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The concept of exponential in real analysis books: a perspective of intuition and rigor point of view

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Abstract: In this paper, we aim to examine the approach adopted in Real Analysis textbooks used by Brazilian universities in the construction of the concept of a^x, and to present a more intuitive construction based on the notions of supremum, infimum and the completeness of the real number system. As a theoretical framework, we discuss the relationship between intuition and rigor in the context of teaching Real Analysis. This qualitative study involves documentary analysis of textbooks. We conclude that the most common constructions in Real Analysis textbooks, although rigorous and elegant, typically rely on formal definitions of exponential and logarithmic functions, which are not very intuitive, because they require many previous concepts. The alternative construction we present prioritizes intuition, offering students greater opportunities for visualization and creativity. It allows them to follow a pedagogical and rigorous path to the definition of the exponential function, which is fundamental for future mathematics teachers and mathematicians.

Keywords: Concept of Exponential Function. Real Analysis Textbooks. Intuition. Rigor.

O conceito de exponencial em livros de Análise Real: um olhar sob o ponto de vista da intuição e do rigor

Resumo: Objetivamos investigar a abordagem de alguns livros didáticos de Análise Real utilizados nas universidades brasileiras na construção do conceito de a^x, e apresentar outra construção mais intuitiva, que se utiliza dos conceitos de supremo e ínfimo, além da completude dos números reais. Como enquadramento teórico, discutimos a relação entre intuição e rigor, à luz do ensino de Análise Real. A metodologia qualitativa contemplou uma pesquisa documental por meio da análise de livros didáticos. Nos resultados, concluímos que as construções mais comuns nos livros didáticos, embora sejam matematicamente rigorosas e elegantes, recorrem a uma definição formal das funções exponencial e logaritmo natural, porém pouco intuitivas, pois demandam muitos outros conceitos prévios. Já a construção apresentada privilegia a intuição, proporcionando visualização e criatividade por parte dos alunos, o que lhes permite trilhar o caminho para uma didática e rigorosa definição do conceito de exponencial, fundamental na formação de professores e bacharéis em Matemática.

Palavras-chave: Conceito de Função Exponencial. Livros Didáticos de Análise Real. Intuição.

Rigor.

El concepto de exponencial en libros de análisis real: una mirada desde el punto de vista de la intuición y el rigor

Resumen: En este artículo, nos proponemos investigar el enfoque de algunos libros de texto de Análisis Real utilizados en universidades brasileñas para la construcción del concepto de a^x, y presentar una construcción alternativa más intuitiva, que se basa en los conceptos de supremo e ínfimo, así como en la completitud de los números reales. Como marco teórico, discutimos la relación entre intuición y rigor, a la luz de la enseñanza del Análisis Real. La metodología, de carácter cualitativo, consistió en una investigación documental mediante el análisis de libros de texto. Como resultado, concluimos que las construcciones más comunes en los manuales, aunque matemáticamente rigurosas y elegantes, recurren a una definición formal de las funciones exponencial y logaritmo natural, la cual resulta poco intuitiva por requerir numerosos conceptos previos. En cambio, la construcción que presentamos privilegia la intuición, favoreciendo la visualización y la creatividad por parte del estudiantado, y permitiéndole recorrer un camino tanto didáctico como riguroso hacia la definición del concepto de exponencial, fundamental en la formación de docentes y licenciados en Matemáticas.

Palabras clave: Concepto de Función Exponencial. Libros de Análisis Real. Intuición. Rigor.

1 Introduction

One of the earliest mentions of the potentiation operation appears in an Egyptian papyrus dated to about 2100-1580 BC, which discusses the calculation of the volume of a quadrangular pyramid (Oliveira & Ponte, 1999). A few millennia have passed since then, until a rigorous formalization of the expression a^x for every $x \in \mathbb{R}$ with restriction $a > 0$. Of course, we will consider, in general, $a \neq 1$ to discuss the bijectivity of the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = a^x$.

A good discussion, from a historical perspective, of the construction of the concept of power can be found in Oliveira and Ponte (1999). Setting a^x to $x \in \mathbb{R}$, with $a > 0$ and $a \neq 1$ requires a long road and goes through the fact that $\mathbb{R}$ is a complete ordered body. However, defining a^x, when x is a rational number, is something relatively simple in the context of Real Analysis. The step-by-step usually starts by defining a^n, when $n \in \mathbb{N}$; in this case, $a^n = a \cdot a \cdots a$ (n times). Note that here there is no restriction for $a \in \mathbb{R}$ and, from this definition, the following properties are obtained almost naturally:

$$a^{n+m} = a^n \cdot a^m \text{ and } (a^n)^m = a^{nm} \qquad (1)$$

To maintain these properties, we define $a^{-n} = \frac{1}{a^n}$. Subsequently, using the notions of supreme and minimum, the *nth* root of $a > 0$ is defined and, from it, $a^{\frac{n}{m}} = \sqrt[m]{a^n}$ is determined, in which n and m are integers. With this definition, the properties presented in (1) are maintained. However, the difficulty in having a good definition of a^x to preserve such properties, lies in the case in which x is irrational.

