**Essential Understandings for the development of students’ Mathematical Reasoning: comprehension presented by early years teachers**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Leandro Quirino dos Anjos**  Universidade Tecnológica Federal do Paraná  Marialva, PR — Brasil | | | **[Desenho de um círculo  Descrição gerada automaticamente com confiança média](https://creativecommons.org/licenses/by-nc-sa/4.0/deed.pt_BR)**  [2238-0345](https://portal.issn.org/resource/ISSN/2238-0345) Brazil / Brésil | ISSN  [10.37001/ripem.v14i4.3890](https://doi.org/10.37001/ripem.v14i4.3890) Logotipo, Ícone  Descrição gerada automaticamente  Received • 18/03/2024  Approved • 13/05/2024  Published • 15/10/2024  Editor • Gilberto Januario [Ícone  Descrição gerada automaticamente](https://orcid.org/0000-0003-0024-2096) |  |
| 🖂 | [leandroquirino2011@gmail.com](mailto:leandroquirino2011@gmail.com) | |
| Ícone  Descrição gerada automaticamente | [0000-0003-0599-3972](https://orcid.org/0000-0003-0599-3972) | |
|  | | |
| **Eliane Maria de Oliveira Araman**  Universidade Tecnológica Federal do Paraná  Londrina, PR — Brasil | | |
| 🖂 | [elianearaman@utfpr.edu.br](mailto:elianearaman@utfpr.edu.br) | |
| Ícone  Descrição gerada automaticamente | [0000-0002-1808-2599](https://orcid.org/0000-0002-1808-2599) | |
|  | | |
| **André Luis Trevisan**  Universidade Tecnológica Federal do Paraná  Londrina, PR — Brasil | | |
| 🖂 | [[andrelt@utfpr.edu.br](mailto:andrelt@utfpr.edu.br)](mailto:geovane.barbosa@ifes.edu.br) | |
| Ícone  Descrição gerada automaticamente | [0000-0001-8732-1912](https://orcid.org/0000-0001-8732-1912) | |
|  | |  |  |  |

***Abstract:*** It is essential to promote initial and continuing training actions for teachers working at different levels of schooling about Mathematical Reasoning (MR), as well as to identify aspects of these training processes that make it possible to infer the participating teachers' understandings this topic. In view of this, this qualitative research aims to understand how the development of a training process can contribute for teachers' understandings about the Essential Understandings for the development of students’ MR, in the context of early years of primary schools. The data were collected during a training process supported by the continuing training model, know as PLOT. The methodology used during this investigation consists of analyzing the dialogues promoted by teachers during the performance of a Professional Learning Task. As a result of this research, we concluded that this process enabled the participating teachers to improve their knowledge of some of the MR processes. We also recognize the teachers' difficulty in understanding the processes of generalization and investigation of why.

***Keywords:*** Mathematics Teaching. Mathematical Reasoning. Mathematical Reasoning Processes. Teachers’ Training.

**Comprensiones Esenciales para el desarrollo del Razonamiento Matemático de los estudiantes: comprensión presentada por docentes de los primeros años**

***Resumen:*** Es fundamental promover acciones de formación inicial y continua de docentes que trabajan en los diferentes niveles de educación en Razonamiento Matemático (RM), así como identificar aspectos de estos procesos de formación que permitan inferir las comprensiones de los docentes participantes sobre este tema. Por lo tanto, esta investigación cualitativa tiene como objetivo comprender cómo el desarrollo de un proceso formativo puede contribuir a la comprensión por parte de los docentes de las Comprensiones Esenciales para el desarrollo de la RM de los estudiantes, en el contexto de los primeros años de la escuela primaria. Los datos fueron recolectados durante un proceso de capacitación apoyado en el modelo de formación continua, conocido como PLOT. La metodología utilizada durante esta investigación consiste en analizar los diálogos promovidos por los docentes durante La realización de una Tarea de Aprendizaje Profesional. Como resultado de esta investigación, concluimos que este proceso permitió a los docentes mejorar sus conocimientos sobre algunos de los procesos de RM. También reconocemos la dificultad de los docentes para comprender los procesos de generalización e investigación el porqué.

***Palabras clave:*** Enseñar Matemáticas. Razonamiento Matemático. Procesos de Razonamiento Matemático. Formación de Profesores.

**Entendimentos Essenciais para o desenvolvimento do Raciocínio Matemático dos alunos: compreensão apresentada por professores dos anos iniciais**

***Resumo:*** É fundamental promover ações de formação inicial e continuada de professores que atuam em diferentes níveis de escolaridade sobre Raciocínio Matemático (RM), bem como identificar aspectos desses processos formativos que possibilitem inferir as compreensões dos professores participantes sobre este tema. Diante disso, esta pesquisa qualitativa tem, como objetivo, compreender como o desenvolvimento de um processo formativo pode contribuir para a compreensão dos professores sobre os Entendimentos Essenciais para o desenvolvimento do RM dos estudantes no contexto dos anos iniciais do ensino fundamental. Os dados foram coletados durante um processo de formação apoiado no modelo de formação continuada, denominado como PLOT. A metodologia utilizada durante esta investigação consiste na análise dos diálogos promovidos pelas professoras durante a realização de uma Tarefa de Aprendizagem Profissional. Como resultado desta pesquisa, concluímos que este processo permitiu às professoras melhorar o seu conhecimento sobre alguns dos processos de RM. Reconhecemos, também, a dificuldade das professoras em relação à compreensão dos processos de generalização e investigação do porquê.

***Palavras-chave:*** Ensino de Matemática. Raciocínio Matemático. Processos de Raciocínio Matemático. Formação de Professores.

1. **Introduction**

The development of students’ mathematical reasoning (MR) is one of the main objectives of Mathematics teaching (Mata-Pereira & Ponte, 2018) and, according to Jeannotte and Kieran (2017), curriculum documents around the world should advocate this development from the earliest years of schooling. However, according to these authors, the way MR is described in these guiding documents “tends to be vague, unsystematic, and even contradictory from one document to the other” (Jeannotte & Kieran, 2017, p. 2). In Brazil, for example, the guidelines for teaching mathematics in primary schools present in the National Common Curriculum Base (Brasil, 2018) do not provide an objective definition of what MR is.

