaper:

Professor Euclides Roxo has forgotten the true purpose of mathematics in secondary school. Its main purpose is not to harvest a more or less abundant amount of practical and isolated knowledge. Mathematics is a discipline of the mind, an inimitable and irreplaceable educator of reasoning to which young people must be subjected. (Almeida Lisboa, 1930³ *apud* Carvalho, 2003, p. 131)

Lisboa's criticism focused on the introduction of intuitive, concrete, and experimental practices in teaching. According to the professor, secondary education should be dedicated to the development of reasoning and logic. For him, deductive geometry was fundamental to this education, constituting, in his view, the essential pillar of secondary school.

The 1931 reform was short-lived and was replaced in 1942 by the Capanema Reform, maintaining the division between intuitive and deductive geometry. In 1951, with the Simões Filho Reform, the teaching of geometry was reduced to the final grades. The three reforms prescribed deductive geometry, that is, the introduction to the study of proofs and demonstrations in geometry for 13-year-old students, as shown in Table 1.

Table 1: Geometry in the prescribed programs for students aged 11 to 14 (1931-1951)

Reform	Organization of Geometry	Guidelines
1931	<i>Introduction to Geometry</i> <i>Geometry</i>	- Start from intuition to <i>gradually reach</i> formal exposition: from experimentation and sensory perception to analytical reasoning
1942	<i>Intuitive Geometry</i> <i>Deductive Geometry</i>	- Intuitive geometry as <i>a smooth transition</i> between experiments with shapes and the deductive conception of geometry
1951	<i>Geometry</i> (13- and 14-year-old students)	- <i>Gradually awakening</i> the feeling of the need for justification, proof, and demonstration

Source: Adapted from Jahn and Leme da Silva (2023)

The analysis in Table 1 suggests that, in the reforms of 1931 and 1942, despite criticism of pedagogical innovations, intuitive geometry was incorporated as a preparation for deductive geometry, in line with the modernization movement. However, in 1951, the experimental and intuitive approach to geometry lost ground in the curriculum. Analysis of the three regulations points to resistance to the introduction of intuitive geometry and the full incorporation of modernizing precepts. In 1951, this resistance culminated in the exclusion of the space previously reserved for preparation based on experiments, which was fundamental to support a formal validation process that included analytical reasoning, deductive concepts, and, above all, the need for proofs and demonstrations.

³ Almeida Lisboa, Joaquim Ignácio de. The Mathematics programs of Colégio Pedro II. *Jornal do Commercio*, Rio de Janeiro, Dec. 21, 1930.

3 Second modernizing movement

According to Kilpatrick (2012), the period between 1950 and 1970 was marked by several attempts to change mathematics education under the name of “modern mathematics” or “new mathematics” (MM), with the aim of aligning school mathematics with 20th-century scientific mathematics. The creation and development of textbooks by *the School Mathematics Study Group* (SMSG) in 1958 in the United States of America (USA), and the *Royaumont Seminar*, organized by the Organization for European Economic Cooperation (OEEC) in 1959 in France, followed by the publication of the Modern Program for Secondary Education, are important milestones in the Brazilian context.

The teaching of geometry was the subject of intense debate, with at least two significant trends emerging. The first, proposed by Birkhoff⁴, advocated modifying Euclid's axioms while preserving the structure of Euclidean geometry. The second trend recommended the development of geometry through geometric transformations, an approach that, in our view, is related to the ideas of Felix Klein⁵.

In 1960, Professor Osvaldo Sangiorgi (1927-2017), author of widely circulated textbooks in Brazil, completed an internship in the United States. Upon his return in 1961, he created the Mathematics Education Study Group (GEEM) in São Paulo, inspired by Birkhoff's perspective. The first collection of textbooks published with the modern approach was developed by Sangiorgi and had a major impact, especially in the state of São Paulo (Valente, 2008). At the same time, the second trend in geometry via geometric transformations (TG) also found its way into Brazilian textbooks. The study by Leme da Silva and Jahn (2024) analyzed four textbooks by authors considered leaders in the Modern Mathematics Movement (MMM) in Brazil and concluded that there was a diversity of approaches: two of them included GT in the appendix; one presented a complete and formal treatment of GT (with one of the main authors having visited several European countries in the 1950s); and a fourth incorporated only two transformations (axial symmetry and homothety), aiming to integrate them into Euclidean geometry, considering their potential to facilitate the study of the notions of congruence and similarity of figures.