When analyzing a set of Real Analysis textbooks, described later, we observed that all adopt the strategy of defining the exponential function, here denoted by exp, and natural logarithm, here denoted by ln and from these functions, establish a^x when $x \in \mathbb{R}$. The constructions of such functions are made using strong results of Real Analysis and employ either in the definition or in obtaining properties of the functions exp and ln, as well as limits, continuity, derivatives, power series, integrals, sequences and series of functions and convergence. Once such functions are built, it is stipulated for $a > 0$ and $a \neq 1$:

$$a^x = exp(xln(a)) \text{ and } log_a x = \frac{ln(x)}{ln(a)} \quad (2)$$

being the second expression of (2) the logarithm of x in base a. The $log_a x$ has the property of being the inverse function of a^x. All properties of a^x and $log_a x$ are obtained from those of exp and ln. Although it is mathematically rigorous and allows us to obtain all the properties of Differential and Integral Calculus, the approach presented in the analyzed books seems to us to be unintuitive, so that we avoid working with the exponential and logarithm functions until all the concepts of Real Analysis are introduced or, in another bias, the functions are used without previously defining them, as happens in Chapter 3 of Figueiredo (1996), in the context of the Taylor series, and in Chapter 3 of Lima (2002b), in an example of infinite limits.

In this article, we discuss the construction of a^x made in Real Analysis textbooks used in Brazilian universities, in addition to another didactic construction using the supremum and the infimum of a set (to be chosen), and the completeness of the set of real numbers. This construction appears as an exercise in Rudin (1976) and Bartle and Sherbert (2011) and seems to us more intuitive, being a shorter way to understand the meaning of a^x, without necessarily resorting to various results of Real Analysis. However, before starting the discussion, we define intuition and explain how it is present in the teaching of Real Analysis, intertwined with the rigor inherent in such teaching. It should also be noted that the analyzed textbooks were chosen not only because their authors are acknowledged in the national and international academic context, but also because they are widely used in Brazilian universities, appearing as basic or complementary bibliography for various subjects involving Real Analysis (Reis, 2009).

2 Intuition and its relationship with rigor in teaching Real Analysis

In the context of university mathematics teaching practice, especially in the teaching of Real Analysis, the relationship between rigor and intuition is conceived as dichotomous, in the sense that, by privileging a more rigorous approach in the presentation of a concept, a definition or a theorem, this necessarily means that intuitive aspects must be abandoned almost entirely or, at least, relegated to a lower plane. However, we defend a complementary relationship between rigor and intuition, in the sense that both processes are equally fundamental to the construction of analytical knowledge. We believe it is important to present our conception of intuition and rigor.

Several authors consider the cognitive aspects related to intuition, characterizing it in different ways. For Tall and Vinner (1981), it is the product of conceptual images, cognitive structures associated with the construction of a concept, including mental processes and mental images. Poincaré (1913) affirms that there are many types of intuitions: first, by the senses and imagination; then, generalization by induction, copied, as it were, from the procedures of the experimental sciences; finally, the intuition of a “pure number”. According to Fischbein (1978), there are two types of intuition: the primary ones, which refer to cognitive beliefs developed by human beings themselves, naturally, before and independently of systematic instruction; and the secondary ones, which are developed as a result of systematic intellectual education.

In the present work, we will also consider Reis’ (2001) dynamic conception of intuition as a movement to establish plausible (sometimes true, sometimes false) reasoning, and of rigor as a movement to conceptualize intuitions, in which some are maintained, and others are refuted. In addition, in the context of Real Analysis teaching, intuition highlights other mental exercises, such as visualization and imagination, which cannot simply be overcome by the association with rigor, for example, formalization and generalization; that is, one must consider what sustains Reis (2009) when reminding us that:

> "Academic rigor", dominant in the world of publications and presentations of works, scientific articles, and others, cannot be transposed in a direct, mechanical, or simplistic way to teaching. This transposition, in fact, should provide a multi-faceted and flexible exploration of concepts, so that they are intuitively meaningful and understandable, with validation and demonstration treatments (i.e., rigor) compatible with the teaching context (institution; degree or licenciate degree; students' prior knowledge; etc.). (p. 93)

It should be noted that "academic rigor", regarding the researcher, is directly related to the Real Analysis teaching, since, according to the researcher, the fundamental topics of the Real Analysis subject are the same as those of Calculus; however, if in Calculus the topics are approached from an applied perspective, with the intuitive interpretation of notions. In Real Analysis, they are usually approached from a logical-formal perspective, with the rigorous definition of the concepts, definitions, and results studied.