Teacher's knowledge on this topic directly influences the actions developed in the classroom (Araman, Serrazina & Ponte, 2019; Loong, Herbert, Bragg & Widjaja, 2017). In this sense, Ponte, Mata-Pereira and Henriques*.* (2012, p. 375) point out that it is “necessary for teachers to know their students' reasoning processes and reflect on them”, especially by analysing situations experienced in the classroom context (Oliveira & Serrazina, 2002).

In addition, Vieira, Rodrigues and Serrazina (2020, p. 14) state that “for teachers to promote the development of reasoning in their students, it is necessary that, during their training, they are confronted with concrete situations that explicitly involve mathematical reasoning, experiment with different strategies and analyse different situations, preferably work done by students”. Thus, it is essential to promote initial and continuing training actions for teachers working at different levels of schooling about MR and the way it is developed by students when solving tasks (Rodrigues, Brunheira & Serrazina, 2021), as well as to identify aspects of these training processes that make it possible to infer the participating teachers' knowledge about the Essential Understandings for the development of students’ MR (Lannin, Ellis & Elliot*.*, 2011).

In the context of continuing education, we rely on the PLOT Model (Professional Learning Opportunities for Teachers) proposed by Ribeiro and Ponte (2020) as a theoretical construct to support the design of the training process, as well as to guide the understanding of professional learning opportunities for the teachers involved (in this case, learning about the essential understandings for the development of students’ MR). This model proposes the work articulated in three domains, namely: the role and actions of the trainer, the discursive interactions between the participants and the use of Professional Teacher Learning Tasks (PTLT).

Based on these assumptions, the aim of this work is to comprehend how the development of a training process can contribute for teachers' knowledge about the Essential Understandings for the development of students’ MR, in the context of early years of primary schools. In order to answer this objective, a training process was organised for teachers who teach mathematics in the early years, focusing on actions for de development of students’ MR, supported by assumptions of the PLOT Model. From this context, data were collected that will subsidies the analyses and discussions presented in the continuity of this article.

1. **Theoretical framework**

According to Oliveira (2008, p. 3), "the expression mathematical reasoning designates a set of complex mental processes through which new propositions (new knowledge) are obtained from known or assumed propositions (prior knowledge)". Similarly, Mata-Pereira and Ponte (2018, p. 781) propose that "mathematical reasoning consists of making justified inferences, that is, using already known mathematical information to justifiably obtain new conclusions". In this same sense, Jeannotte and Kieran (2017, p. 7) define MR “process of communication with others or with oneself that allows for inferring mathematical utterances from other mathematical utterances”. In turn, Lannin *et al.* (2011, p. 10) treat MR as “an evolving process of conjecturing, generalizing, investigating why, and developing and evaluating arguments”, which leads us to the need to better understand each of these aspects and understandings.

According to Jeannotte and Kieran (2017), the central aspects of MR involves the processes of generalising, conjecturing, identifying a pattern, comparing, classifying, justifying, proving and formally proving, some of which are related to the search for similarities and differences and others to validation, as shown in Table 1.

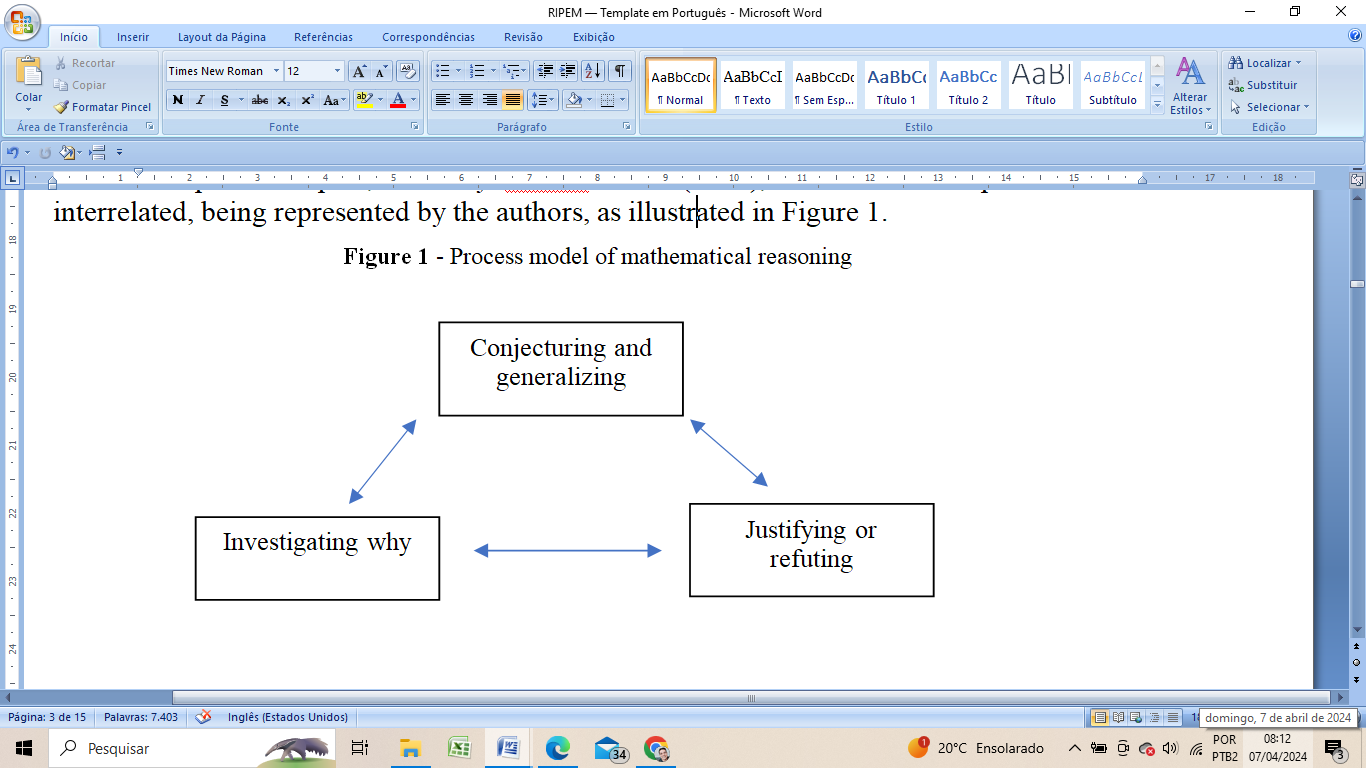
**Table 1:** MR processes and their definitions, based on the model proposed by Jeannotte and Kieran (2017).