Also in 1961, LDB 4.024 decentralized national programs, granting states autonomy to develop their own curricula. In this context, the state of São Paulo was a pioneer in incorporating the MMM's ideas into its program, published in 1965 and discussed within the GEEM. The 1965 program resumed the intuitive study of geometry only in the 1st grade of middle school, maintaining deductive geometry in the 3rd grade, for 13-year-old students.

To investigate how the discussions and proposals of the modernizing movements materialized in the teaching materials of the time, the following section is dedicated to the analysis of four representative books from the period between 1930 and 1967, intended for teaching geometry to 13-year-old students.

4 Analysis of proofs and demonstrations in textbooks

This research examines the approach to proofs and demonstrations in two distinct sets of textbooks, organized chronologically and aligned with the modernization movements in mathematics education in Brazil. The selection of this *corpus* was based on criteria of representativeness and national circulation. Works were selected from authors who played a

⁴ George David Birkhoff (1884-1944): American mathematician; he proposed an alternative axiomatization of Euclidean geometry to that of Hilbert, consolidated in his work *Basic Geometry* (1941).

⁵ Felix Klein (1849-1925): German mathematician; author of the Erlangen Program, which proposed the systematization of geometry through the lens of group theory and geometric transformations.

leading role in the curricular debates of their periods: Euclides Roxo, articulator of modernizing ideas in the 1930s, and Osvaldo Sangiorgi, central leader of the MMM in the 1960s.

The study acknowledges that the selection of works is centered on Rio de Janeiro and São Paulo, the hubs that guided the main reforms of the period. Thus, the research does not intend to exhaust the entirety of national didactic production, nor to ignore the regionalisms and particularities of a country of continental dimensions such as Brazil, which could present distinct dynamics of appropriation. The objective, therefore, is to highlight how validation models were proposed in works of significant normative influence at the time, comparing them with current national guidelines.

The first set comprises the works of Roxo, Mello e Souza, and Thiré (1936, 1944) – hereinafter referred to as Book 01 and Book 02 – which enable us to examine the appropriations of the first modernizing movement. The second set covers two works by Sangiorgi (1958, 1967) – referred to as Book 03 and Book 04 – and allows us to analyze the appropriations brought about by the MMM. The wide circulation of these collections, attested to by their successive editions, gives the sample relevance for understanding the school culture of the time.

Access to the books was achieved through the digitization of the original works held at CEMAT-GHEMAT Brasil⁶ by members of the GEPGE research group⁷. The files obtained in this digitization process were formatted to facilitate reading on electronic devices and with the application of Optical Character Recognition (OCR), making the file searchable. To ensure reliability, the data collection cross-referenced the full qualitative reading of the geometry chapters with systematic searches for key terms ('theorem', 'proof', 'demonstration', 'justification' and their variations) in the digital files. This procedure enabled the exhaustive mapping of validation situations in both the theoretical body and the exercises.

In methodological terms, the analytical procedures were structured into categories that sought to interpret the nature of the validation processes: (i) the mode of introduction of deductive geometry, observing whether the proposition stems from intuitive or experimental activity or from axiomatic presentation; (ii) the proof methods employed, focusing on the language and representations adopted and the formal structure of the proof; (iii) the analysis of an exemplary case, specifically the demonstration of the theorem of the base angles of isosceles triangles; and (iv) the type and quantity of exercises, distinguishing the direct application of formulas from the requirement of hypothetical-deductive reasoning.

5 Textbooks from the 1930s and 1940s

Books 01 and 02 show remarkable similarities in their approach to geometry, devoting approximately half of their content to this topic. The authors present an introduction to the study of deductive geometry, consisting of an explanatory text containing definitions and principles of a deductive axiomatic system for geometry. The terms “proposition,” “axiom,” “postulate,” “theorem,” and “definition” are highlighted. The proof of a theorem “consists of *proving* the proposition, showing, through a chain of reasoning, that the thesis necessarily results from the hypothesis” (Roxo *et al.*, 1936, p. 202, emphasis added).