Ávila (1999), in the preface to his book *Introdução à Análise Matemática* [Introduction to Mathematical Analysis], states that a historical analysis shows that, since the emergence of the Calculus, with Newton and Leibnitz in the seventeenth century, several famous mathematicians, such as the brothers Bernoulli, Euler, D'Alembert, and Lagrange, unsuccessfully tried to give Calculus a rigorous formulation, which was only obtained in the early nineteenth century, with Dedekind and Peano, after the fundamental contributions of Cauchy and Weierstrass. Ávila (1999, p. 22) also presents Dieudonné's argument that imputes the "lack of rigor" of the mathematicians of the eighteenth century to the difficulties they faced in precisely defining the basic notions of Calculus, while highlighting that they had a good intuitive conception of these notions. However, the researcher believes that it was exactly this intuitive conception that led those mathematicians to establish important results, which, according to him, justifies the need for a balance between the rigor necessary for Real Analysis and the intuition so fundamental to the development of mathematical ideas.

Finding the "balance point" that, perhaps, is the watershed in the teaching of Real Analysis — placing, on the one hand, intuitively constructed differential thinking and, on the other, formally constructed analytical thinking — does not seem like an easy task. Reis (2009), when analyzing the subjects of Real Analysis taught at graduation, notes that, in a good part of them, there is an excess of formalism and rigor in the exposition of the themes; in view of this practice, students often remain only to memorize the main results and their demonstrations that should have been intuitively understood in the Calculus.

Again, then, we are faced with the need to better understand how rigor and intuition are explored in Real Analysis textbooks, considering, according to Reis (2001):

The fundamental role that the textbook plays within the configuration of the public curriculum of Real Analysis, because in the programs of the subjects of Real Analysis of any Brazilian university, certainly, one or more books are made available as bibliographic references and, in practice, some of them end up being, in fact, adopted as the main reference for the development of the entire subject;

The undeniable role that the textbook plays as an exponent of the particular knowledge of its authors, since a book carries not only a synthesis of specific knowledge about Real Analysis, but also a huge range of mathematical and pedagogical knowledge of its authors, whose combination culminates in the experience of writing a book of mathematical content that proposes to be didactical.

In this perspective, due to the purposes of this work, we follow with the approach of the concept of exponential in real aalysis textbooks, seeking its relations with exponential functions and logarithms, in such a way that, later, we can compare how the search for rigor was a determinant in each approach, on the one hand, and, on the other hand, how each approach can be considered more or less intuitive, from the dynamic conception of intuition that we are considering.

3 Exponential functions, natural logarithm, a^x and $log_a x$ in Real Analysis books

As we described, we analyzed how the exponential and natural logarithm functions are introduced in the textbooks listed below. The choice for these books was made not only because of our previous experience as Real Analysis professors teaching in undergraduate and graduate subjects, but also because they are considered "classic" books, which, as such, integrate the bibliographic references of programs and, in fact, are adopted as the main reference in subjects of Real Analysis offered in several mathematics courses (degree and licenciate degree) in Brazil.

Table 1: Real Analysis Textbooks

Title	Author(s)	Edition	Year
Análise I [Analysis I]	D. G. de Figueiredo	2nd	1996
Análise Real – Volume 1 [Real Analysis]	E. L. Lima	6th	1996
Curso de Análise – Volume 1 [Analysis Course]	E. L. Lima	10th	1996
Introduction to Real Analysis	R. G. Bartle and D. R. Sherbert	4th	2011
Principles of Mathematical Analysis	W. Rudin	3rd	1976

Source: Prepared by the authors.

The strategy used in the analyzed books is first to define the exponential function or the natural logarithm function and, in a second moment, to obtain the second from the demonstrated properties as the inverse of the first. After constructing the exponential and natural logarithm functions, all the books described in Table 1 define:

$$a^x = e^{xln(a)} \text{ and } log_a x = \frac{ln(x)}{ln(a)},$$

where e^x is the exponential function evaluated in x , and $ln(x)$ is the natural logarithm evaluated in x. From this definition, the properties $a^{x+y} = a^x a^y$, $(a^x)^y = a^{xy}$, $log_a(xy) = log_a x + log_a y$, the derivatives and integrals of a^x and $log_a x$ are obtained using those already obtained for the exponential functions and the natural logarithm.

In the books by Rudin (1976) and Bartle and Sherbert (2011), the authors first define the exponential function; in the books by Lima (2002a, 2002b) and Figueiredo (1996), the authors first define the natural logarithm function. Rudin's (1976), Chapter 3, the author initially introduces the number $e = \sum_{n=0}^{\infty} \frac{1}{n!}$; in Chapter 8, he defines, for any complex number z, the function:

$$Exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

From this definition, it is possible to obtain the properties $Exp(z + w) = Exp(z)Exp(w)$ and $(Exp)'(z) = Exp(z)$ for any z and w complexes.