|  |  |  |
| --- | --- | --- |
| **Process** | | **Definition** |
| **Processes related to the search for similarities and differences Process Definition** | Generalise | By searching for similarities and differences, infers narratives about a set of mathematical objects or a relationship between objects in the set of a subset of this set. |
| Conjecture | By searching for similarities and differences, infers a narrative about some regularity with a probable or true epistemic value and which has the potential for mathematical theorising. |
| Identify a pattern | By searching for similarities and differences, infers a narrative about a recursive relationship between mathematical objects or relationships. |
| Compare | Infers, by searching for similarities and differences, a narrative about mathematical objects or relationships. |
| Classify | Infers, by searching for similarities and differences between mathematical objects, a narrative about a class of objects based on mathematical properties and definitions. |
| **Processes related to validation** | Validate | Aims to change the epistemic value (i.e. the probability or truth) of a mathematical narrative. |
| Justify | By searching for data, collateral and support, it allows to modify the epistemic value of a probable narrative to very probable. |
| Prove | By searching for data, assurance and support, modifies the epistemic value of a narrative from probable to true. |
| Prove formally | By searching for data, assurance and support, modifies the epistemic value of a narrative from probable to true, with greater rigour and degree of formalism than in proving. |

**Source:** Authors´ elaboration (2024)

As the authors emphasise in the proposed model, mathematical reasoning processes are intrinsically linked to the inferences that emerge from the narratives produced in the search for the development of mathematical knowledge, requiring different cognitive skills from students, assuming that students evolve in the investigation involving inductive and deductive reasoning.

When developing MR, it is possible for students to explore different mathematical situations, which, depending on the choice of task and the way it is conducted by the teacher (Ponte, 2005), may have the potential to involve them in the use of different MR processes. Another important aspect, cited by Lannin *et al.* (2011), is that the MR processes are interrelated, being represented by the authors, as illustrated in Figure 1.

**Figure 1** - Process model of mathematical reasoning

**Source:** Lannin *et al.* (2011)

According to Lannin *et al.* (2011), the MR processes are dynamic and allow students to use them at different times of solving a mathematical resolution. To give a better understanding of the various aspects of MR, the authors proposed divided them in nine essential understandings, that are interrelated and integral to the “big ideia” ot MR.

The model proposed by Lannin *et al.* (2011) starts from a general concept of the development of MR, defined by the authors as a big idea that involves the processes of conjecturing, generalising, investigating why, and developing and evaluating arguments. As well as defining the processes that make up the development of MR, the authors point out that there are nine Essential Understandings that subsidise the development of students’ MR, detailed in Table 2.

**Table 2:** Essential Understandings based on Lannin *et al.* (2011)

|  |  |  |
| --- | --- | --- |
| **CONJECTURE AND GENERALISE** | Essential Understanding 1 | From the mathematical concepts and skills they hold, learners can make statements about a mathematical fact, and these statements can be true or false. After further investigation, they can produce a justification that must be accepted or refuted, thus assuming the truth of the conjecture formulated. |
| Essential Understanding 2 | A generalization may or may not involve the formulation of a rule or algebraic expression. This is because, in fact, generalization is related to identifying the similarity between the cases observed in a given situation. |
| Essential Understanding 3 | "Generalizing involves two types of activities-thinking about a relationship, idea, representation, rule, pattern or other mathematical property to identify commonalities and extending the reasoning beyond the domain in which it originated" (Lannin *et al.*, 2011, p. 23). Therefore, it is important that learners develop the ability to identify the domain and boundaries relevant to formulating a generalization, as each generalization applies to a particular domain. |
| Essential Understanding 4 | Mathematical language involves the use of symbols, terms and different representations, when developing the actions of conjecturing and generalising, it is necessary for the teacher to be careful about the statements presented to the students, because, according to Lannin *et al.* (2011, p. 26), "clarifying mathematical language, symbols, and representations is an essential part of the student reasoning". |
| **INVESTIGATE WHY** | Essential Understanding 5 | Investigating why a statement is true or false is an essential part of developing MR, as this process tends to contribute to the validity of a justification or the refutation of conjectures and generalizations. The process of investigating why can occur through research into the factors that explain why a particular mathematical statement is valid, giving "attending to particular features that provide insight into relationships that can explain whether a generalization is true or false" (Lannin *et al*, 2011, p. 30), as well as through explanations of why a general statement is valid or invalid. The process of Investigating why is related to the idea that when a learner investigates why a conjecture or generalization is valid or not, they can use various explanations to formulate their justification. |
| **JUSTIFY AND REFUTE** | Essential Understanding 6 | Starting from the prior knowledge and general statements that learners use to make their generalizations, they can construct valid or invalid justifications. This depends on how they handle argumentation to formulate their conclusion. The production of a justification is broader than checking the validity of a statement, and it is necessary to investigate whether a conjecture or generalization is true beyond the domain being explored. The justification "needs to be extended so that it addresses all aspects, or accounts all elements, of the domain" (Lannin *et al.*, 2011, p. 35), and it is important to use general language, demonstrating that the justification is valid and applies to more than one particular example. |
| Essential Understanding 7 | Refuting and validating are steps that involve the mathematical arguments used during the formulation of a justification and to reach a conclusion, i.e. whether the conjecture is true or false, it is important that students understand the importance of using counterexamples and their role in supporting the argument that a particular statement is false. In this regard, Lannin *et al.* (2011, p. 43) state that "a single counterexample can invalidate a conjecture" as well as a generalization. |
| Essential Understanding 8 | Creating and evaluating arguments, explaining why conjectures are true, refining conjectures based on the arguments that support their formulation, understanding the application of definitions and counterexamples, and showing that conjectures are false, are important factors in mathematical justification. This includes identifying and reviewing errors or misconceptions presented in a justification, whether from a conclusion or just part of the overall statement. Evaluating a justification may involve using different forms of mathematical representation to show that the conjecture is false, reformulating a conjecture in a way that makes it more appropriate or even true, revising a conjecture from the perspective of making sense of the statement made, as well as further analysing the arguments that support the validity of valid conclusions. |
| Essential Understanding 9 | Considering that many of the justifications produced by learners are elaborated from mathematical ideas used or presented previously by the teacher, during the use of examples to check the validity of conjectures, it is essential that learners have the understanding that the validity of a generalization should not be based only on mathematical ideas, testing of examples and opinions about the veracity of a piece of information. In this sense, in the development of justifications in which learners explore the mathematical relations that support and guarantee the validity of the generalization, the learner should present arguments that show that a mathematical relation is valid for all possible cases. |