In Book 01, the authors warn that “it is not possible to establish a general method for proving all theorems in geometry” (Roxo *et al.*, 1936, p. 206), and announce the presentation of various forms of proof (with emphasis on *proof by reduction to absurdity*) in subsequent chapters, thus aiming to promote the development of deductive thinking. In turn, Book 02

⁶ Center for Scientific and Pedagogical Documentation of Mathematics Education (<https://www.cemat-ghemat.com/>)

⁷ Study and Research Group on School Geometry: History and Teacher Training, registered with CNPq and led by Prof. Maria Célia Leme da Silva.

introduces two additional topics on mathematical validations: “Methods of demonstration,” which covers direct and indirect proofs, and “Reduction to absurdity,” illustrated with examples of demonstrations. The distinction in approach between the two books is clearly evident in the exercises proposed to students.

The introductory chapter of Book 01 ends with a list of 12 fundamental propositions accepted without proof. The authors clarify that these include “simple postulates; others are theorems that have already been proven; others, finally, could be proven, but since we cannot be concerned with absolute rigor at this stage of the course, we prefer to accept them without proof” (Roxo *et al.*, 1936, p. 209). In Book 02, eight postulates are presented (two relating to the line, four to the plane, and two to the invariability of figures), assuming the function of a “set of postulates” that underpin the deductive development of geometry in subsequent chapters. As this is an introduction to deductive geometry, in both books, the text is essentially expository in nature, covering definitions and examples.

Due to the space limitations imposed by this paper, we have restricted our detailed analysis to the study of triangles, the inaugural topic that addresses proof in geometry in the books examined. We specifically investigated aspects related to deductive geometry, including the presentation of theorems and proofs, and the presence of validation practices (justifications, arguments, proofs, or demonstrations) in the exercises proposed to students, as summarized quantitatively in Table 2. A notable difference between the two volumes lies in the number of solved and proposed exercises⁸.

Table 2: Theorems and demonstration exercises in the study of triangles

Book	Properties Demonstrated	Demonstration exercises	Total Exercises
Book 01 (1930s)	9	3 proposed and 1 solved	4
Book 02 (1940s)	13	30	32

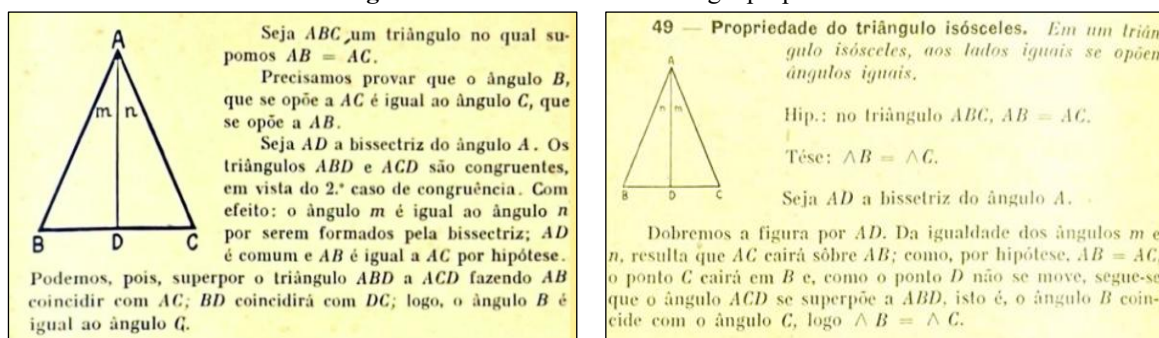
Source: Prepared by the authors.

The comparative analysis between the two books confirms the previous observations: there is an increase in the number of theorems demonstrated and, significantly, an expansion in the set of exercises proposed in general, with an emphasis on those that require the elaboration of proofs or demonstrations. In Book 02, the terms “demonstrate,” “prove,” and “show” are used as synonyms in the statements of exercises that require deductive justification.

In the study of triangles, the notion of congruence is introduced, together with cases of triangle congruence, presented in the form of theorems, all demonstrated in both books. Other properties of triangles are subsequently addressed, with the application of congruence cases being the predominant method for developing their respective proofs. Figure 1 illustrates the proof of the property of equality of base angles in an isosceles triangle, presented comparatively in Book 01 (left) and Book 02 (right).

⁸ Book 01 shows signs of being intended for teachers, while Book 02 more explicitly announces its intended use by students. This difference may be one of the reasons for the reduced number of exercises in Book 01.