Bartle and Sherbert (2011) state, in Chapter 8, the following result:

Theorem 1: There is a single $Exp: \mathbb{R} \to \mathbb{R}$ differentiable function that satisfies:

1. The derivative of E equals the function itself E, i.e. $(Exp)'(x) = Exp(x)$ for every $x \in \mathbb{R}$;

2. $Exp(0) = 1$.

This result is demonstrated using integrals, sequences, series of functions, and convergence concepts. Through this approach, we define $e = Exp(1)$. Using concepts and results of Real Analysis, both Rudin (1976) and Bartle and Sherbert (2011) present the following properties:

(i) $Exp(x) \neq 0$ for every $x \in \mathbb{R}$;

(ii) $Exp(x + y) = Exp(x)\,Exp(y)$;

(iii) $Exp(p) = e^p$ for any p rational.

Rudin's (1976) approach is entirely for complex numbers and, only then is the constraint for real numbers applied. Property (iii) suggests defining $e^x = Exp(x)$.

In both constructions presented, the following theorem can be obtained, either by definition or by results of Real Analysis:

Theorem 2: The following properties apply:

1. e^x is continuously differentiable;

2. $(e^x)' = e^x$;

3. e^x is strictly increasing;

4. $e^{x+y} = e^x e^y$

5. $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.

From properties 1 and 3, there is an inverse function ln, the natural logarithm, which is strictly increasing and differentiable, whose domain is $\{y \in \mathbb{R}: y > 0\} = \mathbb{R}_*^+$. The Inverse Function Theorem ensures that $(ln)'(y) = \frac{1}{y}$. From this definition, all the known properties of the natural logarithm function are obtained.

On the other hand, Figueiredo (1996) and Lima (2002a, 2002b) follow the reverse path, first defining the natural logarithm function, as follows:

Definition: The function $ln: \mathbb{R}_*^+ \to \mathbb{R}$ given by $ln(x) = \int_1^x \frac{1}{t}dt$ is said to be a natural logarithm.

From the properties of the integral, the defined function has a derivative of all orders, is increasing and concave. In addition, it can be directly obtained that $ln(xy) = ln(x) + ln(y)$. From Real Analysis results, we prove that ln is surjective and is defined $exp: \mathbb{R} \to \mathbb{R}_*^+$ as being the inverse of ln. With this approach, the authors can obtain all the known properties of ln and $exp(x) = e^x$. Once these functions are constructed, a^x and $log_a x$ are defined as in (2).

4 A construction of a^x and $log_a x$ via supremum and infimum

Rudin (1976) and Bartle and Sherbert (2011) suggest, as an exercise, a construction of a^x and $log_a x$ via supremum and infimum, which requires less mathematical tooling to define such functions, but such a construction is not much explored by the authors. However, this construction is detailed in the Real Analysis book by the first author of this article, Martins (in preparation), which is about to be released soon, and which we will now detail.

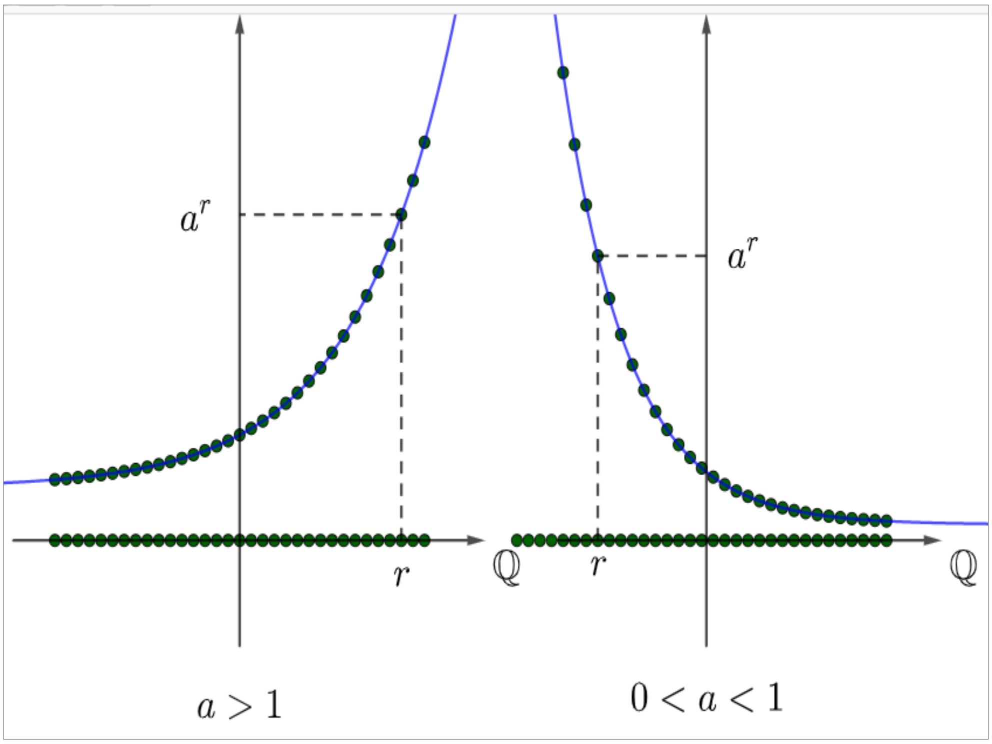
The definition of a^x, when x is a natural number, is made intuitively:a^x is a multiplied by a,x times. To define a^x, when $x \in \mathbb{Z}$, a logical argument is used, in the sense of maintaining the fundamental properties; then, $a^x = \frac{1}{a^{-x}}$. is defined. After designating, via supremum, the notion of nth root, in the sense of ensuring the fundamental properties, $x = \frac{n}{m} \in \mathbb{Q}$, is also determined, in which n and m are integers, $a^x = \sqrt[m]{a^n}$ (let us note that the restriction $a > 0$ is necessary), and the properties of calculation for rational exponents are obtained. For any $r, s \in \mathbb{Q}$, with $r < s$, we have:

1. $a^r < a^s$, if $a > 1$;

2. $a^r > a^s$, if $a < 1$.

Such properties tell us that the function $f: \mathbb{Q} \to \mathbb{R}$ given by $f(x) = a^x$ is increasing, if $a > 1$, and decreasing, if $a < 1$. Figure 1 illustrates an intuitive "graph" of f that is not mathematically rigorous, because, since $\mathbb{Q}$ is dense in $\mathbb{R}$, the axis x cannot geometrically describe the set of rational numbers. The points marked in Figure 1 intuitively illustrate a dense set in the graph of f.

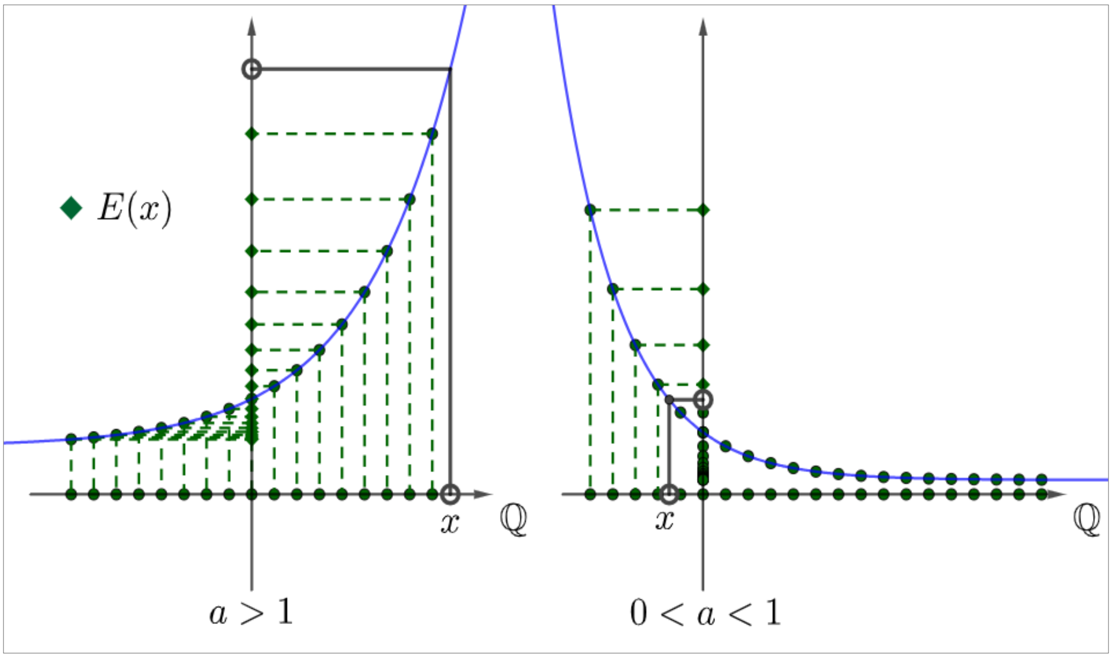
Figure 1: Intuitive illustration of the graph of f



Source: Prepared by the authors.

To obtain a^x using supremum and infimum, we must choose an appropriate set for each $a > 0$ and $x \in \mathbb{R}$, which can be: $E(x) = \{a^s : s \in \mathbb{Q}\, es < x\}$, i.e., $E(x)$ is the set of all powers a^s, in which s is rational and smaller than x, as illustrated in Figure 2.

Figure 2: Intuitive illustration of set $E(x)$ with $x \in \mathbb{R}$



Source: Prepared by the authors.

Thus, for every $r \in \mathbb{Q}$, we have: $a^r = sup\ E(r)$, if $a > 1$ and $a^r = inf\ E(r)$, if $0 < a < 1$.