**Source:** Authors´ elaboration (2024)

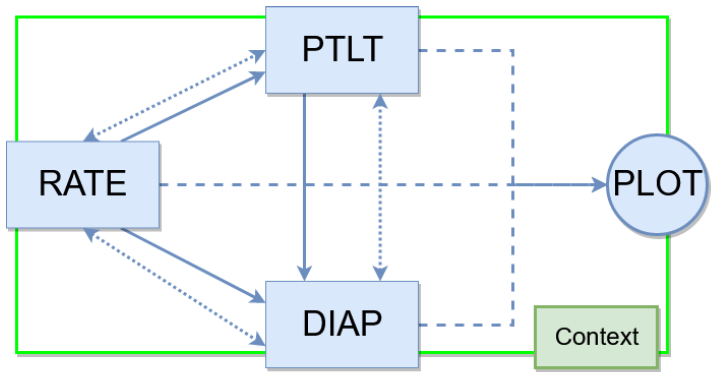
Considering the characteristics presented in each Essential Understanding (Lannin *et al.*, 2011), it is emphasised that their understanding plays a very important role in the development of this research. It is this understanding that we sought to investigate, in order to comprehend how the development of a training process can contribute for teachers' knowledge about the Essential Understandings for the development of students’ MR, in the Elementary School.

1. **Methodological procedures**
   1. **Characterisation and context of the research**

The research from which this article results is qualitative and interpretive in nature (Bogdan & Biklen, 1994). According to the authors, a qualitative investigation involves the experiences lived by certain subjects and the way they interpret them in a context.

The context involving the preparation and development of a training process for teachers was considered. It was assumed that the analysis of mathematical tasks solved by students enables teachers participating in a training process to expand your knowledge about the Essential Understandings for the development of students’ MR. Supported by the PLOT Model (Ribeiro & Ponte, 2020, p. 4), this training process sought to provide Professional Learning Opportunities (PLOT) for teachers through interconnected and interactive actions, involving the three domains that make up the PLOT Model (Figure 2): RATE (Role and Actions of Teacher Education), PTLT (Professional Teacher Learning Tasks) and DIAP (Discursive Interactions Among Participants).

**Figure 2:** PLOT Model (Professional Learning Opportunities for Teachers)



**Source:** Translated from Ribeiro and Ponte (2020, p. 4)

The training actions were based on the use of PTLT (Smith, 2001). According to Ribeiro, Aguiar and Trevisan (2020), PTLT are tasks designed to provide learning to teachers in a specific situation and are characterised, among other aspects, by the use of practice records, such as student resolutions, clippings from curriculum proposals, and teaching plans. For Barboza, Pazuch and Ribeiro (2021, p. 7) such records enable "the formulation of mathematical conjectures, their validation, reformulation and the mobilisation of knowledge necessary for teaching practice". In the case of this research, it is considered the Essential Understandings for the development of students’ MR.

The research was carried in a municipal school of Marialva – PR/Brasil, in person, in 5 meetings of 3 hours each, in the 1st semester of 2022, where the first author of this article worked as a teacher and, during the training process, acted as a facilitator. Ten teachers (fictitious names) from the initial years of primary schools participated, all from the same school where the training was carried out and where the trainer also worked as a teacher. They didn’t aware knowledge of the framework Essential Understanding.

During the training process, the trainer developed three PTLT (called PTLT 1, PTLT 2 and PTLT 3). The first two tasks contribute to expanding teachers' conceptual knowledge regarding the essential understandings for the development of students' MR. In this article, results of PTLT 3 are presented, at which time the teachers analysed some resolutions of a mathematical exploratory task (Ponte, 2005) (Figure 3), applied in a 5th grade class.

**Figure 3**: Exploratory task involving the idea of number sequence

|  |
| --- |
| Observe the following sequence of figures, where white and grey tiles are stacked, following a certain rule.    Source: Mosquito (2008, p. 157).  a) Indicate the number of tiles of each colour and the total number of tiles to construct Figure 5. Explain how you obtained each result.  i. Number of white tiles: \_\_\_\_\_\_\_\_\_\_\_  ii. Number of grey tiles: \_\_\_\_\_\_\_\_\_\_\_\_  iii. Total number of tiles: \_\_\_\_\_\_\_\_\_\_\_\_ |

**Source:** Survey data (2022)

Firstly, the teachers solved the exploratory task individually and then discussed it collectively. Then, in pairs or trios, they analysed the resolutions that some 5th grade students presented for the task, in order to identify the MR processes that they considered to be mobilised by them, justifying them.