Figure 1: Proofs of isosceles triangle properties



Sources: Roxo *et al.* (1936, p. 235); Roxo *et al.* (1944, p. 168)

There are some differences between the proofs. In Book 01, the authors write in natural language, explaining the starting point and what they want to prove. The proof is justified by using the second case of congruence, but it also appeals to the “superposition” of triangles. In Book 02, the terms “hypothesis” and “thesis,” absent in Book 01, are explained, and the proof is presented in a more concise language; to indicate the angles, triangle congruence is not used; the justification is supported by intuition and the use of action resources, as in the expression “Let’s fold the figure” (Roxo *et al.*, 1944, p. 168).

In the exercises in Book 02, before students are asked to perform tests or demonstrations, exercises such as “Separate the hypothesis and thesis in each of the following propositions, translating them symbolically when possible” (Roxo *et al.*, 1944, p. 138), reinforcing the starting point and destination and mathematical language.

In the analysis of the didactic works Book 01 and Book 02, no tasks are identified that are specifically aligned with the approach of *intuitive, lively, and concrete geometry*, as recommended by the educational standards of the period in question. However, the proposal of exercises that required students to separate hypothesis and thesis (1940s) can be interpreted as an indication of preparation for the elaboration of demonstrations. In any case, the processes of validation in geometry are intrinsically linked to the demonstration of theorems, characterized by the logical-deductive chain of arguments that support the validity of the desired result. Another relevant aspect to consider is the reduced number of exercises directed at students in Book 01, which may suggest that the practice of validation in geometry was conceived as an activity intended for teachers, rather than a practice to be internalized by students. In short, in the books analyzed, the emphasis in geometry teaching lies in the validation process, which is materialized in the demonstration of theorems.

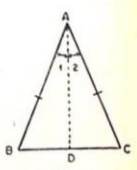
6 Textbooks from the 1950s and 1960s

As already discussed, educational regulations in the 1950s restricted the teaching of geometry to the final two grades of middle school (students aged 13 and above). In Book 03, Sangiorgi (1958) explains that the intuitive or experimental approach to geometry was covered in primary school, with secondary school reserved for more advanced study, namely deductive geometry.

We can observe changes compared to Book 02, from the 1940s. The mathematical language used in the hypothesis and thesis is even more concise, and the proof process is justified by numbered steps. The term “Let us double the figure” (Figure 1, right) is replaced by “Let us draw the bisector of the angle at vertex A ” (Figure 2, right), the justifications are made in parentheses, and at the end, the acronym *c.q.d* (as we wanted to demonstrate). The appeal to the movement and superimposition of triangles is no longer evident; the justifications

highlight definitions and properties already stated.

Figure 2: Proofs of isosceles triangle properties

<p>38. Propriedades do triângulo isósceles.</p> <p>a) <i>Teorema:</i> Em todo triângulo isósceles os ângulos da base são iguais.</p> <p>Seja o triângulo ABC (fig. 72). Temos:</p> <p style="margin-left: 20px;">H $\{ AB = AC$</p> <p style="margin-left: 20px;">T $\{ \hat{B} = \hat{C}$.</p>	 <p style="text-align: center; font-size: small;">FIG. 72</p>	<p>DEMONSTRAÇÃO :</p> <ol style="list-style-type: none"> 1. Tracemos a bissetriz do ângulo do vértice A que encontra BC no ponto D. Logo : $\hat{1} = \hat{2}$ (def. de bissetriz). 2. Os triângulos ABD e ADC são iguais, pelo 1.º caso de congruência (L.A.L.), e, portanto, são necessariamente iguais os ângulos correspondentes \hat{B} e \hat{C}. Logo : <p style="text-align: center; margin-left: 20px;">$\hat{B} = \hat{C}$ c.q.d.</p>
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Source: Sangiorgi (1958, pp. 119-120)

We also identified in Book 03 (1950s) that proofs and demonstrations are presented using symbolic language, with more formal aspects in their content. It is plausible that this increasing formalization contributed to the discussion about the difficulties students had in understanding geometric proofs, a debate that gained ground at the 1957 National Congress on Mathematics Education. In this context, the practice of “memorizing theorems” as a strategy used by students to pass assessments was denounced (Búrigo, 2015).

Leme da Silva and Jahn (2024) conducted a comparative study of Sangiorgi's Books 03 and 04 with the aim of examining how deductive geometry was constituted in each period and found a significant difference between them. In Book 04, there is a reduction in the total number of geometry topics, along with a significant increase in the number of pages. The authors argue that, in Book 04, the same concepts are addressed from a new perspective, both in relation to geometric knowledge and the methodological approaches introduced. While Book 03 maintains an approach similar to that of Book 02, including relative quantitative equivalence in terms of the theorems demonstrated and the exercises and proofs proposed, Book 04 presents substantial differences and a considerable reduction in the number of theorems demonstrated. Regarding the specific topic of the study of triangles, the central focus of this analysis, Table 03 provides a quantitative summary.