In such demonstrations, the definition of supremum and infimum is used, and the fact that, given $a > 0, a \neq 1$, every non-degenerate interval of $\mathbb{R}$ contains a number a^r for some r rational. Proof of this result can be found in the aforementioned book to be released by Martins (in preparation) or in Lima (2013). From the equalities obtained via supremum and infimum, observing Figure 2, we are led to the following definition:

Given $a > 0$ with $a \neq 1$,, for every $x \in \mathbb{R}$, we define:

1. $a^x = sup\ E(x)$, if $a > 1$;

2. $a^x = inf\ E(x)$, if $0 < a < 1$.

Unlike the approach presented by the books described in Table 1, the above definition does not directly provide us with the properties that involve function $f: \mathbb{R} \to \mathbb{R}_*^+$, given $f(x) = a^x$, such as monotonicity, $a^{x+y} = a^x a^y$ and $(a^x)^y = a^{xy}$, that then need to be demonstrated. It is important to note, however, that the construction uses only the completeness of $\mathbb{R}$, the definitions of supremum and infimum, and the visual ideas presented in Figures 1 and 2. We present, below, the demonstration of the properties that can also be found in the aforementioned book to be released by Martins (in preparation):

Theorem 3: Given $a > 0$ and $a \neq 1$, we have $a^{x+y} = a^x a^y$ and $(a^x)^y = a^{xy}$.

Demonstration. We present the demonstration only for $a^{x+y} = a^x a^y$ when $a > 1$, as the other cases are analogous. By definition, if $a > 1$:

$a^{x+y} = sup\ E(x+y) = sup\ \{a^t : t \in \mathbb{Q}\ et < x+y\}$

$a^x = sup\ E(x) = sup\ \{a^r : r \in \mathbb{Q}\ er < x\}$

$a^y = sup\ E(y) = sup\ \{a^s : s \in \mathbb{Q}\ es < x\}$

As $\{a^{r+s} : r < xes < y\} \subset \{a^{r+s} : r+s < x+y\}$, we have

$$sup_{r<x,s<y}\ \{a^{r+s}\} \leq sup_{r+s<x+y}\ \{a^{r+s}\}.$$

Suppose, by absurdity, that $sup_{r<x,s<y}\{a^{r+s}\} < sup_{r+s<x+y}\{a^{r+s}\}$. Thus, there are $p, q \in \mathbb{Q}$ with $p + q < x + y$, such that

$$sup_{r<x,s<y}\{a^r \cdot a^s\} = sup_{r<x,s<y}\{a^{r+s}\} < a^{p+q} \leq sup_{r+s<x+y}\{a^{r+s}\}.$$

By the density of $\mathbb{Q}$ in $\mathbb{R}$, we take $r_1 < x$ and $s_1 < y$, such that $p + q < r_1 + s_1 < x + y$. Using the properties already known for rationals, there is $a^{p+q} < a^{r_1+s_1} = a^{r_1}a^{s_1}$, which contradicts the definition of supremum. Thus, $sup_{r<x,s<y}\{a^{r+s}\} = sup_{r+s<x+y}\{a^{r+s}\}$. Hence,

$$a^x a^y = E(x)E(y) = sup_{r<x,s<y}\{a^{r+s}\} = sup_{r<x,s<y}\{a^r \cdot a^s\} = sup_{r+s<x+y}\{a^{r+s}\}$$
$$= sup_{t<x+y}\{a^t\} = sup\, E(x+y) = a^{x+y}.$$

Theorem 4: If we $x < y$, have

1. $a^x < a^y$, if $a > 1$.

2. $a^x > a^y$, if $0 < a < 1$.

Demonstration. Again, we will demonstrate only case 1, as case 2 is analogous. Let us take$r, s \in \mathbb{Q}$ such that $x < r < s < y$. We start from this definition of supremum: $a^r < a^s \leq a^y$. As a^x is the smallest of the upper quotas of $E(x)$, one must have $a^x \leq a^r$, which proves case 1.

Theorem 4 guarantees that the function $f: \mathbb{R} \to \mathbb{R}_*^+$ given by $f(x) = a^x$, with $a > 0$ and $a \neq 1$ is injective. On the other hand, if $b > 0$ and $X = \{x \in \mathbb{R}: a^x < b\}$, then, $c = sup\, X$ is such that $a^c = b$, i.e.,f is surjective. Thus, the inverse of f can be defined as the logarithm in the base a,, i.e., $g: \mathbb{R}_*^+ \to \mathbb{R}$ is described as $g(y) = log_a y$ in the sense that:

$$x = g(y) = log_a y \Leftrightarrow y = a^x = f(x).$$

The properties of log_a can then be obtained in the traditional way, as is commonly done in high school math textbooks.