Finally, they presented the considerations of their analyses in the form of a plenary session and, based on them, the trainer mediated the collective discussion involving, in the perspective of providing a broader and more comprehensive analysis of the MR processes. All participants could contribute to the discussion by presenting arguments that may not have been identified by the pair or trio previously. During the discussion, the teachers identified which MR processes each student possibly used during the resolution of the task and, for each process cited, presented justifications.

* 1. **Methods for data collection and analysis**

The data produced for the research were collected through audio recordings, later transcribed in full. Afterwards, some excerpts were selected that meet the research objective, that is, moments in which, from the participants' speeches, it was possible to infer how teachers expanded their knowledge about the Essential Understandings for the development of students’ MR.

For the data analysis, the discussions promoted in two moments of PLT 3 were used. The first one is when a trio of teachers discusses the resolution of the task presented by one of the students (Figure 4). The second moment occurs in the plenary, when they present their considerations to the other participants in the training process, with the aim of sharing reflections that had occurred in the other groups.

**Figure 4**: Student's resolution

|  |
| --- |
|  |
| “I saw that figure 1 had 2 white tiles and 3 grey tiles in figure 3, in figure 2, it was the same but there were 3 more grey tiles, so I realised that the rule is 3 in 3 in the grey tiles and the white tiles don't go up or down”. |

**Source:** Survey data (2022)

1. **Analysing the data**

This section presents the analyses separated into the two moments mentioned above. The first highlights the MR processes that the teachers recognised through the student's resolution and, in the second, those that were recognised through the collective discussion.

* 1. **Paula, Carolina and Eloysa's group discussion**

Based on the student's resolution and justification (Figure 4), Eloysa, Paula and Carolina hold the following discussion:

**Paula:** Number of white tiles 2, grey tiles 15. It's for Figure 5 too, right? Total 17.

**Emily:** I did this rule from 3 to 3.

**Paula:** [Reading the student's answer] I saw that Figure 1 had 2 white tiles and 3 grey tiles; Figure 2 had the same thing, but 3 more grey tiles. So I realised that the rule is 3 in 3 for the grey tiles and not up or down for the white.

**Carolina:** I think... I don't know... Could it be a justification?

**Eloysa:** I think it's a conjecture, isn't it?

**Paula:** Here he made the mathematical relationship.

**Eloysa:** Ahem.

**Paula:** The whites stay and the greys go up.

**Eloysa:** It goes up.

**Paula:** Every 3. So, here it's a conjecture.

When analysing the resolution, Emily recognises that the pupil used the same conjecture that she had formulated previously (when she solved the task individually), i.e., that the number of grey bricks increases every 3 years. Paula and Eloysa understand that the pupil identified a mathematical relationship and, from this, elaborated the conjecture that the number of white tiles remains the same and the amount of grey tiles increases by 3 with each new figure represented.

**Facilitator:** Is that a mathematical argument [referring to the pupil's written explanation]?

**Eloysa:** It is, isn't it?

**Facilitator:** Or is it a mathematical relationship?

**Carolina and Eloysa:** It is a relation.

**Facilitator:** So, actually what is this [process]? Because you said so, because you realised that it increases every 3, isn't that it?

**Paula:** The grey ones and the White one remain.

**Facilitator:** And how do we say that? What is he starting to build? What process is he starting to systematise?

**Paula:** Isn't he building a conjecture?

**Facilitator:** Conjecture! He is starting; this is an important process, he is starting, he has observed what we say about patterns and regularities. To reach a generalization, for example, he will need these patterns; he has to look at different cases and see that the same relationship he saw in Figure 2, is valid for Figure 1, is valid for Figure 3.

Paula, Eloysa and Carolina show that they understand that the pupil has identified a mathematical relationship, based on an argument supported by the initial conjecture: "the number of white tiles remains the same and the number of grey tiles increases every 3".

The facilitator, when interacting with the teachers, explains that, at this stage of mathematical argumentation, the pupil has probably started to elaborate his conjecture and, although he recognises that there is a regularity between the cases, he has not used a generalization, because the pupil has only identified the similarity between the cases, and, to be a generalization, there is a need to identify a mathematical relationship to be analysed in more than one case.

**Eloysa:** No. O, in generalization, the pupil starts from a set. Is it?

**Paula:** I think so. I think the conjecture is the last one.

**Eloysa:** Is it? Because here it says that they are going to check that the conjecture is valid and admit that there is a valid mathematical relationship for any situation. I think the generalization is beyond that conjecture.

[...]

**Paula:** So we're trying to find it in the task. The student, from a set of mathematical relations, verifies that a conclusion is valid and admits that there is a mathematical relation valid for any situation [reading the supporting text].

**Eloysa:** But I believe there is too.

**Paula:** Any situation? But this one is not just any situation.

**Eloysa:** So, let's leave it at that.

**Paula:** Because we can make a set of different figures, now it's different from a mathematical rule, in any situation it will be the same rule, right?

Eloysa shows that she comprehends that generalization is a process that will be produced from a set of mathematical relations and that the student used it during her resolution. However, she does not describe which generalization was used. She shows understanding that a generalization is much more than a statement, when she says "I think the generalization is beyond this conjecture".

However, by quoting the expression mathematical relation, Paula and Eloysa show that they do not comprehends that a generalization involves identifying a mathematical relation that is valid in a set of different cases, and that these cases must admit the existence of a similar factor in each of them.

* 1. **Collective discussion**

During the plenary, Paula, Eloysa and Carolina present to the other participants of the formative process, the MR processes that they were able to identify through the student's resolution presented in Figure 4.

**Facilitator:** What processes were you able to identify?

**Paula:** Conjecture.

**Facilitator:** What conjecture?

**Facilitator:** He got it right!

**Paula:** Talking about the conjecture process, from the moment he wrote that the white tile remains the same and the grey ones increase every 3 in each figure.

**Facilitator:** That's cool. Anyone else?

**Rosana:** He did his thinking right.

**Facilitator:** Was that his conjecture?

**Rosana:** He made a conjecture, generalised, justified.

**Facilitator:** Wait a minute! [laughs].

Paula mentions that they identified the occurrence of the formulation of a conjecture, because the student wrote "that the white tile remains the same and the grey ones increase every 3 in each figure", but they did not justify why it was a conjecture. Thus, it is understandable that they should have explained that it is a conjecture because the pupil wrote a statement based on the observation of figures.