Table 03: Theorems and proof tasks in the study of triangles in Books 03 and 04

Book	Properties Demonstrated	Demonstration Exercises	Total Exercises
Book 03 (1950s)	16	25	26
Book 04 (1960s)	07	14	33

Source: Prepared by the authors.

It should be noted that, in Book 04, the study of triangles is done before the introduction of deductive geometry, in a previous chapter devoted to intuitive and experimental geometry. There are no demonstrations of theorems, but rather a predominance of “exploratory exercises,” such as the construction of an isosceles triangle and the exploration of its existence and the measurements of the base angles (cf. Figure 3) so that students can “discover” (or conjecture) the properties involved in an experimental manner.


Another exploratory practice identified in the analyzed work lies in the proposition of exercises in which students are encouraged to construct triangles, varying the conditions of construction (sides and angles), and to conduct systematic comparisons between the figures obtained. The pedagogical objective underlying this practical investigation is for students to empirically verify the congruence of triangles in certain situations, enabling the subsequent acceptance of these cases as axioms and their subsequent application in the deduction of other geometric properties.

Figure 3: Exploratory exercise on isosceles triangles

1. **Construção de triângulos isósceles:** Considere um qualquer segmento AB como base do triângulo isósceles que você quer construir. Escolhido um comprimento dado pela abertura de um compasso, basta determinar um dos pontos C^* , intersecção das circunferências traçadas quando se fixa a ponta do compasso respectivamente nos pontos A e B .

a) Será que existe sempre o ponto C ?
 b) Quando é que "não existe"?
 c) Por que o triângulo obtido é isósceles, quando existe C ?

2. **Um resultado importantíssimo para os triângulos isósceles:**
 De cada triângulo isósceles que você construiu, meça com o transferidor os ângulos da base. Que observou? Se um dos ângulos da base de um triângulo isósceles mede 40° , por exemplo, quanto medirá o outro ângulo da base? Será também 40° ? "Explore" esse resultado com outros triângulos isósceles.



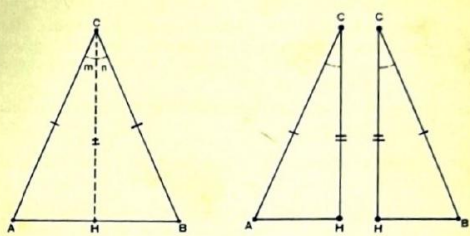
Source: Sangiorgi (1967, p. 207)

The proof of the base angle theorem of an isosceles triangle illustrates the changes proposed by Sangiorgi in Book 04. Only after proposing numerous exploratory exercises is the proof presented. As mentioned, the cases of congruence, proven as theorems in Books 01, 02, and 03, are considered postulates in Book 04, after having been studied in the exploratory exercises.

In addition to the introductory study, the author uses new methodological strategies for the proofs, such as the demonstration on the left in Figure 4, which separates the triangles, organizes the justifications, and at the end, writes "as we wanted to demonstrate"; the demonstration using drawn diagrams (Figure 4, right) and the demonstration in two columns (*statements and justifications*, cf. Figure 4, center).

Figure 4: Demonstrations of the properties of isosceles triangles (Sangiorgi, 1967, pp. 240-241)

Se for a bissetriz, esta irá dividir o ângulo \hat{C} em dois ângulos congruentes, isto é, de medidas iguais ($m = n$), ensejando a formação dos triângulos: ACH e CHB , nos quais figuram os dois ângulos \hat{A} e \hat{B} que nos interessam.



Confrontando esses triângulos, observa-se que eles possuem:

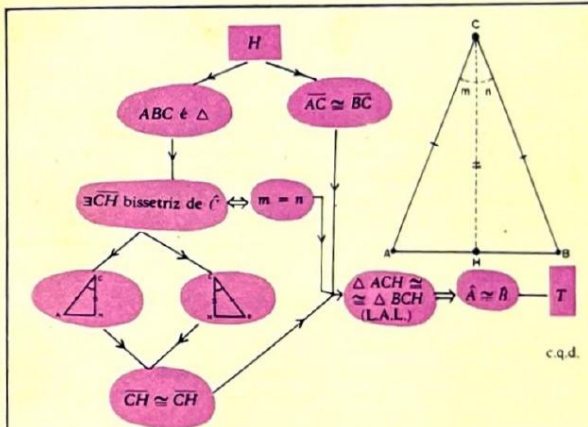
- $\overline{AC} \cong \overline{BC}$ (por hipótese) (L)
- $m = n$ (por construção da bissetriz) (A)
- $\overline{CH} \cong \overline{CH}$ (por ser lado comum) (L)

e, portanto, são triângulos congruentes pelo 1.º Caso (L.A.L.).