5 The pendulum between rigor and intuition in the constructions presented

The approach presented in the Real Analysis textbooks we analyzed is mathematically rigorous and yields good properties, including those already addressed in Differential and Integral Calculus. Undoubtedly, the path of construction of the exponential and natural logarithm functions followed by all authors presents a complete logical-deductive sequence from the point of view of rigor, in the sense that all necessary demonstrations are didactically outlined, as well as some more accessible ones are sent to the reader, a natural practice in mathematics textbooks.

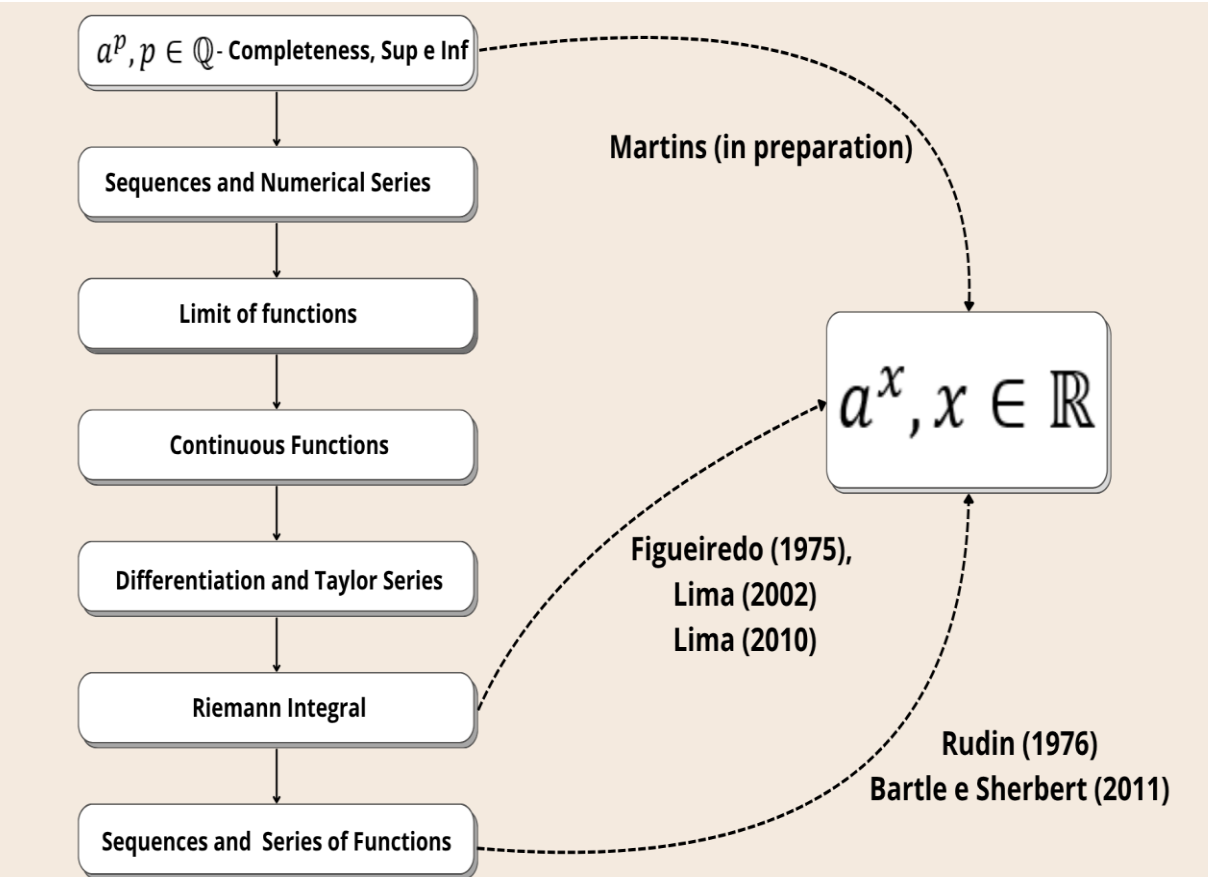
However, regardless of the didactic option of each of the authors to define, at first, the exponential function, as in the case of Rudin (1976) and Bartle and Sherbert (2011) or, initially, the natural logarithm function, as in the case of Lima (2002a, 2002b) and Figueiredo (1996), the construction path seems long and demands a whole tooling that, in general, is only “didactically available” at the end of the Real Analysis subject, which, in its tradition, tracks in a sequential way the classic contents of: real numbers, sequences and numerical series, limits, continuity, derivatives, integrals, sequences and series of functions.

In this context of teaching practice, the construction path presented by Rudin (1976) and Bartle and Sherbert (2011) requires prior work with sequences and series of functions to define the exponential function. On the other hand, what Lima (2002a, 2002b) and Figueiredo (1996) exposed calls for an initial practice with integrals in the definition of the exponential function.