Another aspect that draws attention during the discussion is the moment when Rosana points out that the student "made a conjecture, generalised, justified", since the student got the resolution of the task right and his reasoning is correct. Thus, it is important to emphasise that the fact that the reasoning and resolution are correct does not mean that he used all the MR processes mentioned by Rosana.

**Rosana:** There is the investigation.

**Eloysa:** Then he did the investigation in this case.

**Facilitator:** Why?

**Christina:** He realises the regularity.

**Rosana:** He realises the regularity.

**Facilitator:** And what is the regularity?

**Rosana:** That there are 3 always in grey and the white does not go up or down. He realised the regularity.

**Facilitator:** Yes, he has realised a regularity that is much more valid than the first one, right?

**Eloysa:** Yes, of course.

**Facilitator:** It increases every 3.

**Aline:** It doesn't go up or down.

**Paula:** The white remains.

**Christina:** The white remains.

In this excerpt, the participants show that they understand that one of the factors that makes it possible to identify that there was a process of investigating why, is the fact that the student has identified the regularity existing between the cases. However, this fact is only one of the characteristics, because, in addition, it is necessary for the student to present a mathematical relationship that has the potential to justify why a generalization is true or not.

**Facilitator:** What about investigating why in this case, do they investigate why?

**Christina:** Then they realise that the rule is every third grey.

**Rosana:** And that neither the white goes up nor down.

**Facilitator:** And the total number of tiles?

**Rosana:** That's right.

**Paula:** It is the sum.

**Facilitator:** Do they investigate the mathematical relationship?

**Christina:** Not the total!

**Rosana:** Not in total! Because here we didn't get the total.

Facilitator: Why 15? They say 15 in the grey.

Rosana: Yes. Because he noticed the regularity, always the rule is 3 in 3 in the grey.

**Paula:** But it doesn't justify the total.

**Rosana:** No, he doesn't.

**Paula:** He only justifies the figures. He doesn't justify the total.

Christina and Rosana show that the process of investigating why is justified by the fact that the student has identified and used a mathematical property, which is observed through the sequence of figures. However, it is understandable that this property is only a part of the mathematical argumentation that helps in the formulation and validation of the initial conjecture. However, the trainer seeks to assist in understanding by questioning whether the student has used a mathematical relationship. In this way, Rosana implies that a possible investigation of the why would occur if the student had justified the total value of tiles contained in Figure 5 of the task.

**Facilitator:** In the figure he justified it, right? Trainer: In the picture he justified it, right? So, hold on. He conjectures that the number of grey tiles will increase every 3. The justification, you said there is, why is there a justification?

**Rosana:** Because it has a...

**Eloysa:** Here you are validating his conjecture.

**Facilitator:** It is validating the conjecture; that in a way, his figure is consistent with what he says. He just didn't make it explicit why 15.

**Rosana:** Yes.

**Facilitator:** That there would be a detail to appear to investigate why. Because he understood the relationship, which is 3 by 3.

**Paula:** But he left it.

**Facilitator:** But to investigate why 15? Did he leave why 15?

**Rosana:** He understood the rule, but he didn't explain why it was 15.

**Christina:** The result. How he arrived at Figure 5.

**Rosana:** He would have to say that 12 plus 3 is 15. Because the rule is always.

**Facilitator:** Paula, was there anything else you wanted to say?

**Paula:** That's what is written there, that the rule is.

**Aline:** Every 3 years.

**Rosana:** But he didn't justify the total.

Eloysa recognises that, through the student's resolution, it is possible to identify that he validated the conjecture that the pile of grey tiles increases every three years. Therefore, the trainer's questions help the participants to realise that, in order to mobilise the process of investigating why, there is a need to describe a mathematical relationship that has the potential to explain why, in Figure 5 of the resolution, there are 15 grey tiles.

With this, Rosana states that the student "would have to say that 12 plus 3 is 15". However, this is not true. If the student made this statement, he would not be using a mathematical relation, but a property of the conjecture.

**Facilitator:** It is the conjecture that helps in the justification, but to investigate why, would be to analyse case by case and identify a relationship. He identified that it increases every 3 years from one to the other, but can we say that this is a mathematical relationship?

**Rosana:** Yes.

**Facilitator:** That it increases by 3 from one to the other! What if I asked Figure 100?

**Rosana:** If he did?

**Facilitator:** Yes.

**Paula:** The multiplication table of 3.

Realising that the participants have not yet identified the mathematical relationship that could be used to express the number of grey tiles in any given figure, the trainer poses a question: "what if I asked the [number of tiles in] Figure 100?". Paula shows understanding that to determine the number of grey tiles in Figure 100, she could use the multiplication table of three, i.e. she understands that the mathematical relationship involves determining a number of tiles being a multiple of three.

**Facilitator:** Do you think he would have the courage to go to Figure 100 [referring to the representation using the drawing of the figures]?

**Eloysa:** No.

**Facilitator:** So to get to an investigation of why, he wouldn't need that mathematical relationship? Did he bring this mathematical relationship there?

**Paula:** No.

**Facilitator:** Did you understand? He realised that it increases by 3, but if I ask him Figure 100, maybe he won't want to do it, because drawing it will be a lot of work, so he needs what you said, Paula, even if you can complete it a little bit, but she [the speech] can complete it.

**Paula:** Complete the answer here?

**Facilitator:** Yes. What is 10 x 3, so if it was Figure 100 he would have to think of some number times 3 which would give 100 [grey tiles], plus 2.

**Paula:** In other words, working out the multiplication of 3.

He would have to involve the 3-table in a certain way, to get to 100, and not forget the 2. That's where he starts to investigate why and probably ends up formalising a generalization.