Dêsse modo os ângulos \hat{A} e \hat{B} , como correspondentes de triângulos congruentes, são congruentes, isto é: $\hat{A} \cong \hat{B}$, como queríamos demonstrar.

"SE um triângulo é isósceles, ENTÃO os ângulos da base são congruentes"

sua demonstração será esquematizada da seguinte maneira:



c.q.d.

DEMONSTRAÇÃO:	
Afirmações	Justificações
1) \overline{CH} é bissetriz de \hat{C} , ou seja, $m = n$	1) Todo ângulo admite uma bissetriz e, portanto, pode-se construir \overline{CH} .
2) $\triangle ACH \cong \triangle BCH$	2) Caso L.A.L. $\left\{ \begin{array}{l} \overline{AC} \cong \overline{BC} \text{ (p/hipótese)} \\ m = n \text{ (p/construção)} \\ \overline{CH} \cong \overline{CH} \text{ (lado comum)} \end{array} \right. *$
3) $\hat{A} \cong \hat{B}$	3) Ângulos que se correspondem em triângulos congruentes
	c.q.d.

Source: Sangiorgi (1967, pp. 240-241)

The way Sangiorgi (1967) proposes the study of proofs and demonstrations for 13-year-old students is unprecedented in Brazil and was most likely conceived based on the contact he established in the US with study groups, American textbooks, and researchers involved in Modern Mathematics.

Herbst's (2002) research provides a relevant context for analyzing Sangiorgi's innovation by documenting the consolidation of the two-column proof structure as a common pedagogical practice in American textbooks since the early 20th century, which has remained in use for a long period. In the Brazilian educational scenario, Sangiorgi stands out as one of the pioneers in introducing this specific format. This adoption, most likely aligned with his goal of presenting different ways of constructing a proof, highlights his appropriation of his experiences in the United States. However, it is important to note that the two-column representation, by emphasizing the formalization of deductive reasoning through the explanation of the logical structure of the steps and their respective justifications, represents only one of the innovative strategies employed by Sangiorgi, who also introduced more than one method to demonstrate the same theorem, moving away from rote memorization and seeking to develop students' ability and autonomy to construct proofs.

Sangiorgi also highlights in Book 04: “Don't ‘memorize’ theorem proofs! Value yourself by using any of the methods presented. Add your own ‘touch’ when employing these methods and you will be fulfilling yourself in mathematics!” (Sangiorgi, 1967, p. 258). Certainly, inviting students to construct their own proofs means making it clear that they can create personal ways of proving theorems, other than those found in books or presented by teachers.

An examination of Books 03 and 04 reveals evidence of change in validation practices in geometry. While Book 03 emphasizes more formal processes, presenting a significant increase in the number of theorem demonstrations compared to Books 01 and 02, reiterating the validation process as similar to the practice of mathematicians, with a certain degree of rigor and based on mathematical logic, in a unique and structured way, Book 04 breaks with this model and proposes the insertion of innovative practices, such as exploratory exercises, which invite students to perform “tests” (indicated in quotation marks), which meant validating results found in particular situations. This is what we try to exemplify with the exploratory practices already mentioned for experimentation, through geometric constructions, conditions of existence and properties of isosceles triangles, and cases of triangle congruence.

7 Final considerations

Although regulations from 1930 onwards record the circulation of pedagogical ideas from the first modernizing movement, with an emphasis on the need for an introductory intuitive study, at the end of the 1950s, the legacy of secondary education dominated by a deductive geometry prevailed, offering resistance to the appropriation of new ways of validating geometry. There is evidence that students resorted to memorizing theorems and their proofs, with little or no prior experimental study (Búrigo, 2015).