If we consider, due to our long teaching experience as professors of Real Analysis in undergraduate and graduate subjects, the real possibility that, in some academic periods, the contents of integrals and sequences and series of functions may not even be addressed for a matter of a short time, this imputes to teachers and, consequently, to students the unfeasibility of a rigorous elaboration of the exponential function. Another fact to be considered is that, in some Brazilian universities, the content of sequences and series of functions may be sent to a second subject of analysis (in line and/or in the $\mathbb{R}^n$) which, in general, is mandatory only for degree courses in mathematics.

From this perspective, the didactic option for an equally rigorous construction of the exponential function, but following a path that requires less tooling previously worked in Real Analysis, seems more appropriate, in order to guarantee, in fact, such a construction in the classroom, as in the case of the aforementioned book by Martins (in preparation). To illustrate the main differences in the construction paths of the exponential function followed by the various authors, we present Figure 3.

Figure 3: Path of construction of the exponential function by the various authors



Source: Prepared by the authors.

We also deem it good to analyze some other rigorous and intuitive aspects of the construction made via supremum and infimum that, in addition to using a few (albeit deep) results of Real Analysis, such as the completeness of the line, present a relatively simple visual idea of how to introduce the set $E(x)$ to $x \in \mathbb{R}$ and define a^x, taking the largest of the lower dimensions, when $0 < a < 1$, or the smallest of the upper dimensions, when $a > 0$.

Thus, in line with Reis's (2001) dynamic conception, we understand that the intuitive elements in the illustrations in Figures 1 and 2 contribute to students' visualization, imagination, and creativity. It should be noted that, in addition to conceiving visualization as an important mental process highlighted by intuition, in this article, we understand it in the following terms, established by Arcavi (2003):

> Visualization is the ability, process, and product of creating, interpreting, using reflection on figures, images, diagrams, in our minds, on paper or with technological tools, for the purpose of describing and communicating information, thinking about, and developing previously unknown ideas and advanced understandings. (p. 217).

In this context, the visualization provided by the interpretation of the illustrations in Figures 1 and 2, presented in the classroom, intuitively sediments the path to the formalization and generalization necessary for the rigorous definition of the exponential function, with which the construction proposed in Martins (in preparation) culminates.

It should also be noted that the presentation of the contents in the classroom is elaborated by each Real Analysis professor who has, to a certain extent, freedom to decide the order, time, and form of presentation, especially of those contents that need to be taught in a logical and adequate way for student learning. From this premise, we understand that the pendulum between rigor and intuition in the didactic approach of the various concepts to be constructed moves, sometimes to one side, sometimes to another, largely due to the teacher’s teaching experience and, mainly, their belief in how such an approach will, in fact, contribute to the learning of the highlighted/focused trending concepts.

Thus, we understand that our argumentation for the possibilities of visualization, intuition, formalization, and generalization arising from the construction proposed in Martins (in preparation) does not imply, in any way, disregarding such possibilities in the constructions presented in the Real Analysis textbooks analyzed here, exactly, because we glimpse the potentialities of the pendulum movement between rigor and intuition, as we describe it.

However, it is also worth resuming our consideration that a Real Analysis textbook plays a fundamental role in the configuration of its public curriculum because, in practice, when it is adopted as the main reference for the development of the subject, it ends up directly influencing the didactic approach of the contents in the classroom. Thus, we conclude that, if the book adopted, in its proposed constructions, strives for a rigorous approach to analytical concepts based on intuition, it will be easier for teachers to move pendularly in teaching towards a learning that, in fact, is significant and interpretive, especially by students who, in the future, will also be teachers, at the most varied levels of education.

6 Final Considerations

The constructions presented in the classic Real Analysis textbooks analyzed here have the advantage of allowing us to obtain important properties of Differential and Integral Calculus for the functions $exp(x), ln(x), a^x$ and $log_a x$. However, such constructions require a long path of knowledge of Real Analysis, and the definitions presented in (2), under the dynamic conception we adopted, do not emphasize the intuitive elements that must be worked on in a didactic perspective.

Furthermore, we understand that a rigorous definition of exponential is fundamental for the education of teachers and non-licensed degree mathematicians, and we conceive Real Analysis as a unique opportunity for the construction of nuclear mathematical concepts that collaborate with the development of analytical thinking, so important in the constitution of a broad and solid mathematical thinking. However, we also argue that, in the context of Real Analysis teaching, the centrality of such construction must be considered through didactic paths that provide multiple, flexible explorations of the constructed concepts, seeking a balance between meaning/intuitive understanding and rigorous validation/demonstration, in light of student learning.

Finally, we reaffirm the importance of conducting further research on the presentation of mathematical concepts in textbooks, particularly in subjects with advanced mathematical content. We believe that such research can contribute to teachers reflecting on and even rethinking the teaching of these contents, particularly in higher education, a locus of the expansion of mathematical and pedagogical knowledge, both by textbook authors and by the teachers who adopt them across the most varied subjects.

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