**Paula:** And he also needs to use a mathematical sentence for the total number of tiles.

**Facilitator:** He could. Here he wouldn't need to get to the point of using the letter n, the algebraic structure, but this thought he would already have to use at this point. For example, when looking at Figure 4, Figure 4 has 12, 4 x 3, Figure 3, 9, 3 x 3, that is, he is beginning to observe the regularity of the figure. I can't take it and think from one case to the other; I have to think about the first case, find a regularity, look at the second case, find another regularity, and the regularity that I used for the first is valid for the second; then it involves investigating why. Because I discovered a regularity; the regularity holds in the first, second, third, fourth and fifth, that is, this regularity has the potential to become a generalization.

In this excerpt, the trainer helps the participants to realise the importance of the mathematical relationship between the number of the figure and the total number of tiles, and how to define it from the observation of the figures. When the trainer mentions the question of determining the number of tiles in Figure 100, he recognises that the participants have already comprehended that there is a conjecture, i.e. that the total number of tiles involves a multiple of three, but they have not identified that to obtain this number, one must multiply the number of the figure by three and add the 2 white tiles.

In this way, the facilitator's dialogue allows us to understand that, in order to mobilise the process of investigating why, it is important to first formulate a mathematical relationship to be identified and investigated, with the aim of verifying whether it is valid in each of the cases analysed.

Table 3 below summarises the Essential Understandings for the development of students’ MR mobilised during the discussions on solving the task.

**Table 3:** Synthesis of the Essential Understandings mobilised during the discussions on solving the task.

|  |  |  |
| --- | --- | --- |
| **Excerpts** | **Essential Understandings** | **Justification of why we consider it an understanding** |
| **Paula:** Here he made the mathematical relationship.  ...  **Paula:** 3 in 3. So, here is a conjecture. | Essential Understanding 1 | Paula shows understanding that a conjecture is formulated from a reasoning used to elaborate a statement based on the identified mathematical relationship. |
| **Facilitator:** [...] What is justification?  **Carolina:** It presents mathematical arguments. | Essential Understanding 6 | Carolina shows understanding that the process of justifying, involves using mathematical arguments to prove the ideas that have been understood. |
| **Facilitator:** [...] To come up with a generalization, for example, he will need these patterns; he has to look at different cases and see that the same relationship he saw in Figure 2, holds for Figure 1, holds for Figure 3. | Essential Understanding 2 | The Facilitator cites the idea of similarities between the cases to help formulate the generalization, which in a way involves the use of a conjecture. |
| **Paula:** [...] The student started with the process of building conjectures from the moment he realises that the white tiles remain the same and the grey ones increase every 3 in each figure, right? | Essential Understanding 1 | Paula understands that a conjecture consists in producing a statement based on ideas already assumed to be true. |
| **Paula:** Because we can make a set of different figures, now it's already different from a mathematical rule; in any situation it will be that same rule, right? | Essential Understanding 3 | Paula understands that the application of a mathematical rule must be tested in cases beyond the domain. |
| **Rosana:** That there are always 3 in the grey and the white does not go up or down. He realised the regularity. | Essential Understanding 6 | Rosana presents a justification based on ideas already understood and assumed to be true. |
| **Rosana:** He understood the rule, but he didn't explain why it was 15. | Essential Understanding 6 | Rosana understands that there was no mathematical justification based on ideas already understood. |
| **Facilitator:** So to get to an investigation of why, wouldn't he need that mathematical relationship? Did he bring that mathematical relationship in there? | Essential Understanding 5 | The trainer emphasises that the investigation of why consists of investigating factors that help explain why a generalization is valid. |

**Source:** Authors´ elaboration (2024)

1. **Discussion and Conclusion**

When analysing the discussions promoted by the participants of the formative process during the analysis of the resolution of the task, it is recognised that the teachers had the opportunity to discuss and reflect (Rodrigues *et al.*, 2021) on important aspects that helped in expand their knowledge about Essential Understandings for the development of students’ Mathematical Reasoning (Jeannotte & Kieran, 2017).

It is emphasised that the discussions mainly provided the identification and comprehension of the processes that involve the elaboration of conjectures, justification and refutation (Lannin *et al.*, 2011). It is understood that there were less in-depth reflections involving the processes of generalization and investigation of why, since the resolution analysed presented few elements to support discussions about these two processes.

The analysis of concrete situations that explicitly involved mathematical reasoning enabled teachers to discuss issues involving mathematics teaching, as well as the students' learning process (Vieira *et al.*, 2020). The dialogues provided an opportunity to reflect on how some mathematical knowledge (in this case, mathematical knowledge involving the idea of numerical sequence) is understood by students, thus promoting discussions based on situations experienced in the classroom context (Oliveira & Serrazina, 2002; Ponte *et al.*, 2012).

Through the moments of reflections, it is noted that one of the main difficulties of the teachers is related to the process of generalization and investigation of why (Lannin *et al.*, 2011). This is due to the fact that the teachers use mistaken perceptions about the process of generalising. In some excerpts from the discussions, it can be seen that the teachers showed that they understood that a generalization is related to the moment when the student uses mathematical knowledge to explain how he/she obtained the answer to the task. Thus, during the discussions they, in a way, used few arguments that could contribute to the comprehension of the generalization process. This is evident because, most of the times that this process is discussed, it starts from an approach promoted by the trainer.

Considering the teachers' difficulty in identifying aspects that could contribute to the comprehension of the processes of generalization and investigation of why, it is noted that the continuing education process provided, during the development of PLT (Ribeiro & Ponte, 2020), discussions and reflections with the potential to assist in understanding. However, as the teachers presented few arguments involving the identification of these two processes, they ended up being treated more succinctly and superficially.

Based on the summaries of the knowledge about Essential Understandings for the development of students’ MR mobilised during the discussions involving Student B's resolution, it was found that, during the development of PLT 3, investigated in this research, different discussions and reflections occurred that contributed to the perspective of identifying and comprehension of these Essential Understandings. In this way, Table 4 presents an overview of the moments in which the teachers had the opportunity to discuss and reflect on the aspects that corroborated the comprehension of the Essential Understandings, taking into account the resolutions of some students.