An analysis of textbooks also indicates a significant number of geometric proofs in Books 02 and 03 from the 1940s and 1950s, accompanied by many exercises that required students to provide proofs. In summary, the teaching of geometry in the late 1950s and early 1960s was characterized by a strong emphasis on the logical-deductive process, which, as pointed out by Jahn and Leme da Silva (2023), often culminated in students memorizing theorems.

The second movement, referring to the MMM ideology, seems to have presented

different processes of appropriation in the textbooks analyzed. With regard to the introduction of proofs and demonstrations in geometry, they were not highlighted in the 1965 regulations, nor was there an emphasis on the insertion of intuitive geometry or significant changes in geometric validation processes. On the other hand, Book 04, which announced the modern approach, clearly indicated an attempt to break with established practices of reproducing proofs of countless theorems, without prior experimentation and, possibly, without meaning for students, turning into memorization of “ready-made” theorems, simply “revealed” to them.

When analyzing the period from the 1930s to the 1950s, we identified the circulation of ideas from the first movement, which were incorporated into the programs; however, the persistence of traditional practices in the teaching of geometry, constructed and represented since the 19th century, stands out. An examination of textbooks allows us to infer that Euclidean proofs were considered the standard of rigor, even though intuitive geometry was recommended in the 1930s and 1940s standards. This was a moment of tension between modernizing and traditional conceptions, expressed by teachers, textbook authors, and students, as identified in debates within the educational community, highlighting the complexities of the processes of appropriation and resistance.

Everything indicates that a second movement, evident from 1960 onwards, was necessary for textbooks to be able to offer different approaches to the teaching of proofs and demonstrations. Sangiorgi's book 04 can be considered an innovative manual in terms of the treatment of proofs and demonstrations. Certainly, contact with American culture and the international debate that circulated during the MMM supported Sangiorgi's modernizing proposal, favoring the circulation and appropriation of new practices.

Sangiorgi stood out for introducing an innovative approach by presenting different methods for proving the same theorem, which differed significantly from the traditional emphasis on rote memorization. This methodological flexibility aimed to encourage students to develop the ability to construct proofs as a fundamental skill. Among the teaching strategies he employed, two particularly innovative formats stand out. One was the two-column proof, a model that gained popularity in the United States during the 1960s (Herbst, 2002), which prioritized the explanation of logical steps and their respective justifications, emphasizing the deductive structure of proofs/demonstrations. The other format was the 'drawn diagram', as the author himself mentioned, inspired by the works of Lucienne Félix⁹. Conceived as teaching tools, both practices aimed to promote the activity of proving or demonstrating as a central and meaningful task for students in their learning process. Such innovations by Sangiorgi highlight the unique appropriations he made from his contact with the modernizing ideas in vogue in both the United States and Europe.

The comparative analysis of the textbooks examined therefore points to a gradual but notably non-linear trajectory of incorporation of modernizing elements in the teaching of geometry. Sangiorgi's work emerges as a crucial turning point in this trajectory, proposing and implementing methodologies that prioritized active understanding and the construction of geometric knowledge by students. This approach represented a striking counterpoint to the mere passive reproduction of demonstrations of previously established theorems. The trajectory observed here illustrates the complex dynamics of circulation, the resistance encountered, and the distinct processes of appropriation of pedagogical ideas over time in the Brazilian context. This rich historical narrative offers valuable insights for contemporary discussions about the

⁹ Lucienne Félix (1901-1944), a mathematician who graduated from the École Normale de Sèvres, was a prominent figure in mathematics education in France and in the dissemination of the ideas of Modern Mathematics. She worked at the *Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques* (CIEAEM), a group coordinated by Caleb Gattegno, and played a key role in expanding these ideas, including through visits to Brazil in the 1960s.

teaching of geometry and, in particular, the way in which proofs and demonstrations are approached in mathematics education. This reaffirms the importance of seeking a balance between the logical rigor inherent in mathematics and the need for exploration, intuition, and visualization, which are essential elements for meaningful learning. All of this must be considered in light of the intricate processes of appropriation that unfold in different cultural contexts.

The project continues to investigate validation processes in geometry, particularly after the 1960s, which, based on the results now reported, indicate changes in validation practices in geometry for 13-year-old students. Similarly, other historical moments and segments of education are planned to be investigated, in addition to the expansion and comparison of other sources of historical research.

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Conflicts of Interest

- The authors declare that there are no conflicts of interest that could influence the results of the research presented in this paper.

Data Availability Statement

- The paper was based on a theoretical study, so there is no data to be made available.

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