**Table 4**: General summary of the resolutions that made it possible to mobilise understandings about the Essential Understandings for the development of students’ MR

|  |  |
| --- | --- |
| **Essential Understandings** | **Occurrence** |
| Essential Understanding 1 | X |
| Essential Understanding 2 | X |
| Essential Understanding 3 | X |
| Essential Understanding 4 |  |
| Essential Understanding 5 | X |
| Essential Understanding 6 | X |
| Essential Understanding 7 |  |
| Essential Understanding 8 |  |
| Essential Understanding 9 |  |

**Source:** Authors´ elaboration (2024)

Based on the data presented in Table 4, it is concluded that the student's resolution presented aspects that contributed to the realisation of reflections and discussions involving the understanding of Essential Understanding 1 (recognition that the process of elaborating conjectures consists of formulating a mathematical statement produced by the student). From Student B's resolution, it was possible for the teachers to identify aspects that contributed to the understanding of Essential Understanding 2 (which involves recognising that the process of generalising consists of identifying similarity between cases) and Essential Understanding 6 (that mathematical justification must be produced from logical arguments and based on ideas already understood).

Student B's resolution made it possible to identify arguments that helped to understand Essential Understanding 5 (investigating factors that have the potential to explain why a generalization is true or false). It also enabled discussion of aspects involving the understanding of Essential Understanding 3 (the recognition that the process of generalization involves identifying the applicability of a mathematical relationship by extending it to cases beyond the relevant domain).

As a result, it was conjectured that the analysis of mathematical tasks solved by students enables teachers participating in a training process to comprehend the Essential Understandings for the development of students’ MR. During the development of this research, it was found that the students' resolutions made it possible to identify factors that helped to understand how the MR processes are mobilised through the resolution strategies used by the students. In this way, it was verified that the teachers demonstrated to recognise the MR processes based on the mathematical argumentation used in each resolution, since the arguments used by the students made it possible to identify whether or not a certain MR process was mobilised.

With the results of this research, it is concluded that the development of the continuing education process enabled the participating teachers to improve their knowledge about the MR processes and, thus, presented arguments that contributed to the identification of the Essential Understandings, promoting discussions and reflections that directly impact the mathematics teaching process and also some contributions to professional development.

Regarding the conjecture initially elaborated, it is found to be valid. Therefore, the participants of the continuing education process, when carrying out the CPT involving the identification of the MR processes and the use of some practice records, had the opportunity to recognise and discuss aspects that help in the mobilisation of Essential Understandings for the development of MR.

**References**

Araman, E. M. O., Serrazina, M. L. & Ponte, J. P. (2019). “Eu perguntei se o cinco não tem metade”: ações de uma professora dos primeiros anos que apoiam o raciocínio matemático. *Educação Matemática Pesquisa*, *21*(2), 466–490.

Barboza, L. C. S., Pazuch, V. & Ribeiro, A. J. (2021). Tarefas para aprendizagem de professores que ensinam matemática nos anos iniciais. *Zetetiké*, *29*, 1–25.

Bogdan, R. & Biklen, S. (1994). *Investigação Qualitativa em Educação.*Porto: Porto Editora.

Brasil. (2018). *Base Nacional Comum Curricular*. Brasília: MEC.

Jeannotte, D. & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics,* *96*(1), 1–16.

Lannin, J., Ellis, A. B. & Elliot, R. (2011). *Developing essential understanding of mathematics reasoning for teaching mathematics in prekindergarten-grade 8.* Reston: National Council of Teachers of Mathematics.

Loong, E. Y. K., Vale, C., Herbert, S., Bragg, L. A. & Widjaja, W. (2017). Tracking change in primary teachers’ understanding of mathematical reasoning through demonstration lessons. *Mathematics Teacher Education and Development,* *19*(1), 5–29.

Mata-Pereira, J. & Ponte, J. P. (2018). Promover o Raciocínio Matemático dos alunos: uma investigação baseada em design. *Bolema*, *32*(62), 781–801.

Mosquito, E. M. L. (2008). *Práticas Lectivas dos Professores de Matemática do 3º Ciclo do ensino básico.* Dissertação (Mestrado em Educação Especialidade de Didática da Matemática) – Universidade de Lisboa, Lisboa.

Oliveira, P. (2008). O raciocínio matemático à luz de uma epistemologia soft. *Educação e Matemática, 100,* 3–9.

Oliveira, I, & Serrazina, L. (2002). A reflexão e o professor como investigador. In: GTI (Org.), *Reflectir e investigar sobre a prática profissional*. Lisboa: APM.

Ponte, J. P. (2005). Gestão curricular em Matemática. In: GTI (Ed.), *O professor e o desenvolvimento curricular* (pp. 11-34). Lisboa: APM.

Ponte, J.P., Mata-Pereira, J. & Henriques, A. (2012). O raciocínio matemático nos alunos do Ensino Básico e do Ensino Superior. *Práxis Educativa, 7*(2), 355–377.

Ribeiro, A. J., Aguiar, M. & Trevisan, A. L. (2020). Oportunidades de aprendizagem vivenciadas por professores ao discutir coletivamente uma aula sobre padrões e regularidades. *Quadrante, 29*, 52–73.

Ribeiro, A. J. & Ponte, J. P. M. (2020). Um modelo teórico para organizar e compreender as oportunidades de aprendizagem de professores para ensinar matemática. *Zetetiké,* *28*, 1–20.

Rodrigues, M., Brunheira, L. & Serrazina, L. (2021). A framework for prospective primary teachers’ knowledge of mathematical reasoning processes. *International Journal of Educational Research,* *107*, 1–11.

Smith, M. S. (2001). *Practice-Basead Professional Development for Teachers of Mathematics.* Reston, Virgínia: National Council of Teachers of Mathematics.

Vieira, W., Rodrigues, M. & Serrazina, L. (2020). O conhecimento de futuros professores sobre os processos de raciocínio matemático antes e depois de uma experiência de formação. *Quadrante,* *29*(1), 8–35.